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I. Ya. Kmit
(Inst. Appl. Problems Mech. and Math. Nat. Acad. Sci. Ukraine, Lviv and Humboldt Univ. Berlin, Germany),
L. Recke (Humboldt Univ. Berlin, Germany),
V. I. Tkachenko (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

ROBUSTNESS OF EXPONENTIAL DICHOTOMIES OF BOUNDARY-VALUE PROBLEMS FOR GENERAL FIRST-ORDER HYPERBOLIC SYSTEMS*

ГРУБІСТЬ ЕКСПОНЕНЦІАЛЬНИХ ДИХОТОМІЙ КРАЙОВИХ ЗАДАЧ ДЛЯ ЗАГАЛЬНИХ ГІПЕРБОЛІЧНИХ СИСТЕМ ПЕРШОГО ПОРЯДКУ

We examine the robustness of exponential dichotomies of boundary-value problems for general linear first-order one-dimensional hyperbolic systems. It is assumed that the boundary conditions guarantee an increase in the smoothness of solutions in a finite time interval, which includes reflection boundary conditions. We show that the dichotomy survives in the space of continuous functions under small perturbations of all coefficients in the differential equations.

Вивчається грубість експоненціальної дихотомії для крайових задач для загальних лінійних гіперболічних систем першого порядку. Припускається, що крайові умови забезпечують підвищення гладкості розв'язків за скінченний проміжок часу, що дозволяє також розглядати умови відбиття від межі області. Показано, що дихотомія зберігається у просторі неперервних функцій при малих збуреннях всіх коефіцієнтів диференціальних рівнянь.

1. Introduction and main results. The concept of exponential dichotomy plays a crucial role in various aspects of the perturbation and the stability theory [3, 4, 17 – 19]. An important problem here is robustness of the exponential dichotomy of a system, i.e., its stability with respect to small perturbations in the system. This problem is extensively studied in the literature, e.g., in [6, 13, 14, 20] for finite-dimensional case and in [2, 7, 15] for infinite-dimensional case. It should be noted that the hyperbolic case (see, e.g., [16]) seems more complicated here in comparison to ODEs and parabolic PDEs, mostly due to worse regularity properties of hyperbolic operators.

We address the issue of stability of exponential dichotomies for general linear one-dimensional first-order hyperbolic systems

$$\big(\partial_t + a(x,t,\varepsilon)\partial_x + b(x,t,\varepsilon)\big)u = 0, \quad x \in (0,1), \tag{1.1}$$

subjected to (nonlocal) boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=m+1}^{n} p_{jk}(t)u_k(0,t) + \sum_{k=1}^{m} p_{jk}(t)u_k(1,t), \quad 1 \le j \le m, \\ u_j(1,t) &= \sum_{k=1}^{m} q_{jk}(t)u_k(0,t) + \sum_{k=m+1}^{n} q_{jk}(t)u_k(1,t), \quad m < j \le n. \end{aligned} \tag{1.2}$$

Here $u = (u_1, \ldots, u_n)$ is a vector of real-valued functions, $a = \operatorname{diag}(a_1, \ldots, a_n)$ and $b = \{b_{jk}\}_{j,k=1}^n$ are matrices of real-valued functions, and $0 \le m \le n$ are fixed integers.

Set

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$$\Pi = \{(x,t)\colon 0 < x < 1,\ -\infty < t < \infty\}.$$

Assume that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ and all $(x,t) \in \overline{\Pi}$ the following conditions are fulfilled:

$$a_j,\ b_{jk},\ p_{jk},\ q_{jk} \quad \text{are continuously differentiable in} \quad x,\ t,\ \varepsilon \quad \text{for all} \quad j,k \le n, \tag{1.3}$$

$$a_j > 0 \quad \text{for all} \quad j \le m \quad \text{and} \quad a_j < 0 \quad \text{for all} \quad j > m, \tag{1.4}$$

$$\inf_{x,t} |a_j| > 0 \quad \text{for all} \quad j \le n, \tag{1.5}$$

$$\sup_{x,t} \{|a_j|, |\partial_x a_j|, |\partial_t a_j|, |\partial_\varepsilon a_j|\} < \infty \quad \text{for all} \quad j \le n, \tag{1.6}$$

$$\sup_{x,t} \{|p_{jk}|, |q_{jk}|, |b_{jk}|, |\partial_\varepsilon b_{jk}|, |\partial_t b_{jk}|\} < \infty \quad \text{for all} \quad j,k \le n, \tag{1.7}$$

$$\begin{gathered} \text{for all} \quad 1 \le j \ne k \le n \quad \text{there exist} \quad \beta_{jk}, \gamma_{jk} \in C^1\left([0,1] \times \mathbb{R} \times [0, \varepsilon_0)\right) \\ \text{such that} \quad b_{jk}(x,t,0) = \beta_{jk}(x,t,\varepsilon)\left(a_k(x,t,\varepsilon) - a_j(x,t,0)\right) \\ \text{and} \quad b_{jk}(x,t,\varepsilon) = \gamma_{jk}(x,t,\varepsilon)\left(a_k(x,t,\varepsilon) - a_j(x,t,\varepsilon)\right), \end{gathered} \tag{1.8}$$

and

$$\sup_{x,t} \left\{|\partial_x \beta_{jk}|, |\partial_t \beta_{jk}|, |\partial_x \gamma_{jk}|, |\partial_t \gamma_{jk}|\right\} < \infty \quad \text{for all} \quad j \ne k. \tag{1.9}$$

Given $s \in \mathbb{R}$, set

$$\Pi_s = \left\{(x,t)\colon 0 < x < 1, s < t < \infty\right\}.$$

We subject the system (1.1), (1.2) by the initial conditions at time $t = s$:

$$u(x,s) = \varphi(x), \quad x \in [0,1], \tag{1.10}$$

and consider the initial boundary-value problem (1.1), (1.2), (1.10) in Π_s for arbitrarily fixed $s \in \mathbb{R}$. Now we intend to switch to a weak formulation of the latter using integration along characteristic curves: For given $j \le n$, $x \in [0,1]$, $t \in \mathbb{R}$, and $\varepsilon \in [0, \varepsilon_0]$ the j-th characteristic of (1.1) passing through the point $(x,t) \in \overline{\Pi}_s$ is defined as the solution $\xi \in [0,1] \mapsto \omega_j(\xi; x,t,\varepsilon) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \omega_j(\xi; x,t,\varepsilon) = \frac{1}{a_j(\xi, \omega_j(\xi; x,t,\varepsilon), \varepsilon)}, \quad \omega_j(x; x,t,\varepsilon) = t. \tag{1.11}$$

Write

$$c_j(\xi, x, t, \varepsilon) = \exp \int_x^\xi \left(\frac{b_{jj}}{a_j}\right)(\eta, \omega_j(\eta; x,t,\varepsilon), \varepsilon)\, d\eta, \qquad d_j(\xi, x, t, \varepsilon) = \frac{c_j(\xi, x, t, \varepsilon)}{a_j(\xi, \omega_j(\xi; x,t,\varepsilon), \varepsilon)}.$$

Due to (1.5), the characteristic curve $\tau = \omega_j(\xi; x, t, \varepsilon)$ reaches the boundary of Π_s in two points with distinct ordinates. Let $x_j(x, t, \varepsilon)$ denote the abscissa of that point whose ordinate is smaller. Let us introduce linear bounded operators $R\colon C\left(\overline{\Pi}_s\right)^n \mapsto C\left([s,\infty)\right)^n$ and $B^\varepsilon\colon C\left(\overline{\Pi}_s\right)^n \mapsto C\left(\overline{\Pi}_s\right)^n$ and an affine bounded operator $S\colon C\left(\overline{\Pi}_s\right)^n \mapsto C\left(\overline{\Pi}_s\right)^n$ by

$$\begin{aligned} (Ru)_j(t) &= \sum_{k=m+1}^{n} p_{jk}(t)u_k(0,t) + \sum_{k=1}^{m} p_{jk}(t)u_k(1,t), \quad 1 \le j \le m, \\ (Ru)_j(t) &= \sum_{k=1}^{m} q_{jk}(t)u_k(0,t) + \sum_{k=m+1}^{n} q_{jk}(t)u_k(1,t), \quad m < j \le n, \end{aligned} \tag{1.12}$$

$$(B^\varepsilon u)_j(x,t) = c_j(x_j(x,t,\varepsilon), x, t, \varepsilon) u_j\left(x_j(x,t,\varepsilon), \omega_j(x_j(x,t,\varepsilon); x, t, \varepsilon)\right), \tag{1.13}$$

and

$$(Su)_j(x,t) = \begin{cases} (Ru)_j(t) & \text{if } \ t > s, \\ \varphi_j(x) & \text{if } \ t = s. \end{cases} \tag{1.14}$$

By abuse of notation, we did not indicate the dependence of the above operators on s; in fact, in the consideration below the value of $s \in \mathbb{R}$ will be arbitrarily fixed.

Straightforward calculations show that a C^1-map $u\colon [0,1] \times [0,\infty) \to \mathbb{R}^n$ is a solution to (1.1), (1.2), (1.10) if and only if it satisfies the following system of integral equations:

$$u_j(x,t) = (B^\varepsilon S u)_j(x,t) -$$

$$- \int\limits_{x_j(x,t,\varepsilon)}^{x} d_j(\xi, x, t, \varepsilon) \sum_{\substack{k=1 \\ k \ne j}}^{n} b_{jk}(\xi, \omega_j(\xi; x, t, \varepsilon), \varepsilon) u_k(\xi, \omega_j(\xi; x, t, \varepsilon)) d\xi, \quad j \le n. \tag{1.15}$$

Now, the notion of weak (continuous) solution in Π_s can be naturally defined as follows.

Definition 1.1. *A continuous function u is called a continuous solution to* (1.1), (1.2), (1.10) *in $\overline{\Pi}_s$ if it satisfies* (1.15).

For given $\varepsilon > 0$, denote by $U^\varepsilon(t,s)\colon C([0,1])^n \mapsto C([0,1])^n$ the evolution operator of the system (1.1), (1.2) whose existence is given by Theorem 2.1, i.e., a bounded operator mapping the values of solutions at time s into their values at time t and satisfying the properties $U^\varepsilon(s,s) = I$ and $U^\varepsilon(t,s)U^\varepsilon(s,\tau) = U^\varepsilon(t,\tau)$ for all $t \ge s \ge \tau$.

We examine robustness of exponential dichotomies for a range of boundary operators ensuring that smoothness of solutions increases in finite time. With this aim we will assume that the system (1.1), (1.2) has a smoothing property studied in [9, 10].

Definition 1.2. *Let $\varepsilon > 0$. The evolution operator $U^\varepsilon(t,s)$ to the problem* (1.1), (1.2) *is called smoothing if, for every $s \in \mathbb{R}$, there exists $t > s$ such that $U^\varepsilon(t,s)\varphi \in C^1\left([0,1]\right)^n$ for every $\varphi \in C\left([0,1]\right)^n$.*

In the following definition the range of an operator P will be denoted by $\operatorname{Im} P$.

Definition 1.3. *Let* $\varepsilon > 0$. *We say that the system* (1.1), (1.2) *has an exponential dichotomy on* $\mathbb{R}$ *with exponent* $\beta > 0$ *and bound* M *if there exist projections* $P^\varepsilon(t)$, $t \in \mathbb{R}$, *such that*

(i) $U^\varepsilon(t,s)P^\varepsilon(s) = P^\varepsilon(t)U^\varepsilon(t,s)$, $t \geq s$;

(ii) $U^\varepsilon(t,s)|_{\mathrm{Im}(P^\varepsilon(s))}$ *for* $t \geq s$ *is an isomorphism on* $\mathrm{Im}(P^\varepsilon(s))$, *then* $U^\varepsilon(s,t)$ *is defined as an inverse map from* $\mathrm{Im}(P^\varepsilon(t))$ *to* $\mathrm{Im}(P^\varepsilon(s))$;

(iii) $\|U^\varepsilon(t,s)(1 - P^\varepsilon(s))\| \leq Me^{-\beta(t-s)}$, $t \geq s$;

(iv) $\|U^\varepsilon(t,s)P^\varepsilon(s)\| \leq Me^{\beta(t-s)}$, $t \leq s$.

Here and below by $\|\cdot\|$ we denote the operator norm in $\mathcal{L}\left(C([0,1])^n\right)$.

Now we formulate our main result.

Theorem 1.1. *Suppose that the system* (1.1), (1.2) *with* $\varepsilon = 0$ *has an exponential dichotomy and the corresponding evolution operator* $U^0(t,s)$ *is bounded*:

$$\sup_{0\leq t-s\leq 1} \|U^0(t,s)\| < \infty. \tag{1.16}$$

Moreover, assume that there is $\varepsilon_0 > 0$ *such that the following conditions are fulfilled*: (1.3) – (1.9) *and*

$$\text{there exists} \quad k \in \mathbb{N} \quad \text{such that} \quad (B^\varepsilon R)^k = 0 \quad \text{for all} \quad \varepsilon \leq \varepsilon_0. \tag{1.17}$$

Then there exists $\varepsilon' \leq \varepsilon_0$ *such that for all* $\varepsilon \leq \varepsilon'$ *the system* (1.1), (1.2) *has an exponential dichotomy.*

***Remark* 1.1.** Note that the boundary conditions (1.2) together with the property (1.17) generalize boundary conditions appearing in models of chemical kinetics [1, 21].

2. Basic facts. The first fact follows from the results obtained in [8, 11] and entails, in particular, the existence of an evolution operator.

Theorem 2.1. *Under the conditions* (1.3) – (1.9), *for given* $\varepsilon > 0$, $s \in \mathbb{R}$, $T > 0$, *and* $\varphi \in C\left([0,1]\right)^n$ *fulfilling the zero-order compatibility conditions*

$$\begin{aligned} \varphi_j(0) &= (R\varphi)_j(s), \quad 1 \leq j \leq m, \\ \varphi_j(1) &= (R\varphi)_j(s), \quad m < j \leq n, \end{aligned} \tag{2.1}$$

the initial boundary-value problem (1.1), (1.2), (1.10) *has a unique continuous solution in* Π_s *and this solution satisfies the apriori estimate*

$$\|u\|_{C\left(\overline{\Pi}_s \setminus \Pi_{s+T}\right)^n} \leq C(T)\|\varphi\|_{C([0,1])^n} \tag{2.2}$$

with a constant $C(T) > 0$ *depending on* T, *but not on* s, φ, *and* $\varepsilon \leq \varepsilon_0$.

The second fact can be readily obtained by [10] (Theorem 2.7) and the argument used in its proof. It states the smoothing property of the evolution operator as well as the fact that the time at which the continuous solution to (1.1), (1.2), (1.10) reaches the C^1-regularity does not exceed a fix number d, whatsoever initial time $s \in \mathbb{R}$.

Lemma 2.1. *Under the conditions* (1.3) – (1.9) *and* (1.17) *the evolution operator is smoothing and satisfies the following property*:

$$\begin{aligned} &\text{there exists} \quad d > 0 \quad \text{such that for any} \quad s \in \mathbb{R} \quad \text{and} \quad t \quad \text{as in Definition 1.2}, \\ &\text{the inequality} \quad |t - s| \leq d \quad \text{is true for all} \quad \varepsilon \leq \varepsilon_0. \end{aligned} \tag{2.3}$$

The third fact is a variant of [5] (Theorem 7.6.10).

Theorem 2.2. *Assume that the evolution operator $U^0(t,s)$ has an exponential dichotomy on $\mathbb{R}$ and satisfies* (1.16). *Then there exists $\eta > 0$ such that for all $\varepsilon > 0$ with*

$$\|U^0(t,s) - U^\varepsilon(t,s)\| < \eta, \quad \textit{whenever} \quad t - s = 2d$$

the evolution operator $U^\varepsilon(t,s)$ has an exponential dichotomy on $\mathbb{R}$ also.

Proof. Given $s \in \mathbb{R}$ and $\varepsilon > 0$, set

$$t_n = s + 2dn, \qquad T_n^\varepsilon = U^\varepsilon(t_0 + 2d(n+1), t_0 + 2dn) \qquad \text{for} \quad n \in \mathbb{Z}.$$

If the evolution operator $U^0(t,s)$ has an exponential dichotomy, then the sequence $\{T_n^0\}$ has a discrete dichotomy in the sense of [5] (Definition 7.6.4).

By [5] (Theorem 7.6.7), there exists $\eta > 0$ such that for all $\varepsilon > 0$ with

$$\sup_n \|T_n^0 - T_n^\varepsilon\| \le \eta$$

$\{T_n^\varepsilon\}$ has a discrete dichotomy.

Now we are in the conditions of [5, p. 229, 230], Excersise 10 (see also a more general statement [7], Theorem 4.1), what finishes the proof.

3. Proof of Theorem 1.1. Given $s \in \mathbb{R}$ and $\varepsilon > 0$, let us introduce linear bounded operators D^ε, $F^\varepsilon \colon C\left(\overline{\Pi}_s\right)^n \to C\left(\overline{\Pi}_s\right)^n$ by

$$\left(D^\varepsilon w\right)_j(x,t) = -\int\limits_{x_j(x,t,\varepsilon)}^{x} d_j(\xi,x,t,\varepsilon) \sum_{\substack{k=1\\ k\neq j}}^{n} b_{jk}(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon) w_k(\xi,\omega_j(\xi;x,t,\varepsilon))\, d\xi,$$

$$\left(F^\varepsilon f\right)_j(x,t) = \int\limits_{x_j(x,t,\varepsilon)}^{x} d_j(\xi,x,t,\varepsilon) f_j(\xi,\omega_j(\xi;x,t,\varepsilon)) d\xi.$$

Here again we dropped the dependence of D^ε and F^ε on s, as throughout the proof $s \in \mathbb{R}$ is arbitrarily fixed. To simplify further notation, set

$$a(x,t) = a(x,t,0), \qquad b(x,t) = b(x,t,0), \qquad c_j(x,t) = c_j(x,t,0), \qquad d_j(x,t) = d_j(x,t,0),$$

$$a^\varepsilon(x,t) = a(x,t,\varepsilon), \qquad b^\varepsilon(x,t) = b(x,t,\varepsilon), \qquad \beta_{jk}^\varepsilon(x,t) = \beta_{jk}(x,t,\varepsilon),$$

$$\omega_j(\xi;x,t) = \omega_j(\xi;x,t,0), \qquad x_j(x,t) = x_j(x,t,0), \quad D = D^0, \quad F = F^0. \tag{3.1}$$

Fix arbitrary values $s \in \mathbb{R}$ and $\varepsilon \le \varepsilon_0$ and an arbitrary initial function $\varphi \in C\left([0,1]\right)^n$ in (1.10). Let u and v be the continuous solutions to the problem (1.1), (1.2), (1.10) with $\varepsilon = 0$ and ε, respectively. By Lemma 2.1, the evolution operator $U^\varepsilon(t,s)$ is smoothing with the time of smoothing not exceeding d. This means that starting at $t = s + d$ the solutions u and v are continuously differentiable and, therefore, satisfy the system (1.1) pointwise. Hence, the difference $u - v$ fulfills the equation

$$(\partial_t + a(x,t)\partial_x + b(x,t))(u-v) = (a^\varepsilon(x,t) - a(x,t))\,\partial_x v + (b^\varepsilon(x,t) - b(x,t))\,v, \quad (x,t) \in \Pi_{s+d}, \tag{3.2}$$

and the boundary conditions

$$\begin{aligned} (u_j - v_j)(0,t) &= (R(u-v))_j\,(t), \quad 1 \le j \le m, \quad t \ge s, \\ (u_j - v_j)(1,t) &= (R(u-v))_j\,(t), \quad m < j \le n, \quad t \ge s, \end{aligned} \tag{3.3}$$

or, the same, the operator equation

$$(u-v)\big|_{\overline{\Pi}_{s+d}} = BR(u-v) + D(u-v) + F\left((a^\varepsilon - a)\,\partial_x v\right) + F\left((b^\varepsilon - b)\,v\right). \tag{3.4}$$

A similar equation is true for $u - v$ under the operator BR, what entails

$$(u-v)\big|_{\overline{\Pi}_{s+d}} = (BR)^2(u-v) + (I + BR)D(u-v) +$$

$$+(I+BR)F\left((a^\varepsilon - a)\,\partial_x v\right) + (I+BR)F\left((b^\varepsilon - b)\,v\right).$$

Doing this iteration, on the k-th step we meet the property (see (1.17))

$$(BR)^k\,(u-v) \equiv 0 \tag{3.5}$$

and, hence, get the formula

$$(u-v)\big|_{\overline{\Pi}_{s+d}} = \sum_{i=0}^{k-1}(BR)^i D(u-v) + \sum_{i=0}^{k-1}(BR)^i F\left((a^\varepsilon - a)\,\partial_x v\right) + \sum_{i=0}^{k-1}(BR)^i F\left((b^\varepsilon - b)\,v\right).$$

In particular,

$$(u-v)(x, s+2d) = \sum_{i=0}^{k-1}\left[(BR)^i D(u-v)\right](x, s+2d) +$$

$$+\sum_{i=0}^{k-1}\left[(BR)^i F\left((a^\varepsilon - a)\,\partial_x v\right)\right](x, s+2d) + \sum_{i=0}^{k-1}\left[(BR)^i F\left((b^\varepsilon - b)\,v\right)\right](x, s+2d). \tag{3.6}$$

Therefore, on the account of Theorem 2.2, we are done if we show that, given $\eta > 0$, there is $\varepsilon' \le \varepsilon_0$ such that

$$\|(u-v)(\cdot, s+2d)\|_{C([0,1])^n} \le \eta \|\varphi\|_{C([0,1])^n}, \tag{3.7}$$

the bound being uniform in $s \in \mathbb{R}$, $\varepsilon \le \varepsilon'$, and $\varphi \in C\left([0,1]\right)^n$. To derive (3.7), we estimate each of the three sums in (3.6) separately.

To obtain the desired estimate for the first sum in (3.6), we first derive the formula for $D(u-v)$ contributing into this summand. To this end, use the operator representation for u and v, namely,

$$u = BSu + Du, \qquad v = B^\varepsilon S v + D^\varepsilon v,$$

where the functions u and v are restricted to $\overline{\Pi}_s \setminus \Pi_{s+2d}$ and the operators B^ε, S, and D^ε are restricted to $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n$. Note that, as it follows from the definition, B^ε, S, and D^ε map

$C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n$ into $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n$. Thus, for the difference we have

$$u - v = BS(u - v) + (B - B^\varepsilon)\, Sv + D(u - v) + (D - D^\varepsilon)v, \tag{3.8}$$

hence

$$D(u - v) = DBS(u - v) + D\,(B - B^\varepsilon)\, Sv + D^2(u - v) + D(D - D^\varepsilon)v. \tag{3.9}$$

Our next objective is to rewrite the last equation with respect to the new variable

$$w = D(u - v). \tag{3.10}$$

With this aim we substitute (3.8) into the first summand in the right-hand side of (3.9) and get

$$w = D(BS)^2(u - v) + D(I + BS)\,(B - B^\varepsilon)\, Sv + D(I + BS)w + D(I + BS)(D - D^\varepsilon)v. \tag{3.11}$$

Continuing in this fashion (again substituting (3.8) into the first summand in the right-hand side of (3.11)), on the r-th step we arrive at the formula

$$w = D(BS)^r(u - v) + D\sum_{i=0}^{r-1}(BS)^i\,(B - B^\varepsilon)\, Sv +$$

$$+ D\sum_{i=0}^{r-1}(BS)^i w + D\sum_{i=0}^{r-1}(BS)^i(D - D^\varepsilon)v. \tag{3.12}$$

Since $(u - v)(\cdot, s) \equiv 0$ on $[0, 1]$, there exists $r_0 \in \mathbb{N}$ such that $(BS)^{r_0}(u - v) = 0$. Therefore, the resulting equation for w restricted to $\overline{\Pi}_s \setminus \Pi_{s+2d}$ can be written as

$$w = D\sum_{i=0}^{r_0-1}(BS)^i\,(B - B^\varepsilon)\, Sv + D\sum_{i=0}^{r_0-1}(BS)^i(D - D^\varepsilon)v + D\sum_{i=0}^{r_0-1}(BS)^i w. \tag{3.13}$$

Our goal now is to show the existence of a function $\alpha\colon [0, 1] \to \mathbb{R}$ with $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ for which we have

$$\|w\|_{C\left(\overline{\Pi}_s\setminus\Pi_{s+2d}\right)^n} \le \alpha(\varepsilon)\|\varphi\|_{C([0,1])^n}, \tag{3.14}$$

the estimate being uniform in $s \in \mathbb{R}$ and $\varphi \in C([0, 1])^n$ satisfying the zero-order compatibility conditions (2.1). With this aim we first show that there is a function $\tilde{\alpha}(\varepsilon)$ meeting the same properties as $\alpha(\varepsilon)$ such that the $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n$-norm of the first two summands in the right-hand side of (3.13) is bounded from above by $\tilde{\alpha}(\varepsilon)\|\varphi\|_{C([0,1])^n}$. Afterwords, we use the boundedness of the operators B, S, D restricted to $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n$, then apply Gronwall's inequality to (3.13), and this way derive (3.14). To this end, observe that the integral operator D can be considered as Volterra operator of the second kind. This follows from the fact that D can be equivalently defined by the formula

$$(Dw)_j\,(x, t) = -\int\limits_{t_j(x,t)}^{t} \tilde{d}_j(\tau, x, t) \sum_{\substack{k=1\\k\ne j}}^{n} b_{jk}(\tilde{\omega}_j(\tau; x, t), \tau) w_k(\tilde{\omega}_j(\tau; x, t), \tau)\, d\tau,$$

where $\tau \in \mathbb{R} \mapsto \tilde{\omega}_j(\tau; x, t) \in [0, 1]$ is the inverse form of the j-th characteristic of (1.1) passing through the point $(x, t) \in \overline{\Pi}$, $t_j(x, t)$ is a minimum value of τ at which the characteristic $\tau = \tilde{\omega}_j(\tau; x, t)$ reaches $\partial\Pi_s$, and

$$\tilde{d}_j(\tau, x, t) = \exp \int_t^\tau b_{jj}(\tilde{\omega}_j(\eta; x, t), \eta)\, d\eta.$$

Thus, the estimate (3.14) will be proved as soon as we derive the upper bound $\tilde{\alpha}(\varepsilon)\|\varphi\|_{C([0,1])^n}$ for the absolute value of the first two summands in (3.13). The idea behind the proof is a smoothing property of the operators representing those summands. We prove this only for one summand in each sum (when $i = 0$). For all other summands we apply similar argument.

Thus, to get the desired estimate for the summand $D(D - D^\varepsilon)v$, it suffices to show that, given $j \le n$, the function $(DD^\varepsilon v)_j(x, t)$ is continuously differentiable in ε and that the derivative is bounded on $\overline{\Pi}_s \setminus \Pi_{s+2d}$ uniformly in $s \in \mathbb{R}$ and $\varepsilon \le \varepsilon_0$. Indeed, following the techniques from [9], fix a sequence $v^l \in C^1\left(\overline{\Pi}\right)^n$ such that

$$v^l \to v \quad \text{in} \quad C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)^n \quad \text{as} \quad l \to \infty. \tag{3.15}$$

We are done if we prove that $\partial_\varepsilon \left[\left(DD^\varepsilon v^l\right)_j(x, t)\right]$ converges in $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)$ as $l \to \infty$ and that the limit function is bounded on $\overline{\Pi}_s \setminus \Pi_{s+2d}$ uniformly in $s \in \mathbb{R}$ and $\varepsilon \le \varepsilon_0$. Consider the following expression for $\left(DD^\varepsilon v^l\right)_j(x, t)$:

$$\left(DD^\varepsilon v^l\right)_j(x, t) = \sum_{\substack{k=1\\k\ne j}}^n \sum_{\substack{i=1\\i\ne k}}^n \int\limits_{x_j(x,t)}^{x} \int\limits_{x_k(\xi, \omega_j(\xi; x,t), \varepsilon)}^{\xi} d_{jki}(\xi, \eta, x, t, \varepsilon) b_{jk}(\xi, \omega_j(\xi; x, t)) \times$$

$$\times v_i^l(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t), \varepsilon))\, d\eta d\xi \tag{3.16}$$

with

$$d_{jki}(\xi, \eta, x, t, \varepsilon) = d_j(\xi, x, t) d_k(\eta, \xi, \omega_j(\xi; x, t), \varepsilon) b_{ki}(\eta, \omega_k(\eta; \xi, \omega_j(\xi; x, t), \varepsilon), \varepsilon).$$

Let $x_{jk}(x, t, \varepsilon)$ denote the x-coordinate of the point (if any) where the characteristics $\omega_j(\xi; x, t)$ and $\omega_k(\xi; 0, s, \varepsilon)$ if $k \le m$ and the characteristics $\omega_j(\xi; x, t, \varepsilon)$ and $\omega_k(\xi; 1, s, \varepsilon)$ if $k > m$ intersect. Hence, $x_{jk}(x, t, \varepsilon)$ satisfies the equation

$$\omega_j(x_{jk}(x, t, \varepsilon); x, t) = \omega_k(x_{jk}(x, t, \varepsilon); 0, s, \varepsilon) \tag{3.17}$$

if $k \le m$, and the equation

$$\omega_j(x_{jk}(x, t, \varepsilon); x, t) = \omega_k(x_{jk}(x, t, \varepsilon); 1, s, \varepsilon) \tag{3.18}$$

if $k > m$. Suppose for definiteness that $j \le m$ and $k > m$ (similar argument works for all other $j \ne k$). Thus, if $x_{jk}(x, t, \varepsilon)$ exists for some (x, t, ε), then the integrals in (3.16) admit the decomposition

$$\int\limits_{x_j(x,t)}^{x} \int\limits_{x_k(\xi,\omega_j(\xi;x,t),\varepsilon)}^{\xi} d\eta d\xi = \int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} \int\limits_{x_k(\xi,\omega_j(\xi;x,t),\varepsilon)}^{\xi} d\eta d\xi + \int\limits_{x_{jk}(x,t,\varepsilon)}^{x} \int\limits_{1}^{\xi} d\eta d\xi, \tag{3.19}$$

where the function $x_k(\xi, \omega_j(\xi; x, t), \varepsilon)$ in the right-hand side satisfies the equality

$$\omega_k(x_k(\xi, \omega_j(\xi; x, t), \varepsilon); \xi, \omega_j(\xi; x, t), \varepsilon) = s. \tag{3.20}$$

Now we intend to show that the derivatives $\partial_\varepsilon x_k(\xi, \omega_j(\xi; x, t), \varepsilon)$ and $\partial_\varepsilon x_{jk}(x, t, \varepsilon)$ exist. With this aim we introduce a couple of useful formulas

$$\partial_x \omega_j(\xi; x, t, \varepsilon) = -\frac{1}{a_j(x, t, \varepsilon)} \exp \int\limits_{\xi}^{x} \left(\frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t, \varepsilon), \varepsilon) d\eta, \tag{3.21}$$

$$\partial_t \omega_j(\xi; x, t, \varepsilon) = \exp \int\limits_{\xi}^{x} \left(\frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t, \varepsilon), \varepsilon) d\eta, \tag{3.22}$$

$$\partial_\varepsilon \omega_j(\xi; x, t, \varepsilon) = \exp \int\limits_{\xi}^{x} \left(\frac{\partial_t a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t, \varepsilon), \varepsilon) d\eta \times$$

$$\times \int\limits_{\xi}^{x} \left(\frac{\partial_\varepsilon a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t, \varepsilon), \varepsilon) \times$$

$$\times \exp \int\limits_{x}^{\eta} \left(\frac{\partial_t a_j}{a_j^2} \right) (\eta_1, \omega_j(\eta_1; x, t, \varepsilon), \varepsilon) \, d\eta_1 \, d\eta. \tag{3.23}$$

Then the existence of the derivatives $\partial_\varepsilon x_k(\xi, \omega_j(\xi; x, t), \varepsilon)$ and $\partial_\varepsilon x_{jk}(x, t, \varepsilon)$ follow from the equalities (3.20) and (3.18), respectively. Furthermore, we derive the formulas

$$\partial_\varepsilon x_k(\xi, \omega_j(\xi; x, t), \varepsilon) = -a_k(x_k(\xi, \omega_j(\xi; x, t), \varepsilon), s) \partial_4 \omega_k(x_k(\xi, \omega_j(\xi; x, t), \varepsilon); \xi, \omega_j(\xi; x, t), \varepsilon) \tag{3.24}$$

and

$$\partial_\varepsilon x_{jk}(x, t, \varepsilon) \left(\frac{a_k^\varepsilon - a_j}{a_j a_k^\varepsilon} \right) (x_{jk}(x, t, \varepsilon), \omega_j(x_{jk}(x, t, \varepsilon); x, t)) = \partial_4 \omega_k(x_{jk}(x, t, \varepsilon); 1, s, \varepsilon). \tag{3.25}$$

Hence, on the account of the assumption (1.8), from the last equality we get

$$\partial_\varepsilon x_{jk}(x, t, \varepsilon) b_{jk}(x_{jk}(x, t, \varepsilon), \omega_j(x_{jk}(x, t, \varepsilon); x, t)) =$$

$$= \left(\beta_{jk}^\varepsilon a_j a_k^\varepsilon \right) (x_{jk}(x, t, \varepsilon), \omega_j(x_{jk}(x, t, \varepsilon); x, t)) \partial_4 \omega_k(x_{jk}(x, t, \varepsilon); 1, s, \varepsilon). \tag{3.26}$$

Now, using the regularity assumption (1.3), we are able to compute the derivative

$$\partial_\varepsilon \left[(DD^\varepsilon v^l)_j(x,t) \right] =$$

$$= \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \int\limits_{x_j(x,t)}^{x} \int\limits_{x_k(\xi,\omega_j(\xi;x,t),\varepsilon)}^{x} \partial_\varepsilon \Big[d_{jki}(\xi,\eta,x,t,\varepsilon) b_{jk}(\xi,\omega_j(\xi;x,t)) \Big] \times$$

$$\times v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))\, d\eta d\xi +$$

$$+ \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \int\limits_{x_j(x,t)}^{x} \int\limits_{x_k(\xi,\omega_j(\xi;x,t),\varepsilon)}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon) b_{jk}(\xi,\omega_j(\xi;x,t)) \times$$

$$\times \partial_\varepsilon \omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon) \partial_2 v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))\, d\eta d\xi +$$

$$+ \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \left(\beta_{jk}^\varepsilon a_j a_k^\varepsilon \right) (x_{jk}(x,t,\varepsilon), \omega_j(x_{jk}(x,t,\varepsilon);x,t)) \partial_4 \omega_j(x_{jk}(x,t,\varepsilon);1,s,\varepsilon) \times$$

$$\times \int\limits_{x_k(x_{jk}(x,t,\varepsilon),\omega_j(x_{jk}(x,t,\varepsilon);x,t),\varepsilon)}^{\xi} \Big[d_{jki}(\xi,\eta,x,t,\varepsilon) v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)) \Big]_{\xi=x_{jk}(x,t,\varepsilon)} d\eta -$$

$$- \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} \partial_\varepsilon x_k(\xi,\omega_j(\xi;x,t),\varepsilon) b_{jk}(\xi,\omega_j(\xi;x,t)) \times$$

$$\times \Big[d_{jki}(\xi,\eta,x,t,\varepsilon) v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)) \Big]_{\eta=x_k(\xi,\omega_j(\xi;x,t),\varepsilon)} d\xi -$$

$$- \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \left(\beta_{jk}^\varepsilon a_j a_k^\varepsilon \right) (x_{jk}(x,t,\varepsilon), \omega_j(x_{jk}(x,t,\varepsilon);x,t)) \partial_4 \omega_j(x_{jk}(x,t,\varepsilon);1,s,\varepsilon) \times$$

$$\times \int\limits_{1}^{\xi} \Big[d_{jki}(\xi,\eta,x,t,\varepsilon) v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)) \Big]_{\xi=x_{jk}(x,t,\varepsilon)} d\eta, \tag{3.27}$$

where $\partial_r g$ here and below denotes the derivative of g with respect to the r-th argument. Note that $x_k(x_{jk}(x,t,\varepsilon), \omega_j(x_{jk}(x,t,\varepsilon);x,t),\varepsilon) = 1$, hence the third and the fifth summands in the right-hand side cancel out. The first and the fourth summands converge in $C\left(\overline{\Pi}_s \setminus \Pi_{s+2d}\right)$ as $l \to \infty$. Our task is therefore reduced to show the uniform convergence of all integrals in the second summand. For this purpose we will transform the integrals as follows: Changing the order of integration and using (1.3) and (1.8), we get (to simplify notation in the calculation below we drop the dependence of x_j on x and t)

$$\int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon) b_{jk}(\xi,\omega_j(\xi;x,t))\partial_\varepsilon\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_2 v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))\,d\xi d\eta=$$

$$=\int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_\varepsilon\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon) b_{jk}(\xi,\omega_j(\xi;x,t))\times$$

$$\times\Big[\big(\partial_\xi\omega_k\big)(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\Big]^{-1}\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi d\eta=$$

$$=\int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_\varepsilon\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))\Big(\beta_{jk}^\varepsilon a_j a_k^\varepsilon\Big)(\xi,\omega_j(\xi;x,t))\times$$

$$\times\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi d\eta=$$

$$=\int_{x_j}^{x}\int_{\eta}^{x} \tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi d\eta=$$

$$=-\int_{x_j}^{x}\int_{\eta}^{x} \partial_\xi\tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)v_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\right)d\xi d\eta+$$

$$+\int_{x_j}^{x}\Big[\tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)v_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\right)\Big]_{\xi=\eta}^{\xi=x}d\eta. \tag{3.28}$$

Here

$$\tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)=d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_\varepsilon\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t),\varepsilon)\left(\beta_{jk}^\varepsilon a_j a_k^\varepsilon\right)(\xi,\omega_j(\xi;x,t)).$$

Now, the desired convergence follows from (3.15) and (3.21) – (3.23). The desired boundedness of the limit function is a consequence of the assumptions (1.6), (1.7), and (1.9).

Returning to the formula (3.13), similar argument works also for the operators contributing into the first sum: Again, for $i=0$, on the account of the definition of the operators D and B^ε, we have to show that the ε-derivative of

$$(DB^\varepsilon v^l)_j(x,t)=$$

$$=\sum_{\substack{k=1\\k\neq j}}^{n}\int_{x}^{x_j(x,t)} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\omega_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon);\xi,\omega_j(\xi;x,t),\varepsilon)d\xi \tag{3.29}$$

converges uniformly on $\overline{\Pi}_s \setminus \Pi_{s+2d}$ and that the limit function is bounded uniformly in $s \in \mathbb{R}$ and $\varepsilon \le \varepsilon_0$. To show this, we differentiate (3.29) in ε, use (1.8), and integrate by parts. To be more precise, fix arbitrary $j \le m$ and $k > m$ (similarly for all other $j \ne k$) and rewrite the k-th summand in the right-hand side of (3.29) as (up to the sign)

$$\int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+\int\limits_{x_{jk}(x,t,\varepsilon)}^{x} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(1,\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi. \tag{3.30}$$

Then the ε-derivative of this expression equals

$$\int\limits_{x_j(x,t)}^{x} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))\partial_\varepsilon c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+\int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_\varepsilon x_k(\xi,\omega_j(\xi;x,t),\varepsilon)\partial_1 v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+\int\limits_{x_{jk}(x,t,\varepsilon)}^{x} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(1,\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_\varepsilon\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)\partial_2 v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi. \tag{3.31}$$

For the first summand the desired convergence and the uniform boundedness of the limit function is obvious. The last two summands are equal to

$$\int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_\varepsilon x_k(\xi,\omega_j(\xi;x,t),\varepsilon)\left[\partial_\xi x_k(\xi,\omega_j(\xi;x,t),\varepsilon)\right]^{-1}\partial_\xi v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+\int\limits_{x_{jk}(x,t,\varepsilon)}^{x} d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi;x,t))c_k(1,\xi,\omega_j(\xi;x,t),\varepsilon)\partial_\varepsilon\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\left[\partial_\xi\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))\right]^{-1}\partial_\xi v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi. \tag{3.32}$$

Next we use the formulas (3.20), (3.21), and (3.22) and calculate

$$\partial_\xi x_k(\xi,\omega_j(\xi;x,t),\varepsilon)=a_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)\left(\frac{a_k^\varepsilon-a_j}{a_ja_k^\varepsilon}\right)(\xi,\omega_j(\xi;x,t))\times$$

$$\times\partial_3\omega_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon);\xi,\omega_j(\xi;x,t),\varepsilon),$$

$$\partial_\xi\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)=\left(\frac{a_k^\varepsilon-a_j}{a_ja_k^\varepsilon}\right)(\xi,\omega_j(\xi;x,t))\partial_3\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon).$$

Now, due to the assumptions (1.3) and (1.8), we are in a position to bring the expression (3.32) to a desirable form

$$\int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)} d_j(\xi,x,t)c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\partial_\varepsilon x_k(\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times a_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)\partial_3\omega_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon);\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\left(a_ja_k^\varepsilon\beta_{jk}^\varepsilon\right)(\xi,\omega_j(\xi;x,t))\partial_\xi v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+\int\limits_{x_{jk}(x,t,\varepsilon)}^{x} d_j(\xi,x,t)c_k(1,\xi,\omega_j(\xi;x,t),\varepsilon)\partial_\varepsilon\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\partial_3\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)\left(a_ja_k\beta_{jk}^\varepsilon\right)(\xi,\omega_j(\xi;x,t))\partial_\xi v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi=$$

$$=-\int\limits_{x_j(x,t)}^{x_{jk}(x,t,\varepsilon)}\partial_\xi e_{jk}(\xi,x,t,\varepsilon)v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)d\xi+$$

$$+e_{jk}(\xi,x,t,\varepsilon)v_k^l(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)\Big|_{\xi=x_j(x,t)}^{x_{jk}(x,t,\varepsilon)}-$$

$$-\int\limits_{x_{jk}(x,t,\varepsilon)}^{x}\partial_\xi\tilde{e}_{jk}(\xi,x,t,\varepsilon)v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))d\xi+$$

$$+\tilde{e}_{jk}(\xi,x,t,\varepsilon)v_k^l(1,\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon))\Big|_{\xi=x_{jk}(x,t,\varepsilon)}^{x}, \tag{3.33}$$

where

$$e_{jk}(\xi,x,t,\varepsilon)=d_j(\xi,x,t)c_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),\xi,\omega_j(\xi;x,t),\varepsilon)\partial_\varepsilon x_k(\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times a_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon),s)\partial_3\omega_k(x_k(\xi,\omega_j(\xi;x,t),\varepsilon);\xi,\omega_j(\xi;x,t),\varepsilon)\left(a_j a_k^\varepsilon\beta_{jk}^\varepsilon\right)(\xi,\omega_j(\xi;x,t)),$$

$$\tilde{e}_{jk}(\xi,x,t,\varepsilon)=d_j(\xi,x,t)c_k(1,\xi,\omega_j(\xi;x,t),\varepsilon)\partial_\varepsilon\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon)\times$$

$$\times\left(a_j a_k^\varepsilon\beta_{jk}\right)(\xi,\omega_j(\xi;x,t))\partial_3\omega_k(1;\xi,\omega_j(\xi;x,t),\varepsilon).$$

To finish with the first summand in (3.6) it remains, similarly to (3.28), apply the conditions (1.6), (1.7), (1.9), (3.15), and the formulas (3.21) – (3.23).

The last two summands in (3.6) are treated by means of the assumptions (1.5), (1.6), (1.9) (entailing, in particular, the uniform boundedness of the operators B and F restricted to $C\left(\overline{\Pi}_s\setminus\Pi_{s+2d}\right)$) as well as by the smoothing apriori estimate

$$\|v\|_{C\left(\overline{\Pi}_{s+d}\setminus\Pi_{s+2d}\right)^n}+\|\partial_x v\|_{C\left(\overline{\Pi}_{s+d}\setminus\Pi_{s+2d}\right)^n}\le C\|\varphi\|_{C([0,1])^n}, \tag{3.34}$$

where the constant $C>0$ depends on d but not on $\varepsilon\le\varepsilon_0$ and $s\in\mathbb{R}$. We are therefore reduced to prove the estimate (3.34). To this end, we start with the operator representation of v in $\overline{\Pi}_{s+d}\setminus\Pi_{s+2d}$, namely,

$$v=B^\varepsilon Rv+D^\varepsilon v.$$

After a number of iterations we derive the following formula suitable for our purposes:

$$v=\sum_{i=0}^{k-1}(B^\varepsilon R)^i\left(D^\varepsilon B^\varepsilon R+(D^\varepsilon)^2\right)v. \tag{3.35}$$

The estimate (3.34) now readily follows from the smoothing property in x of the operators $D^\varepsilon B^\varepsilon$ and $(D^\varepsilon)^2$ and the apriori estimate (2.2) with $2d$ in place of T. Showing the smoothing property of the operators $D^\varepsilon B^\varepsilon$ and $(D^\varepsilon)^2$ in x, we follow a similar argument as in the proof above of the smoothing property in ε. We illustrate this by example of the operator $(D^\varepsilon)^2$ (and similarly for $D^\varepsilon B^\varepsilon$): We take into account that $\left[(D^\varepsilon)^2v^l\right]_j(x,t)$ on $\overline{\Pi}_{s+d}\setminus\Pi_{s+2d}$ is given by the formula (3.16) where b_{jk} is replaced by b_{jk}^ε; $x_j(x,t)\equiv0$ if $j\le m$; and $x_j(x,t)\equiv1$ if $j>m$. Below we therefore drop the dependence of x_j on x and t. Changing the order of integration, we have

$$\partial_x\left[\left((D^\varepsilon)^2v^l\right)_j(x,t)\right]=$$

$$=\sum_{\substack{k=1\\k\ne j}}^{n}\sum_{\substack{i=1\\i\ne k}}^{n}\int\limits_{x_j}^{x}\int\limits_{\eta}^{x}\partial_x\left[d_{jki}(\xi,\eta,x,t,\varepsilon)b_{jk}(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\right]v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta+$$

$$+\sum_{\substack{k=1\\k\neq j}}^{n}\sum_{\substack{i=1\\i\neq k}}^{n}\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)b_{jk}(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\times$$

$$\times\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\partial_x\omega_j(\xi;x,t,\varepsilon)\partial_2 v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta. \tag{3.36}$$

Let us transform the second summand similarly to (3.28): For given $k\neq j$ and $i\neq k$ we have (using the assumptions (1.3) and (1.8))

$$\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)b_{jk}(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\times$$

$$\times\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\partial_x\omega_j(\xi;x,t,\varepsilon)\partial_2 v_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta=$$

$$=\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\partial_x\omega_j(\xi;x,t,\varepsilon)\times$$

$$\times b_{jk}(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\Big[\big(\partial_\xi\omega_k\big)(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\Big]^{-1}\times$$

$$\times\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta=$$

$$=\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_x\omega_j(\xi;x,t,\varepsilon)\big(a_k a_j\gamma_{jk}\big)(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\times$$

$$\times\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta=$$

$$=\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} \tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)\big(\partial_\xi v_i^l\big)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon))d\xi d\eta=$$

$$=-\int\limits_{x_j}^{x}\int\limits_{\eta}^{x} \partial_\xi\tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)v_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\right)d\xi d\eta+$$

$$+\int\limits_{x_j}^{x}\Big[\tilde{d}_{jki}(\xi,\eta,x,t,\varepsilon)v_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon)\right)\Big]_{\xi=\eta}^{\xi=x}d\eta,$$

where

$$\tilde{d}_{jki}(\xi,\eta,x,t)=d_{jki}(\xi,\eta,x,t,\varepsilon)\partial_x\omega_j(\xi;x,t,\varepsilon)\left(a_k a_j\gamma_{jk}\right)(\xi,\omega_j(\xi;x,t,\varepsilon),\varepsilon).$$

Now in (3.36) we can pass to the limit as $l\to\infty$ and then to the right-hand side apply the apriori estimate (2.2). Combining the resulting inequality with the formula (3.35) and the apriori estimate (2.2) gives (3.34).

Theorem 1.1 is proved.

1. *Akramov T. A., Belonosov V. S., Zelenyak T. I., Lavrentev M. M. (Jr.), Slinko M. G., Sheplev V. S.* Mathematical foundations of modeling of catalytic processes: a review // Theor. Found. Chem. Eng. – 2000. – **34**, № 3. – P. 295 – 306.
2. *Barreira L., Valls C.* Smooth robustness of exponential dichotomies // Proc. Amer. Math. Soc. – 2011. – **139**. – P. 999 – 1012.
3. *Coppel W. A.* Dichotomies in stability theory // Lect. Notes Math. – 1978. – **629**. – 98 p.
4. *Daleckiy Yu., Krein M.* Stability of solutions of differential equations in Banach space. – Providence: Amer. Math. Soc., 1974. – 392 p.
5. *Henry D.* Geometric theory of semilinear parabolic equations // Lect. Notes Math. – 1981. – 840. – 352 p.
6. *Johnson R., Sell G.* Smoothness of spectral subbundles and reducibility of quasiperiodic linear differential systems // J. Different. Equat. – 1981. – **41**. – P. 262 – 288.
7. *Chow S.-N., Leiva H.* Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces // J. Different. Equat. – 1995. – **120**. – P. 429 – 477.
8. *Kmit I.* Classical solvability of nonlinear initial boundary problems for first-order hyperbolic systems // Int. J. Dynam. Syst. Different. Equat. – 2008. – **1**, № 3. – P. 191 – 195.
9. *Kmit I.* Smoothing effect and Fredholm property for first-order hyperbolic PDEs // Oper. Theory: Adv. and Appl. – 2013. – **231**. – P. 219 – 238.
10. *Kmit I.* Smoothing solutions to initial boundary problems for first-order hyperbolic systems // Appl. Anal. – 2011. – **90**, № 11. – P. 1609 – 1634.
11. *Kmit I., Hörmann G.* Semilinear hyperbolic systems with nonlocal boundary conditions: reflection of singularities and delta waves // J. Anal. Appl. – 2001. – **20**, № 3. – P. 637 – 659.
12. *Lichtner M., Radziunas M., Recke L.* Well-posedness, smooth dependence and center manifold reduction for a semilinear hyperbolic system from laser dynamics // Math. Meth. Appl. Sci. – 2007. – **30**. – P. 931 – 960.
13. *Naulin R., Pinto M.* Admissible perturbations of exponential dichotomy roughness // Nonlinear Anal. – 1998. – **31**. – P. 559 – 571.
14. *Palmer K. J.* A perturbation theorem for exponential dichotomies // Proc. Roy. Soc. Edinburgh A. – 1987. – **106**. – P. 25 – 37.
15. *Pliss V. A., Sell G. R.* Robustness of exponential dichotomies in infinite-dimensional dynamical systems // J. Dynam. Different. Equat. – 1999. – **11**. – P. 471 – 513.
16. *Romanovskii R. K., Bel'gart L. V.* On the exponential dichotomy of solutions of the Cauchy problem for a hyperbolic system on a plane // Differents. Uravn. – 2010. – **46**, № 8. – P. 1125 – 1134.
17. *Sacker R., Sell G.* Dichotomies for linear evolutionary equations in Banach spaces // J. Different. Equat. – 1994. – **113**. – P. 17 – 67.
18. *Samoilenko A. M.* Elements of the mathematical theory of multi-frequency oscillations. – Dordrecht: Kluwer, 1991. – 327 p.
19. *Tkachenko V. I.* On the exponential dichotomy of pulse evolution systems // Ukr. Math. J. – 1994. – **46**, № 4. – P. 441 – 448.
20. *Yi Y.* A generalized integral manifold theorem // J. Different. Equat. – 1993. – **102**. – P. 153 – 187.
21. *Zelenyak T. I.* On stationary solutions of mixed problems relating to the study of certain chemical processes // Different. Equat. – 1966. – **2**. – P. 98 – 102.

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