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ON THE MAXIMAL OPERATOR OF (C, α)-MEANS OF WALSH – KACZMARZ – FOURIER SERIES

ПРО МАКСИМАЛЬНИЙ ОПЕРАТОР (C, α)-СЕРЕДНІХ РЯДІВ УОЛША – КАЧМАЖА – ФУР'Є

Simon [J. Approxim. Theory. – 2004. – **127**. – P. 39 – 60] proved that the maximal operator $\sigma^{\alpha,\kappa,*}$ of the (C, α)-means of the Walsh – Kaczmarz – Fourier series is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/(1+\alpha)$, $0 < \alpha \le 1$.

Recently, Gát and Goginava proved that this boundedness result does not hold if $p \le 1/(1+\alpha)$. However, in the endpoint case $p = 1/(1+\alpha)$ the maximal operator $\sigma^{\alpha,\kappa,*}$ is bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ to the space weak-$L_{1/(1+\alpha)}$.

The main aim of this paper is to prove a stronger result, that is for any $0 < p \le 1/(1+\alpha)$ there exists a martingale $f \in H_p$ such that the maximal operator $\sigma^{\alpha,\kappa,*}f$ does not belong to the space L_p.

Saймон довів [див. J. Approxim. Theory. – 2004. – **127**. – P. 39 – 60], що максимальний оператор $\sigma^{\alpha,\kappa,*}$ (C, α)-середніх рядів Уолша – Качмажа – Фур'є є обмеженим з мартингального простору Харді H_p до простору L_p для $p > 1/(1+\alpha)$, $0 < \alpha \le 1$.

Нещодавно Гат і Гогінава довели, що цей результат про обмеженість не виконується, якщо $p \le 1/(1+\alpha)$. Однак у випадку кінцевої точки $p = 1/(1+\alpha)$ максимальний оператор $\sigma^{\alpha,\kappa,*}$ є обмеженим з мартингального простору Харді $H_{1/(1+\alpha)}$ до простору слабкого-$L_{1/(1+\alpha)}$.

Головна мета даної статті — довести більш вагомий результат, тобто довести, що для будь-якого $0 < p \le 1/(1+\alpha)$ існує мартингал $f \in H_p$ такий, що максимальний оператор $\sigma^{\alpha,\kappa,*}f$ не належить простору L_p.

1. Introduction. In 1948 Šneider [1] introduced the Walsh – Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \ \ge\ C\ >\ 0$$

holds a.e. In 1974 Schipp [2] and Young [3] proved that the Walsh – Kaczmarz system is a convergence system. Skvortsov in 1981 [4] showed that the Fejér means with respect to the Walsh – Kaczmarz system converge uniformly to f for any continuous functions f. Gát [5] proved, for any integrable functions, that the Fejér means with respect to the Walsh – Kaczmarz system converge almost everywhere to the function and Gát proved that $\left\|\sigma^{\kappa*}f\right\|_1 \le C\left\|f\right\|_{H_1}$. Gát's result was extended to the Hardy space by Simon [6], who proved that $\sigma^{\kappa*}$ is of type (H_p, L_p) for $p > 1/2$. Weisz [7] showed that in endpoint case $p = 1/2$ the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

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In paper [8] Simon proved the (H_p, L_p)-boundedness of the maximal operator of (C, α)-means of Walsh – Kaczmarz – Fourier series, where $0 < \alpha \leq 1$ and $1/(1 + \alpha) < p \leq 1$.

In the paper [9] Gát and Goginava proved that in theorem of Simon the assumption $p > 1/(1 + \alpha)$ is essential, namely, this boundedness result does not hold if $p \leq 1/(1 + \alpha)$. However, in the endpoint case $p = 1/(1 + \alpha)$ the maximal operator $\sigma^{\alpha,\kappa,*}$ is bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ to the space weak-$L_{1/(1+\alpha)}$.

The main aim of this paper is to prove a stronger result, for any $0 < p \leq 1/(1 + \alpha)$ there exists a martingale $f \in H_p$ such that

$$\left\| \sigma^{\alpha,\kappa,*} f \right\|_p = +\infty.$$

2. Dyadic Hardy space and (C, α)-means. Now, we give a brief introduction to the theory of dyadic analysis [10]. Let denote by Z_2 the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on Z_2 is given in the way that the measure of a singleton is $1/2$. Let $G := \times_{k=0}^{\infty} Z_2$, G be called the Walsh group. The elements of G are sequences $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$, $k \in \mathrm{N}$.

The group operation on G is the coordinate-wise addition (denoted by +), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervals are defined by

$$I_0(x) := G, \qquad I_n(x) := \{ y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots) \}$$

for $x \in G$, $n \in \mathrm{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathrm{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathrm{N}$.

Let L_p denote the usual Lebesgue spaces on G (with the corresponding norm or quasinorm $\|\cdot\|_p$).

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k}, \qquad x \in G, \qquad k \in \mathrm{N}.$$

Let the Walsh – Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \qquad n_i \in \{0, 1\}, \qquad i \in \mathrm{N},$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{ j \in \mathrm{N} : n_j \neq 0 \}$. Walsh – Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh – Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x)\prod_{k=0}^{|n|-1}\left(r_{|n|-1-k}(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh – Kaczmarz functions and the set of Walsh – Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \le n < 2^{k+1}\} = \{w_n : 2^k \le n < 2^{k+1}\}$$

for all $k \in \mathrm{P}$ and $\kappa_0 = w_0$.

Skvortsov (see [4]) gave a relation between the Walsh – Kaczmarz functions and the Walsh – Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)$$

for $A \in \mathrm{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x)w_{n-2^{|n|}}(\tau_{|n|}(x)), \qquad n \in \mathrm{N}, \qquad x \in G.$$

The Dirichlet kernels are defined by

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k,$$

where $\psi_n = w_n$ or κ_n, $n \in \mathrm{P}$, $D_0^{\alpha} := 0$. The 2^nth Dirichlet kernels have a closed form (see, e.g., [10])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if} \quad x \notin I_n, \\ 2^n, & \text{if} \quad x \in I_n. \end{cases}$$

If $f \in L_1(G)$, then the number

$$\hat{f}^{\psi}(n) = \int_G f\psi_n$$

is said to the nth Walsh – (Kaczmarz) – Fourier coefficient.

Denote by S_n^{ψ} the nth partial sums of the Walsh – (Kaczmarz) – Fourier series of a function f, namely

$$S_n^{\psi}(f; x) = \sum_{k=0}^{n-1} \hat{f}^{\psi}(k)\psi_k.$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k, $k \in \mathrm{N}$.

Denote by $f = (f^{(n)}, n \in \mathrm{N})$ a martingale with respect to $(F_n, n \in \mathrm{N})$ (for details see, e. g., [11]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathrm{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n\in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)d\mu(u) \right|, \qquad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \ldots)$ then the Walsh – (Kaczmarz) – Fourier coefficients must be defined in a little bit different way:

$$\hat{f}(i) = \lim_{k\to\infty} \int_G f^{(k)}(x)\psi_i(x)d\mu(x) \qquad (\psi = w \text{ or } \kappa).$$

The Walsh – (Kaczmarz) – Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbb{N})$ obtained from f.

Set $A_n^\alpha := \dfrac{(1+\alpha)\ldots(n+\alpha)}{n!}$ for any $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha \neq -1, -2, \ldots$. It is known that $A_n^\alpha \sim n^\alpha$. For $n = 1, 2, \ldots$ and a martingale f the (C, α)-means of the Walsh – (Kaczmarz) – Fourier series of the function f is given by

$$\sigma_n^{\alpha,\psi} f(x) = \frac{1}{A_{n-1}^\alpha} \sum_{j=1}^{n} A_{n-j}^{\alpha-1} S_j^\psi(f; x) \qquad (\psi = w \text{ or } \kappa).$$

For a martingale f we consider the maximal operator

$$\sigma^{\alpha,\psi,*} f = \sup_{n\in \mathbb{P}} \left| \sigma_n^{\alpha,\psi} f(x) \right| \qquad (\psi = w \text{ or } \kappa).$$

The nth (C, α)-kernel of the Walsh – (Kaczmarz) – Fourier series defined by

$$K_n^{\alpha,\psi}(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k^\psi(x) \qquad (\psi = w \text{ or } \kappa).$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a\, d\mu = 0$;

b) $\|a\|_\infty \leq \mu(I)^{-1/p}$;

c) $\operatorname{supp} a \subset I$.

The basic result of atomic decomposition is the following one.

Theorem A [11]. *A martingale* $f = (f^{(n)} : n \in \mathbb{N})$ *is in* H_p, $0 < p \leq 1$, *if and only if there exists a sequence* $(a_k, k \in \mathbb{N})$ *of* p*-atoms and a sequence* $(\mu_k, k \in \mathbb{N})$ *of real numbers such that for every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)},$$
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{1}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (1).

In the paper [8] Simon proved the following theorem.

Theorem B. *Let $0 < \alpha \le 1$ and $1/(1+\alpha) < p \le 1$. Then there exists a constant C such that*

$$\|\sigma^{\alpha,\kappa,*} f\|_p \le C \|f\|_{H_p}$$

for all $f \in H_p(G)$.

In this paper we prove that in theorem of Simon the assumption $p > 1/(1+\alpha)$ is essential. Moreover, we prove that the following is true.

Theorem 1. *Let $0 < \alpha \le 1$ and $0 < p \le 1/(1+\alpha)$. Then there exists a martingale $f \in H_p(G)$ such that*

$$\|\sigma^{\alpha,\kappa,*} f\|_p = +\infty.$$

3. Proof of main result. ***Proof.*** Let $(m_k : k \in \mathrm{N})$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^p} < \infty, \tag{2}$$

$$\sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{m_l} < \frac{2^{2m_k/p}}{m_k}, \tag{3}$$

$$\frac{2^{2m_{k-1}/p}}{m_{k-1}} \le \frac{2^{m_k}}{m_k}. \tag{4}$$

Let

$$f^{(A)}(x) := \sum_{k, 2m_k < A} \lambda_k a_k, \quad \text{where} \quad \lambda_k := \frac{2}{m_k}$$

and

$$a_k(x) := 2^{2(1/p-1)m_k - 1}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right).$$

The martingale $f := (f^{(0)}, f^{(1)}, \ldots, f^{(A)}, \ldots)$ is in $H_p(G)$. Indeed, since

$$\|a_k\|_\infty = 2^{2m_k(1/p-1)-1}2^{2m_k+1} = (\operatorname{supp} a_k)^{-1/p},$$

$$S_{2^A}a_k(x) = \begin{cases} 0, & \text{if} \quad A \le 2m_k, \\ a_k(x), & \text{if} \quad A > 2m_k, \end{cases}$$

and

$$f^{(A)}(x) = \sum_{k:2m_k<A} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A} a_k(x)$$

by (2) and Theorem A we conclude that $f \in H_p(G)$.

Now, we investigate the Fourier coefficients.

Let $j \in \{2^{2m_k}, \ldots, 2^{2m_k+1}-1\}$ for some $k = 0, 1, 2, \ldots$. Then it is evident that

$$\hat{f}^{\kappa}(j) := \lim_{A\to\infty} \widehat{f^{(A)}}^{\kappa}(j) = \frac{2^{2m_k(1/p-1)}}{m_k}$$

and $\hat{f}^{\kappa}(j) = 0$, if $j \notin \{2^{2m_k}, \ldots, 2^{2m_k+1}-1\}$, $k = 0, 1, 2, \ldots$.

Set $q_{A,s} := 2^{2A} + 2^{2s}$ for any $A > s$. Now, we decompose the $q_{m_k,s}$ th Walsh – Kaczmarz (C,α)-means as follows

$$\sigma_{q_{m_k,s}}^{\alpha,\kappa} f(x) = \frac{1}{A_{q_{m_k,s}-1}} \sum_{j=1}^{2^{2m_k}-1} A_{q_{m_k,s}-j}^{\alpha-1} S_j^{\kappa} f(x) +$$

$$+ \frac{1}{A_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A_{q_{m_k,s}-j}^{\alpha-1} S_j^{\kappa} f(x) = I + II.$$

Let $j < 2^{2m_k}$. Then (3) gives that

$$\left|S_j^{\kappa} f(x)\right| \le \sum_{l=0}^{k-1} \sum_{\upsilon=2^{2m_l}}^{2^{2m_l+1}-1} \left|\hat{f}^{\kappa}(\upsilon)\right| \le \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} 2^{2m_l} < 2\frac{2^{2m_{k-1}/p}}{m_{k-1}}$$

and

$$|I| \le c\frac{1}{A_{q_{m_k,s}-1}} \sum_{j=1}^{2^{2m_k}-1} A_{q_{m_k,s}-j}^{\alpha-1} \left|S_j^{\kappa} f(x)\right| \le c(\alpha)\frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{5}$$

Now, we discuss II.

For $2^{2m_k} \le j < q_{m_k,s}$ we have the following:

$$S_j^{\kappa} f(x) = \sum_{\upsilon=0}^{2^{2m_{k-1}+1}-1} \hat{f}^{\kappa}(\upsilon)\kappa_\upsilon(x) + \sum_{\upsilon=2^{2m_k}}^{j-1} \hat{f}^{\kappa}(\upsilon)\kappa_\upsilon(x) =$$

$$= \sum_{l=0}^{k-1} \sum_{\upsilon=2^{2m_l}}^{2^{2m_l+1}-1} \hat{f}^{\kappa}(\upsilon)\kappa_\upsilon(x) + \sum_{\upsilon=2^{2m_k}}^{j-1} \hat{f}^{\kappa}(\upsilon)\kappa_\upsilon(x) =$$

$$= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l(1/p-1)}}{m_l} \kappa_v(x) + \frac{2^{2m_k(1/p-1)}}{m_k} \sum_{v=2^{2m_k}}^{j-1} \kappa_v(x) =$$

$$= \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) + \frac{2^{2m_k(1/p-1)}}{m_k} \left(D_j^{\kappa}(x) - D_{2^{2m_k}}(x)\right).$$

This gives that

$$II = \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) +$$

$$+ \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \left(D_j^{\kappa}(x) - D_{2^{2m_k}}(x)\right) =: II_1 + II_2.$$

To discuss II_1, we use (3) and $\left|D_{2^n}(x)\right| \le 2^n$. Thus, we can write

$$\left|II_1\right| \le c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} 2^{2m_l+1} \le c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{m_l} < c(\alpha) \frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{6}$$

From $\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x) = I + II_1 + II_2$ and (5), (6) we have

$$\left|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)\right| \ge \left|II_2\right| - \left|I\right| - \left|II_1\right| \ge \left|II_2\right| - c \frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{7}$$

Now, we discuss II_2. We can write the nth Dirichlet kernel with respect to the Walsh – Kaczmarz system in the following form:

$$D_n^{\kappa}(x) = D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}\left(\tau_{|k|}(x)\right) =$$

$$= D_{2^{|n|}}(x) + r_{|n|}(x) D^{w}_{n-2^{|n|}}\left(\tau_{|n|}(x)\right).$$

By the help of this, we immediately get

$$\left|II_2\right| = \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \left| \sum_{j=1}^{2^{2s}} A^{\alpha-1}_{q_{m_k,s}-j-2^{2m_k}} \left(D^{\kappa}_{j+2^{2m_k}}(x) - D_{2^{2m_k}}(x)\right) \right| =$$

$$= \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \left| r_{2m_k}(x) \sum_{j=1}^{2^{2s}} A^{\alpha-1}_{2^{2s}-j} D_j^{w}\left(\tau_{2m_k}(x)\right) \right| =$$

$$= \frac{2^{2m_k(1/p-1)}}{m_k} \frac{A^{\alpha}_{2^{2s}-1}}{A^{\alpha}_{q_{m_k,s}-1}} \left| K^{\alpha,w}_{2^{2s}}\left(\tau_{2m_k}(x)\right) \right| \ge$$

$$\ge c(\alpha) \frac{2^{2m_k(1/p-1)-2m_k\alpha} A^{\alpha}_{2^{2s}-1}}{m_k} \left| K^{\alpha,w}_{2^{2s}}\left(\tau_{2m_k}(x)\right) \right|.$$

Thus, from (7) and (4) we have

$$\left|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)\right| \geq c\,\frac{2^{2m_k(1/p-1)-2m_k\alpha} A^{\alpha}_{2^{2s}-1}}{m_k}\left|K^{\alpha,w}_{2^{2s}}\left(\tau_{2m_k}(x)\right)\right| - c\,\frac{2^{m_k}}{m_k}.$$

On the set I_{2s}

$$A^{\alpha}_{2^{2s}-1} K^{\alpha,w}_{2^{2s}} = \sum_{l=0}^{2^{2s}-1} A^{\alpha-1}_{2^{2s}-l}\, l \geq C 2^{2s(1+\alpha)}$$

and

$$\left|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)\right| \geq C\,\frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k} - c\,\frac{2^{m_k}}{m_k}.$$

We decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{t=0}^{A-1} J_t^A,$$

where $A > t \geq 1$ and $J_t^A := \{x \in G : x_{A-1} = \ldots = x_{A-t} = 0, x_{A-t-1} = 1\}$, $J_0^A :=$ $:= \{x \in G : x_{A-1} = 1\}$. Notice that, by the definition of τ_A we have $\tau_A(J_t^A) = I_t \backslash I_{t+1}$.

Therefore, we can write

$$\int_G \left|\sigma^{\alpha,\kappa,*} f\right|^p d\mu \geq \sum_{t=1}^{2m_k-1} \int_{J_t^{2m_k}} \left|\sigma^{\alpha,\kappa,*} f\right|^p d\mu \geq$$

$$\geq \sum_{s=[m_k/2]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} \left|\sigma^{\alpha,\kappa,*} f\right|^p d\mu \geq \sum_{s=[m_k/2]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} \left|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f\right|^p d\mu \geq$$

$$\geq \sum_{s=[m_k/2]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} \left(c\,\frac{2^{2m_k(1/p-(1+\alpha))} A^{\alpha}_{2^{2s}-1}}{m_k}\left|K^{\alpha,w}_{2^{2s}} \circ \tau_{2m_k}\right| - c\,\frac{2^{m_k}}{m_k}\right)^p d\mu \geq$$

$$\geq \sum_{s=[m_k/2]+1}^{m_k-1} \int_{I_{2s}\backslash I_{2s+1}} \left(c\,\frac{2^{2m_k(1/p-(1+\alpha))} A^{\alpha}_{2^{2s}-1}}{m_k}\left|K^{\alpha,w}_{2^{2s}}\right| - c\,\frac{2^{m_k}}{m_k}\right)^p d\mu \geq$$

$$\geq \sum_{s=[m_k/2]+1}^{m_k-1} \int_{I_{2s}\backslash I_{2s+1}} \left(C\,\frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k} - c\,\frac{2^{m_k}}{m_k}\right)^p d\mu$$

and

$$\int_G \left|\sigma^{\alpha,\kappa,*} f\right|^p d\mu \geq c \sum_{s=[m_k/2]+1}^{m_k-1} \int_{I_{2s}\backslash I_{2s+1}} \left|\frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k}\right|^p d\mu \geq$$

$$\geq c \sum_{s=[m_k/2]+1}^{m_k-1} \frac{2^{2s((1+\alpha)p-1)} 2^{2m_k(1-p(1+\alpha))}}{m_k^p} \geq$$

$$\geq \begin{cases} c m_k^{1-p}, & p = \dfrac{1}{1+\alpha}, \\ c \dfrac{2^{m_k(1-p(1+\alpha))}}{m_k^p}, & 0 < p < \dfrac{1}{1+\alpha}. \end{cases}$$

That is $\left\| \sigma^{\alpha,\kappa,*} f \right\|_p = +\infty$ for $0 < p \leq 1/(1+\alpha)$. The proof is complete.

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