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ON THE ASYMPTOTIC EQUILIBRIUM AND ASYMPTOTIC EQUIVALENCE OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

ПРО АСИМПТОТИЧНУ РІВНОВАГУ ТА АСИМПТОТИЧНУ ЕКВІВАЛЕНТНІСТЬ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ У БАНАХОВИХ ПРОСТОРАХ

We present some conditions for the asymptotic equilibrium of nonlinear differential equations in Banach spaces, in particular, of the linear nonhomogenous equation. We also discuss analogous problems for the linear equation with a nonbounded operator. Some obtained results are applied to problems of asymptotic equivalence.

Наведено деякі умови асимптотичної рівноваги нелінійних диференціальних рівнянь у банахових просторах і, зокрема, лінійного неоднорідного рівняння. Також розглянуто аналогічні питання для лінійного рівняння із необмеженим оператором. Деякі отримані результати застосовано до задач асимптотичної еквівалентності.

1. Introduction. Asymptotic equilibrium and asymptotic equivalence of differential equation systems in R^n were investigated in papers [1 – 4]. Some extensions for the case of linear differential equations in Banach spaces were given in [5]. This paper studies the same problem for nonlinear differential equations and, particularly, for the nonhomogenous linear equation in Banach spaces E. We also discuss analogous problems for the linear equation with a nonbounded operator. At last, we apply some obtained results to problems of asymptotic equivalence.

2. Asymptotic equilibrium for nonhomogenous linear equations.

Definition 1. *We say that the equation*

$$\dot{x} = f(t, x) \tag{1}$$

has an asymptotic equilibrium if every its solution has a finite limit at the infinity and for each $u_0 \in E$ *there exists a solution* $x(t)$ *of* (1) *such that* $x(t) \to u_0$ *as* $t \to +\infty$.

Here and in the following, E denotes a Banach space. I, $L(E)$, $L_1([a, b], E)$, $C([0, T], E), \ldots$ are well-known notations. For the linear equation

$$\dot{x} = A(t)x, \tag{2}$$

where $A(t)$ is a linear bounded operator strongly continuous on $[0, \infty)$, the following statement was proved in [5].

Theorem 1. *Equation* (2) *has a linear asymptotic equilibrium if and only if the equation*

$$\frac{dU}{dt} = A(t)U \tag{2'}$$

considered in the space of all linear bounded operators $L(E)$ *has a solution* $V(t)$ *which strongly tends to* I *as* $t \to +\infty$ *and which has* $V^{-1}(t) \in L(E)$ *for* $t \geq t_0 \geq 0$.

We consider now the nonhomogenous linear equation

$$\dot{x} = A(t)x + f(t), \tag{3}$$

where $f(t)$ is a function continuous on $[0, \infty)$. Suppose that equation (2) has a linear asymptotic equilibrium and let $V(t)$ be the operator mentioned in Theorem 1. It is easy to verify that

$$x(t) = V(t)V^{-1}(t_0)x_0 + \int_{t_0}^{t} V(t)V^{-1}(\tau)f(\tau)d\tau, \quad t_0 \geq 0, \tag{4}$$

is a solution of equation (3) which satisfies condition $x(t_0) = x_0$. Let $f(t)$ be such that integral $\int_0^{+\infty} V^{-1}(\tau)f(\tau)d\tau$ converges. By virtue of properties of $V(t)$ and from the formula (4), we can state that there exists $\lim_{t\to+\infty} x(t) := x(+\infty)$. We show now that, for $u_0 \in E$, the solution $x(t)$ of (3) satisfying condition $x(t_0) = x_0$ with

$$x_0 = V(t_0)u_0 - \int_{t_0}^{+\infty} V(t_0)V^{-1}(\tau)f(\tau)d\tau \tag{5}$$

tends to u_0 as $t \to +\infty$.

In fact, replacing the expression of x_0 from (5) into (4), we obtain

$$x(t) = V(t)u_0 - \int_{t}^{+\infty} V(t)V^{-1}(\tau)f(\tau)d\tau.$$

Now, our statement is implied from the property of $V(t)$ and the Banach – Steinhass theorem. Thus, we have prove the following statement.

Theorem 2. *Let equation* (2) *have a linear asymptotic equilibrium and let continuous function* $f(t)$ *be such that integral* $\int_0^{+\infty} V^{-1}(t)f(t)dt$ *converges. Then equation* (3) *has an asymptotic equilibrium.*

We note that $\int_0^{+\infty} V^{-1}(t)f(t)dt$ converges if $\|V^{-1}(t)\| \leq M \quad \forall\, t \geq 0$ (for some $M > 0$) and $f \in L_1([0,\infty), E)$. In particular, if the operator function $A(t)$ satisfies the condition of Theorem 3 in [5], then equation (3) has an asymptotic equilibrium if $f \in L_1([0,\infty), E)$. In fact, in this case there exists a solution $V(t)$ of equation (2′) which tends to I by norm of the space $L(E)$ as $t \to +\infty$. Consequently, it is easy to verify that $\|V^{-1}(t)\| \leq M$ for $t \geq 0$.

3. The case of nonlinear differential equations. We consider now the equation

$$\dot{x} = f(t, x), \tag{6}$$

where $f\colon [0,+\infty) \times E \to E$. Further, we need the following statement (see [6]).

Proposition 1. *If* $f\colon [0,T] \times E \to E$ *is a compact operator, then the operator* $F\colon [0,T] \times D \to C([0,T], E)$, *defined by the formula*

$$(Fx)(t) := x_0 + \int_0^t f(\tau, x(\tau))d\tau, \qquad t \in [0,T], \quad x \in D,$$

is also a compact operator, where D *is a set of continuous functions* $x\colon [0,T] \to E$.

Theorem 3. *Let the compact operator* $f(t,x)$ *satisfy the following conditions:*

$$\|f(t,x)\| \leq g(t)h(\|x\|), \quad (t,x) \in [0,\infty) \times E,$$

where $\int_0^{+\infty} g(t)dt < +\infty$; $h(u)$ *is a positive continuous nondecreasing function such that*

$$\int_{u_0}^{+\infty} \frac{du}{h(u)} = +\infty, \quad u_0 > 0.$$

Then equation (6) *has an asymptotic equilibrium.*

Proof. Let $x(t)$ be an arbitrary solution of (6) satisfying the condition $x(t_0) = x_0$. Then

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\,d\tau. \tag{7}$$

Hence,

$$\|x(t)\| \le \|x_0\| + \int_{t_0}^{t} g(\tau)h(\|x(\tau)\|)\,d\tau.$$

According to the theorem about the integral inequality, we have then $\|x(t)\| \le y(t)$, where $y(t)$ is a solution of the problem

$$\begin{aligned} \dot{y} &= g(t)h(y), \\ y(t_0) &= \|x_0\|. \end{aligned} \tag{8}$$

From (8) we have

$$\int_{\|x_0\|}^{y(t)} \frac{du}{h(u)} = \int_{t_0}^{t} g(\tau)\,d\tau \le \int_{t_0}^{+\infty} g(\tau)\,d\tau < +\infty.$$

This shows that $y(t)$ is upper bounded. Hence, $\|x(t)\| \le M$ for $t \ge t_0$. Let now t_1, $t_2 > t_0$ satisfy the inequality

$$\left| \int_{t_1}^{t_2} g(\tau)\,d\tau \right| < \frac{\varepsilon}{h(M)}.$$

Then

$$\|x(t_1) - x(t_2)\| = \left\| \int_{t_1}^{t_2} f(\tau, x(\tau))\,d\tau \right\| \le \left| \int_{t_1}^{t_2} g(\tau)h(\|x(\tau)\|)\,d\tau \right| \le$$

$$\le h(M) \left| \int_{t_1}^{t_2} g(\tau)\,d\tau \right| < \varepsilon.$$

This means that there exists $\lim\limits_{t\to+\infty} x(t)$. Let now $u_0 \in E$ be an arbitrary element of E. Let $x(t)$ be a solution of (6) which tends to u_0 as $t \to +\infty$. Then

$$u_0 = x_0 + \int_{t_0}^{+\infty} f(\tau, x(\tau))\,d\tau. \tag{9}$$

From (7), (9) we obtain

$$x(t) = u_0 - \int_{t}^{+\infty} f(\tau, x(\tau))\,d\tau. \tag{10}$$

Thus, $x(t)$ is a solution of integral equation (10). Consequently, it remains only to prove the existence of solutions for integral equation (10). For this purpose, we denote by Ω the set of continuous functions $x(t)$ satisfying inequality $\|x(t)\| \le R$ for $t \ge$

$\geq t_0 \geq 0$, where R is large enough. Clearly, Ω is closed, bounded, and convex. Define now a map F by

$$(Fx)(t) := u_0 - \int_t^{+\infty} f(\tau, x(\tau))\,d\tau, \quad x \in \Omega, \quad t \geq t_0,$$

t_0 is large enough,

$$\|(Fx)(t)\| \leq \|u_0\| + \left\| \int_{t_0}^{+\infty} f(\tau, x(\tau))\,d\tau \right\| \leq \|u_0\| + h(R) \int_{t_0}^{+\infty} g(\tau)\,d\tau.$$

We choose $R > 2\|u_0\|$ and t_0 be large enough such that

$$\int_{t_0}^{+\infty} g(t)\,dt < \frac{R}{2h(R)}.$$

Then $\|(Fx)(t)\| \leq R$. This shows that $F: \Omega \to \Omega$.

We prove now that F is a compact operator. In fact,

$$(Fx)(t) = u_0 - \int_t^{T} f(\tau, x(\tau))\,d\tau - \int_T^{+\infty} f(\tau, x(\tau))\,d\tau = (Gx)(t) + (Hx)(t), \tag{11}$$

where

$$(Gx)(t) := u_0 - \int_t^{T} f(\tau, x(\tau))\,d\tau, \quad t \geq t_0,$$

$$(Hx)(t) := -\int_T^{+\infty} f(\tau, x(\tau))\,d\tau.$$

Choosing $T > t_0$ such that

$$\int_T^{+\infty} g(\tau)\,d\tau < \frac{\varepsilon}{4h(R)}$$

we get

$$\|(Hx)(t)\| \leq \int_T^{+\infty} h(\|x(\tau)\|)\,g(\tau)\,d\tau < h(R) \int_T^{+\infty} g(\tau)\,d\tau < \frac{\varepsilon}{4}.$$

By proposition mentioned above, operator G is compact. Consequently, sequence $\{(Gx_n)(t)\}$ contains a subsequence $\left\{(Gx_{n_j})(t)\right\}$ which converges. This means that there exists a number $K > 0$ such that

$$\left\| \left(Gx_{n_j}\right)(t) - \left(Gx_{n_{j+p}}\right)(t) \right\| < \frac{\varepsilon}{2} \quad \forall j > K, \quad p \in N.$$

From (11), we obtain

$$\left\| \left(Fx_{n_j}\right)(t) - \left(Fx_{n_{j+p}}\right)(t) \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall t \geq t_0.$$

This shows that $F: \Omega \to \Omega$ is a compact operator. According to the Schauder theorem, there exists an element $x \in \Omega$ such that $x = F(x)$ or

$$x(t) = u_0 - \int_t^{+\infty} f(\tau, x(\tau))\,d\tau.$$

It is easy to verify that $x(t)$ is a solution of (6) which tends to u_0 as $t \to +\infty$.

Theorem is completely proved.

***Corollary* 1.** *If the compact operator $f(t, x)$ satisfies conditions*

$$\|f(t,x)\| \le g(t)\|x\|^{\alpha}, \quad 0 < \alpha \le 1,$$

$$\int_{t_0}^{+\infty} g(t)\,dt < +\infty, \quad t_0 \ge 0,$$

then equation (6) *have an asymptotic equilibrium.*

Theorem 4. *Let the compact operator $f(t, x)$ satisfy the condition*

$$\|f(t,x) - f(t,y)\| \le g(t)h(\|x-y\|), \quad x, y \in E, \quad t \ge 0,$$

where

$$\int_0^{+\infty} g(t)\,dt < +\infty$$

and the positive continuous and nondecreasing function $h(u)$ satisfies the condition

$$\int_{u_0}^{+\infty} \frac{du}{h(u)} = +\infty, \quad u_0 > 0.$$

Then equation (6) *has an asymptotic equilibrium.*

The proof of this theorem is analogous to that of Theorem 3.

4. The case of linear equations with nonbounded linear operator. In this section, we consider the equation

$$\dot{x} = A(t)x \tag{12}$$

in the Hilbert space H. $A(t)$ is a linear operator defined in $D(A) \subseteq H$. We suppose that $D(A)$ does not depend on $t \in [0, +\infty)$ and that $D(A)$ is everywhere dense in H. Moreover, we suppose that the Cauchy problem $x(0) = x_0$, $x_0 \in D(A)$, has a solution defined on $[0, +\infty)$.

Theorem 5. *Let, for each $h \in D(A)$, $\|A(t)h\| \in L_1[0, +\infty)$ and let the operator $A(t)$ be self-adjoint. Then every bounded solution of equation* (12) *has a weak finite limit at the infinity. Moreover, if the inclusion $\|A(t)h\| \in L_1[0, +\infty)$ is uniform for $h \in S(0,1) \cap D(A)$ (see* [5]), *then every bounded solution of* (12) *has a strong finite limit at the infinity.*

Proof. Let $x(t)$ be any bounded solution of (12), i.e., there is $M > 0$ such that $\|x(t)\| \le M$ $\forall\, t \ge 0$. Then, for any $h \in D(A)$, we have

$$\langle x(t), h\rangle = \langle x_0, h\rangle + \int_{t_0}^{t} \langle A(\tau)x(\tau), h\rangle\,d\tau = \langle x_0, h\rangle + \int_{t_0}^{t} \langle x(\tau), A(\tau)h\rangle\,d\tau, \tag{13}$$

where $x_0 = x(t_0)$. Hence,

$$|\langle x(t_1) - x(t_2),\ h\rangle| = \left|\int_{t_1}^{t_2} \langle x(\tau),\ A(\tau)h\rangle\,d\tau\right| \le M\left|\int_{t_1}^{t_2} \|A(\tau)h\|\,d\tau\right| < \varepsilon$$

if $t_1, t_2 > T$, where T is large enough. This shows that there exists $\lim_{t\to+\infty}\langle x(t), h\rangle$ for all $h \in D(A)$. Because of the denseness of $D(A)$ and the boundedness of $x(t)$, we easily prove that this limit exists for all $h \in H$. Thus, the first statement is proved. Since H is weakly complete, there exists $h_0 \in H$ such that

$$\lim_{t\to+\infty}\langle x(t), h\rangle = \langle h_0, h\rangle, \quad h \in H.$$

By virtue of (13), we have

$$\langle h_0, h\rangle = \langle x_0, h\rangle + \int_{t_0}^{+\infty}\langle x(\tau), A(\tau)h\rangle d\tau. \tag{14}$$

From (13), (14) we obtain

$$\langle x(t), h\rangle = \langle h_0, h\rangle - \int_{t}^{+\infty}\langle x(\tau), A(\tau)h\rangle d\tau, \quad h \in D(A). \tag{15}$$

Hence,

$$|\langle x(t), h\rangle| \le |\langle h_0, h\rangle| + M\int_{t_0}^{+\infty}\|A(\tau)h\| d\tau < |\langle h_0, h\rangle| + \varepsilon \tag{16}$$

if t_0 large enough. By virtue of (16) and the denseness of $D(A)$, we have

$$\|x(t)\| \le \|h_0\| + \varepsilon \tag{17}$$

for t large enough. On the other hand, by theorem about the weak convergence,

$$\|h_0\| \le \|x(t)\| + \varepsilon \tag{18}$$

for t large enough. Inequalities (17), (18) show that $\lim_{t\to+\infty}\|x(t)\| = \|h_0\|$. Since $x(t)$ weakly tends to h_0, we obtain that $\lim_{t\to+\infty} x(t) = h_0$.

Theorem is proved.

We extend now the notion “solution”.

Definition 2. *Let* $A(t) = A^*(t)$, $t \ge t_0 \ge 0$, $x(t)$ *is said to be an extended solution of the equation* (12) *if it satisfies the relation*

$$\frac{d}{dt}\langle x(t), y\rangle = \langle x(t), A(t)y\rangle \quad \forall\, y \in D(A), \quad t \ge t_0 \ge 0.$$

This definition of solution is given in [7].

Theorem 6. *Let* $\|A(t)h\| \in L_1[0, +\infty)$ *uniformly for* $h \in S(0,1) \cap D(A)$; $A(t) = A^*(t)$. *Then for each* $h_0 \in D(A)$ *there exists an extended solution* $x(t)$ *of equation* (16) *such that*

$$\lim_{t\to+\infty} x(t) = h_0. \tag{19}$$

Proof. Consider the functional

$$\zeta_1(t, h) = \langle h_0, h\rangle - \int_{t}^{+\infty}\langle A(\tau)x_0(\tau), h\rangle d\tau,$$

where $t \ge t_0$, $h \in D(A)$, $x_0(t) \equiv h_0$,

$$|\zeta_1(t,h)| \;\le\; \|h_0\|\|h\| + \int_t^{+\infty} \|x_0(\tau)\|\|A(\tau)h\|d\tau \;\le\; \|h_0\|(\|h\|+q), \tag{20}$$

where $q = \int_{t_0}^{+\infty} \|A(\tau)h\|d\tau$. We choose t_0 be large enough such that $0 < q < 1$. Inequality (20) shows that $\zeta_1(t,h)$ is a linear continuous functional defined in $D(A)$. Because of the denseness of $D(A)$ on H, we can extend continuously this functional in H with the norm preserving. We denote the extended functional also by $\zeta_1(t,h)$. According to the Riesz theorem, there exists an element $x_1(t)$ in H such that $\zeta_1(t,h) = \langle x_1(t), h\rangle$.

Clearly, $\|x_1(t)\| \le (1+q)\|h_0\|$. Consider now the functional

$$\zeta_2(t,h) \;:=\; \langle h_0, h\rangle - \int_t^{+\infty} \langle x_1(\tau), A(\tau)h\rangle d\tau, \quad h \in D(A).$$

By the analogous proof, we obtain that $\zeta_2(t,h)$ is a linear continuous functional defined in H. Consequently,

$$\zeta_2(t,h) \;=\; \langle x_2(t), h\rangle,$$

where $\|x_2(t)\| \;\le\; (1+q+q^2)\|h_0\|$. Continuing this process, we have the linear continuous functional

$$\zeta_n(t,h) \;:=\; \langle h_0, h\rangle - \int_t^{+\infty} \langle x_{n-1}(\tau), A(\tau)h\rangle d\tau, \tag{21}$$

defined in $D(A)$. The continuous extension of this functional has a form

$$\zeta_n(t,h) \;=\; \langle x_n(t), h\rangle, \tag{22}$$

$$\|x_n(t)\| \;\le\; \left(1+q+\ldots+q^n\right)\|h_0\| \;\le\; \frac{\|h_0\|}{1-q}. \tag{23}$$

We show now that the sequence $\{x_n(t)\}$ uniformly converges on $[t_0, +\infty)$. To prove this statement, it suffices to show that

$$\|x_n(t) - x_{n-1}(t)\| \;\le\; \|h_0\|q^n. \tag{24}$$

In fact, for $n = 1$ we have

$$\|x_1(t) - x_0(t)\| \;\le\; \sup_{\|h\|\le 1} |\langle x_1(t) - x_0(t), h\rangle| \;=\; \sup_{h\in S(0,1)\cap D(A)} |\langle x_1(t) - x_0(t), h\rangle| \;\le$$

$$\le\; \sup_{h\in S(0,1)\cap D(A)} \left| \int_t^{+\infty} \|A(\tau)h\|\|x_0(\tau)\|d\tau \right| \;\le\; \|h_0\|q,$$

i.e., formula (24) is true for $n = 1$. Let us now assume that (24) is true for n. Then

$$\|x_{n+1}(t) - x_n(t)\| \;=\; \sup_{\|h\|\le 1} \left|\langle x_{n+1}(t) - x_n(t), h\rangle\right| \;=$$

$$=\; \sup_{h\in S(0,1)\cap D(A)} \left| \int_{t_0}^{+\infty} \langle x_n(\tau) - x_{n-1}(\tau), A(\tau)h\rangle d\tau \right| \;\le$$

$$\leq \int_{t_0}^{+\infty} \left\| x_n(\tau) - x_{n-1}(\tau) \right\| \| A(\tau) h \| d\tau \leq \| h_0 \| q^{n+1},$$

i.e., formula (15) is valid for $n+1$. Since $0 < q < 1$, inequality (24) shows that sequence $\{x_n(t)\}$ uniformly converges on $[t_0, +\infty)$.

Setting $x(t) = \lim_{n \to +\infty} x_n(t)$ and tending $n \to +\infty$ in (21), (22), we obtain

$$\langle x(t), h \rangle = \langle h_0, h \rangle - \int_t^{+\infty} \langle x(\tau), A(\tau) h \rangle d\tau, \quad h \in D(A). \tag{25}$$

This show that $x(t)$ is an extended solution of (12) and that $x(t)$ weakly tends to h_0 as $t \to +\infty$. We prove now that $x(t)$ strongly tends to h_0 as $t \to +\infty$. By virtue of the uniform convergence of $\{x_n(t)\}$, it suffices to show that $x_n(t) \to h_0$ as $t \to +\infty$. In fact, we have

$$|\langle x_n(t) - h_0, h \rangle| < \int_{t_0}^{+\infty} \left\| x_{n-1}(\tau) \right\| \| A(\tau) h \| d\tau \leq \frac{\| h_0 \|}{1-q} \int_{t_0}^{+\infty} \| A(\tau) h \| d\tau.$$

Hence,

$$\| x_n(t) - h_0 \| \leq \frac{\| h_0 \| q}{1-q}.$$

Since $q \to 0$ as $t \to +\infty$, our statement is proved.

5. Asymptotic equivalence. In this section, we consider equations

$$\dot{y} = A(t) y, \tag{26}$$

$$\dot{x} = A(t) x + f(t, x). \tag{27}$$

Definition 3. *Equations* (26), (27) *are said to be asymptotically equivalent if to each solution* $x(t)$ *of* (27) *there exists a solution* $y(t)$ *of* (26) *such that*

$$\lim_{t \to +\infty} \| x(t) - y(t) \| = 0 \tag{28}$$

and conversely, to each solution $y(t)$ *of* (26) *there exists a solution* $x(t)$ *of* (27) *satisfying* (28).

We assume throughout that $A(t) \in L(E)$ for $t \geq 0$ and $A(t)$ is strongly continuous on $[0, +\infty)$; $f\colon [0, +\infty) \times E \to E$ is a continuous operator. We denote by $U(t)$ the Cauchy operator of (26) satisfying $U(0) = I$. Consider the equation

$$\dot{z} = U^{-1}(t) f[t, U(t) z]. \tag{29}$$

Theorem 7. *Let equation* (26) *be stable and consequently* $\| U(t) \| \leq M$. *Moreover, we suppose that equation* (29) *has an asymptotic equilibrium. Then equations* (26), (27) *are asymptotically equivalent.*

Proof. I.et $x(t)$ be an arbitrary solution of (27). It is easy to verify that $z(t) = U^{-1}(t) x(t)$ is a solution of (29). By virtue of the assumptions, there exists $z_{+\infty} = \lim_{t \to +\infty} z(t)$. Setting $y(t) = U(t) z_{+\infty}$, we easily verify that $y(t)$ is a solution of (26) which satisfies relation (28). Conversely, let now $y(t)$ be an arbitrary solution of (26) satisfying condition $y(0) = y_0$. Then $y(t) = U(t) y_0$. According to the assumption, there exists a solution $z(t)$ of (29) such that $\lim_{t \to +\infty} z(t) = y_0$. Let $x(t) = U(t) z(t)$. It is easy to verify that $x(t)$ is a solution of (27) and

$$\lim_{t\to+\infty} \| x(t) - y(t) \| \le M \lim_{t\to+\infty} \| z(t) - y_0 \| = 0.$$

Theorem is proved.

Remark. We have proved that, in the condition of stability of equation (26), the asymptotic equilibrium of equation (29) is a sufficient condition for the asymptotic equivalence of equations (26), (27). In general, this condition is not necessary.

Example. Consider the following example:

$$\dot{x} = Ax + B(t)x,$$

$$\dot{y} = Ay,$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{bmatrix}.$$

In this case,

$$U(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}, \qquad U^{-1}(t) = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{bmatrix}, \qquad U^{-1}(t)B(t)U(t) = \begin{bmatrix} 0 & e^{-2t} \\ 1 & 0 \end{bmatrix}.$$

By the Levison theorem (see [8, p. 159]), above equations are asymptotically equivalent. However, equation

$$\dot{z} = U^{-1}(t)B(t)U(t)z$$

has not an asymptotic equilibrium. In fact, this equation can be written in the form

$$\dot{z}_1 = e^{-2t}z_2,$$

$$\dot{z}_2 = z_1.$$

Suppose that this system has an asymptotic equilibrium. Then for $h_0 = (1, 1)$, there exists a solution $(z_1(t), z_2(t))$ such that $z_1(t) \to 1$; $z_2(t) \to 1$ as $t \to +\infty$. Hence, $\dot{z}_2(t) \to 1$ as $t \to +\infty$. Therefore,

$$1 - \varepsilon < \dot{z}_2(t) < 1 + \varepsilon \quad \forall t \ge T > 0.$$

Consequently,

$$z_2(t) > z_2(T) + (1-\varepsilon)(t-T).$$

Tending $t \to +\infty$, we obtain a contracdition.

However, we have the following theorem.

Theorem 8. *Let equation* (26) *be bistable* (*see* [9, *p.* 165]). *Then the asymptotic equilibrium of equation* (29) *is a necessary and sufficient conditions for the asymptotic equivalence of* (26), (27).

Proof. According to assumptions, we have

$$\| U(t) \| \le M, \qquad \left\| U^{-1}(t) \right\| \le M \quad \forall t \ge 0.$$

Obviously, we remain to prove the necessary condition. Let $y_0 \in E$ and $y(t) = U(t)y_0$ be a solution of (26). According to our assumption, there exists a solution $x(t)$ of (27) such that $\| x(t) - y(t) \| \to 0$ as $t \to +\infty$. Consider $z(t) = U^{-1}(t)x(t)$. It is a solution of (29) and

$$\| z(t) - y_0 \| \leq \left\| U^{-1}(t) \right\| \| x(t) - y(t) \| \leq M \| x(t) - y(t) \|.$$

Therefore, $z(t) \to y_0$ as $t \to +\infty$. Let now $z(t)$ be an arbitrary solution of (29). Then $x(t) = U(t) z(t)$ is a solution of (27). According to our assumption, there exists a solution $y(t) = U(t) y_0$ $(y_0 = y(0))$ of (26) such that $\| x(t) - y(t) \| \to 0$ as $t \to +\infty$. Consequently, we have that $\| z(t) - y_0 \| \leq \left\| U^{-1}(t) \right\| \| x(t) - y(t) \| \leq M \| x(t) - y(t) \| \to 0$ as $t \to +\infty$. This shows that $z(t) \to y_0$ as $t \to +\infty$. Thus, equation (29) has an asymptotic equilibrium.

Theorem 9. *Let equation* (26) *be bistable. The compact operator* $f(t, x)$ *satisfies conditions of Theorem* 3 *or Theorem* 4. *Then equations* (26), (27) *are asymptotically equivalent*.

In fact, in this case conditions of Theorem 3 or Theorem 4 are satisfied for equation (29). Hence, it has an asymptotic equilibrium. By virtue of Theorem 6, we obtain the assertion of this theorem.

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