UDC 517.9

D. Jaruszewska-Walczak (Inst. Math. Univ. Gdańsk, Poland)

UNIQUENESS OF SOLUTIONS OF IMPULSIVE HYPERBOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

ЄДИНІСТЬ РОЗВ'ЯЗКІВ ГІПЕРБОЛІЧНИХ ДИФЕРЕНЦІАЛЬНО-ФУНКЦІОНАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСАМИ

For impulsive partial differential-functional equations, we prove theorems on the existence and uniqueness of solutions and their continuous dependence on the right-hand sides of the equations.

Для функціонально-диференціальних рівнянь з частинними похідними та імпульсною дією доведені теореми існування і єдиності розв'язків та їх неперервної залежності від правої частини.

1. Introduction. Numerous problems in the theory of differential or differential-functional equations have been solved by means of differential inequalities. The classical theory of partial differential inequalities is described in detail in the monographs [1] and [2]. The main applications of the theory deal with such questions as estimates of solutions of partial differential equations, estimates for the domain of existence of solutions, criteria of the uniqueness of solutions, stability criteria, continuous dependence of a solution on initial data and on the right hand side of the equation, etc. A similar role in the theory of differential-functional equations with the first-order partial derivatives is played by ordinary differential-functional inequalities. Some results in this field can be found in [3, 4]. All results in [3, 4] concern initial-value problems for the first-order partial differential-functional equations.

In the present paper, we give a generalization of the well-known theorems on differential inequalities with initial conditions [1, 2] or differential-functional inequalities with initial and boundary conditions [5, 6] to the case of impulsive differential-functional inequalities.

The theory of impulsive ordinary differential equations was described in [7, 8]. First, the theory of impulsive partial differential equations was introduced in [9 – 11] and [12]. In [13], impulsive differential inequalities and uniqueness criteria for the Cauchy problem for nonlinear impulsive partial differential equation of the first-order were considered.

In the present paper, we show that impulsive ordinary differential-functional inequalities can be applied to the proof of a comparison theorem and in the uniqueness theory of impulsive partial differential-functional equations.

We now formulate the problem.

Let $C(X, Y)$ denote the class of all continuous mappings from X into Y, where X and Y are metric spaces.

We introduce the following assumption:

Assumption H_0. Assume that a region $\Omega \subset R^{1+n}$ satisfies the following conditions:

i) Ω is open and $\Omega \subset (-h, a) \times R^n$, where $h \geq 0$ and $a > 0$;

ii) let $\Omega_x = \{(\xi, \eta) = (\xi, \eta_1, \ldots, \eta_n) \in \overline{\Omega} : \xi \leq x\}$. Assume that Ω_x is bounded for each $x \in [-h, a)$;

iii) let $S_t = \{y : (t, y) \in \overline{\Omega}\}$. Assume that S_t is nonempty for each $t \in [-h, a)$;

iv) for each $(x, y) \in \overline{\Omega}$ and each sequence $\{x^{(\nu)}\}$ such that $x^{(\nu)} \in [-h, a)$, $\nu = 1, 2, \ldots$, and $\lim_{\nu \to +\infty} x^{(\nu)} = x$, there exists a sequence $y^{(\nu)}$ such that $y^{(\nu)} \in S_{x^{(\nu)}}$, $\nu = 1, 2, \ldots$, and $\lim_{\nu \to +\infty} y^{(\nu)} = y$.

Suppose that Ω satisfies Assumption H_0. Then we define $E_0 = \{(x, y) \in \overline{\Omega} : x \in [-h, 0]\}$.

Assume that $0 < x_1 < \ldots < x_k < a$ are given numbers. Let $I_{\text{imp}} = \{x_1, \ldots, x_k\}$, $P_i = (x_i, x_{i+1})$, $i = 0, 1, \ldots, k$, where $x_0 = 0$, $x_{k+1} = a$, and $\Omega_i = \{(x, y) \in \Omega : x \in \in P_i\}$.

Let sets Z and Z_0 be such that $Z \cap Z_0 = \varnothing$ and $Z \cup Z_0 = \{(x, y) \in \operatorname{Fr} \Omega : x \neq \neq x_i,\ i = 0, 1, \ldots, k+1\}$ where $\operatorname{Fr} \Omega$ is the boundary of Ω. We assume that the differential equation is satisfied on Z and consider a boundary condition on Z_0. In the next parts of the paper, we formulate additional conditions on the sets Z and Z_0.

Let $E = \Omega \cup Z$, $E_{\text{imp}} = \{(x, y) \in \Omega : x \in I_{\text{imp}}\}$ and $E^* = \bigcup_{x \in [-h, a)} \Omega_x$.

Let $C_{\text{imp}}(E^*, R)$ denote the class of all functions $z : E^* \to R$ that satisfy the following conditions:

i) $z|_{E_0}$, $z|_{\Omega_i}$ are continuous for $i = 0, 1, \ldots, k$;

ii) for each i, j, $1 \le i \le k$, $0 \le j \le k$, there exist the limits

$$\lim_{(t,s) \to (x_i^-, y)} z(t, s) = z(x_i^-, y), \quad y \in S_{x_i},$$

and

$$\lim_{(t,s) \to (x_j^+, y)} z(t, s) = z(x_j^+, y), \quad y \in S_{x_j}, \quad \text{and} \quad z(x_j^+, y) = z(x_j, y), \quad y \in S_{x_j}.$$

We denote by $C^*_{\text{imp}}(E^*, R)$ the set of functions $z \in C_{\text{imp}}(E^*, R)$ such that z has partial derivatives $D_x z$ and $D_y z = (D_{y_1} z, \ldots, D_{y_n})$ on Ω_i, $i = 0, 1, \ldots, k$, and there exists the total derivative of z on the set Z.

Let $\Gamma = (E \setminus E_{\text{imp}}) \times R \times C_{\text{imp}}(E^*, R) \times R^n$ and $\tilde{\Gamma} = E_{\text{imp}} \times R \times C^*_{\text{imp}}(E^*, R)$. Assume that $f : \Gamma \to R$, $g : \tilde{\Gamma} \to R$, $\varphi : E_0 \to R$, and $\psi : Z_0 \to R$ are given functions. We consider the impulsive differential-functional equation

$$D_x z(x, y) = f(x, y, z(x, y), z, D_y z(x, y)), \qquad (x, y) \in E \setminus E_{\text{imp}}, \tag{1}$$

$$\Delta z(x, y) = g(x, y, z(x^-, y), z), \qquad (x, y) \in E_{\text{imp}}, \tag{2}$$

where $y = (y_1, \ldots, y_n)$, $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$, and $\Delta z(x, y) = z(x, y) - z(x^-, y)$, with the initial-boundary conditions

$$z(x, y) = \varphi(x, y) \quad \text{on } E_0, \qquad z(x, y) = \psi(x, y) \quad \text{on } Z_0. \tag{3}$$

In the differential equation (1), there is a dependence on the point (x, y), on the values of unknown functions of z and their derivatives at the point (x, y), and on the function z. The differential-integral equation

$$D_x z(x, y) = F_1\left(x, y, z(x, y), \int\limits_{H(x,y)} z(t, s)\, dt\, ds,\ D_y z(x, y)\right)$$

or the differential equation with retarded argument

$$D_x z(x, y) = F_2(x, y, z(x, y), z(A(x, y), B(x, y)), D_y z(x, y))$$

are examples of Eq. (1). Similar examples can be given for Eq. (2).

The function $u : E^* \to R$ is a solution of problem (1)–(3) if $u \in C^*_{\text{imp}}(E^*, R)$ and u satisfies Eqs. (1), (2) and the initial-boundary conditions (3).

2. Comparison theorem. Let $C_{\text{imp}}([-h, a), R_+)$ denote the set of all $\alpha : [-h, a) \to R_+$ such that

i) $\alpha|_{[-h,0]}$, $\alpha|_{P_i}$ are continuous for each $i = 0, 1, \ldots, k$;

ii) for each i, j, $1 \le i \le k$, $0 \le j \le k$, there exist the limits

$$\lim_{x \to x_i^-} \alpha(x) = \alpha(x_i^-), \quad \text{and} \quad \lim_{x \to x_j^+} \alpha(x) = \alpha(x_j^+), \quad \text{and} \quad \alpha(x_j^+) = \alpha(x_j).$$

For $w \in C^*_{\text{imp}}(E^*, R)$, we define $Tw : [-h, a) \to R$ in the following way:

$$(Tw)(x) = \max(|u(x, y)| : y \in S_x\}.$$

Assume that $\lambda = (\lambda_1, \ldots, \lambda_n) : E \backslash E_{\text{imp}} \to R^n$, and

$$\sigma;\ ((0, a) \backslash I_{\text{imp}}) \times R_+ \times C_{\text{imp}}([-h, a), R_+) \to R_+,$$
$$\tilde{\sigma} : I_{\text{imp}} \times R_+ \times C_{\text{imp}}([-h, a), R_+) \to R_+$$

are given functions. We consider the impulsive differential-functional inequalities

$$\left| D_x z(x, y) - \sum_{i=1}^{n} \lambda_i(x, y) D_{y_i} z(x, y) \right| \le \sigma(x, |z(x, y)|, Tz), \qquad (x, y) \in E \backslash E_{\text{imp}},$$

$$|\Delta z(x, y)| \le \tilde{\sigma}(x, |z(x^-, y)|, Tz), \qquad (x, y) \in E_{\text{imp}}.$$

We prove that the functions satisfying these inequalities can be estimated by solutions of ordinary impulsive differential-functional equations.

Assumption H_1. Assume that the function $\lambda = (\lambda_1, \ldots, \lambda_n) : E \backslash E_{\text{imp}} \to R^n$ has the following property: for each $(x, y) \in Z$, there exists $\varepsilon_0 > 0$ such that $(x - \tau, y + \tau \lambda(x, y)) \in \overline{\Omega}$ for $\tau \in [0, \varepsilon_0]$.

Assumption H_2. Assume that

1) the function $\sigma : ((0, a) \backslash I_{\text{imp}}) \times R_+ \times C_{\text{imp}}([-h, a), R_+) \to R_+$ is continuous and nondecreasing with respect to the functional argument and satisfies the Volterra condition, i.e., if $(x, p) \in ((0, a) \backslash I_{\text{imp}}) \times R_+$, $z, \bar{z} \in C_{\text{imp}}([-h, a), R_+)$ and $z(\xi) = \bar{z}(\xi)$ for $\xi \le x$, then $\sigma(x, p, z) = \sigma(x, p, \bar{z})$;

2) the function $\tilde{\sigma} : I_{\text{imp}} \times R_+ \times C_{\text{imp}}([-h, a), R_+) \to R_+$ is continuous and nondecreasing with respect to the last arguments and satisfies the following left-hand-side Volterra condition: if $z(\xi) = \bar{z}(\xi)$ for $\xi < x_i$, then $\tilde{\sigma}(x_i, p, z) = \tilde{\sigma}(x_i, p, \bar{z})$, $i = 1, 2, \ldots, k$, $p \in R_+$;

3) for each $\gamma \in C([-h, a), R_+)$, the maximum solution of the problem

$$\alpha'(x) = \sigma(x, \alpha(x), \alpha), \qquad x \in (0, a) \backslash I_{\text{imp}},$$
$$\Delta\alpha(x) = \tilde{\sigma}(x, \alpha(x^-), \alpha), \qquad x \in I_{\text{imp}}, \tag{4}$$
$$\alpha(x) = \gamma(x), \qquad x \in [-h, 0],$$

where $\Delta\alpha(x) = \alpha(x) - \alpha(x^-)$, exists on $[-h, a)$.

Lemma. *Suppose that Assumption H_0 is satisfied and*

1) *$\rho \in C_{\text{imp}}([-h, a), R_+)$ and $\rho(t) \le \gamma(t)$ for $t \in [-h, 0]$, where $\gamma \in C([-h, 0], R_+)$;*

2) *$\omega(\cdot, \gamma)$ is the maximum solution of (4) and $T_+ = \{t \in (0, a) : \rho(t) > \omega(t, \gamma)\}$;*

3) *for* $x \in T_+ \setminus I_{\text{imp}}$, *we have*

$$D_-\rho(x) \le \sigma(x, \rho(x), \rho),$$

and for $x \in I_{\text{imp}} \cap T_+$

$$\Delta\rho(x) \le \tilde{\sigma}(x, \rho(x^-), \rho).$$

Then $\rho(x) \le \omega(x, \gamma)$, $x \in [-h, a)$.

In the proof of Lemma, we use the classical methods. Here, we omit details (see [1] or [2] for the case without functional variable).

Theorem 1. *Suppose that Assumption* H_0 *and Assumption* H_1 *are satisfied and*

1) *the function* $u \in C^*_{\text{imp}}(E^*, R)$ *satisfies the initial inequality* $|u(x, y)| \le \le \gamma(x)$, $(x, y) \in E_0$, *where* $\gamma \in C([-h, 0], R_+)$;

2) *the functions* σ *and* $\tilde{\sigma}$ *satisfy Assumption* H_2 *and* $\omega(\cdot, \gamma)$ *is the maximum solution of* (4);

3) *the boundary estimate* $|u(x, y)| \le \omega(x, \gamma)$, $(x, y) \in Z_0$, *is true;*

4) *the inequalities*

$$\left| D_x u(x, y) - \sum_{i=1}^{n} \lambda_i(x, y) D_{y_i} u(x, y) \right| \le$$

$$\le \sigma(x, |u(x, y)|, Tu), \qquad (x, y) \in E \setminus E_{\text{imp}},$$

and

$$|\Delta u(x, y)| \le \tilde{\sigma}(x, |u(x^-, y)|, Tu), \qquad (x, y) \in E_{\text{imp}},$$

are true.

Under these assumptions, we have

$$|u(x, y)| \le \omega(x, \gamma), \qquad (x, y) \in E^*.$$

Proof. Let $\rho(x) = \max\{|u(x, y)| : y \in S_x\}$, $x \in [-h, a)$. Then $\rho \in \in C_{\text{imp}}([-h, a), R_+)$ and $\rho(x) \le \gamma(x)$ for $x \in [-h, 0]$. Let $x^* \in (0, a)$ be a point such that $\rho(x^*) > \omega(x^*, \gamma)$. There exists $y^* \in S_{x^*}$ such that $\rho(x^*) = |u(x^*, y^*)|$. We have $(x^*, y^*) \in E$.

Assume that $(x^*, y^*) \in \operatorname{Int} E \setminus E_{\text{imp}}$. Then $D_x u(x^*, y^*) = 0$ and we get

$$D_-\rho(x^*) \le |D_x u(x^*, y^*)| =$$

$$= \left| D_x u(x^*, y^*) - \sum_{i=1}^{n} \lambda_i(x^*, y^*) D_{y_i} u(x^*, y^*) \right| \le$$

$$\le \sigma(x^*, |u(x^*, y^*)|, Tu) = \sigma(x^*, \rho(x^*), \rho).$$

Assume that $(x^*, y^*) \in Z$. Then there exists $\varepsilon_0 > 0$ such that $(x^* - \tau, y^* + + \tau\lambda(x^*, y^*)) \in \overline{\Omega}$ for $\tau \in [0, \varepsilon_0]$. Since $\rho(x^*) = |u(x^*, y^*)|$, we have a) $\rho(x^*) = u(x^*, y^*)$ or b) $\rho(x^*) = -u(x^*, y^*)$.

Let us consider the case a). We define $\overline{u}(\tau) = u(x^* - \tau, y^* + \tau\lambda(x^*, y^*))$, $\tau \in \in [0, \varepsilon_0]$. As a result, we get

$$D_-\rho(x^*) \le -\frac{d}{d\tau}\overline{u}(0).$$

Since

$$\frac{d}{d\tau}\overline{u}(\tau) = -D_x u(x^* - \tau, y^* + \tau\lambda(x^*, y^*)) +$$
$$+ \sum_{i=1}^{n} D_{y_i} u(x^* - \tau, y^* + \tau\lambda(x^*, y^*))\lambda_i(x^*, y^*)$$

we have

$$D_-\rho(x^*) \le -\frac{d}{d\tau}\overline{u}(0) =$$
$$= D_x u(x^*, y^*) + \sum_{i=1}^{n} D_{y_i} u(x^*, y^*)\lambda_i(x^*, y^*) \le$$
$$\le \sigma(x^*, |u(x^*, y^*)|, Tu) = \sigma(x^*, \rho(x^*), \rho).$$

In a similar way, we obtain the inequality $D_-\rho(x^*) \le \sigma(x^*, \rho(x^*), \rho)$ if $(x^*, y^*) \in Z$ and the possibility b) holds.

Since $|u(x_i, y)| \le |u(x_i^-, y)| + \tilde{\sigma}(x_i, |u(x_i^-, y)|, Tu)$ and $|u(x_i^-, y)| \le \rho(x_i^-)$ for $y \in S_{x_i}$, we have

$$\rho(x_i^*) = |u(x_i, y^*)| \le \rho(x_i^-) + \tilde{\sigma}(x_i, \rho(x_i^-), \rho)$$

or

$$\Delta\rho(x_i) \le \tilde{\sigma}(x_i, \rho(x_i^-), \rho), \quad i = 1, 2, \ldots, k.$$

By Lemma, we get $\rho(x) \le \omega(x, \gamma)$, $x \in [-h, a)$, which completes the proof of Theorem 1.

3. Estimation of the difference between solutions of two problems. Uniqueness criteria. We now introduce Assumption H_3 for the functions $f\colon \Gamma \to R$ and $g : \tilde{\Gamma} \to \to R$.

Assumption H_3. Suppose that

1) a function $f\colon \Gamma \to R$ of variables (x, y, p, z, q) satisfies the Volterra condition on Γ, i.e., if $(x, y, p, q) \in (E \setminus E_{\text{imp}}) \times R \times R^n$, z, $\overline{z} \in C_{\text{imp}}(E^*, R)$, and $z(\xi, \eta) = = \overline{z}(\xi, \eta)$ for $(\xi, \eta) \in E \setminus E_{\text{imp}}$, $\xi \le x$, then $f(x, y, p, z, q) = f(x, y, p, \overline{z}, q)$;

2) there exists partial derivatives $(D_{q_1} f, \ldots, D_{q_n} f) = D_q f$ on Γ and for each $(x, y) \in Z$, there is $\varepsilon_0 > 0$ such that $(x - \tau, y + \tau D_q f(x, y, p, z, q)) \in \overline{\Omega}$ for $\tau \in \in [0, \varepsilon_0]$, $(p, z, q) \in C_{\text{imp}}(E^*, R) \times R^n$;

3) for each $(x, y, p, z) \in (E \setminus E_{\text{imp}}) \times R \times C_{\text{imp}}(E^*, R)$ we have $D_q f(x, y, p, z, q) \in C(R^n, R^n)$;

4) the function $g : \tilde{\Gamma} \to R$ of variables (x, y, p, z) satisfies the left-hand-side Volterra condition.

Let us consider problem (1)–(3) and the problem

$$D_x z(x, y) = \tilde{f}(x, y, z(x, y), z, D_y z(x, y)), \quad (x, y) \in E \setminus E_{\text{imp}},$$
$$\Delta z(x, y) = \tilde{g}(x, y, z(x^-, y), z), \quad (x, y) \in E_{\text{imp}}, \tag{5}$$
$$z(x, y) = \tilde{\varphi}(x, y), \quad (x, y) \in E_0; \quad z(x, y) = \tilde{\psi}(x, y), \quad (x, y) \in Z_0,$$

where $\tilde{f} : \Gamma \to R$, $\tilde{\varphi} : E_0 \to R$, $\tilde{\psi} : Z_0 \to R$, and $\tilde{g} : \tilde{\Gamma} \to R$ are given functions.

The following theorem allows us to estimate the difference between solutions of the problems (1)–(3) and (5).

Theorem 2. *Suppose that Assumptions* H_0, H_2, *and* H_3 *are satisfied and*

1) $\tilde{f}:\Gamma\to R$ *and the estimate*

$$|f(x,y,p,z,q)-\tilde{f}(x,y,\overline{p},\overline{z},q)|\le\sigma(x,|p-\overline{p}|,T(z-\overline{z}))$$

hold for (x,y,p,z,q), $(x,y,\overline{p},\overline{z},q)\in\Gamma$;

2) $|\varphi(x,y)-\tilde{\varphi}(x,y)|\le\gamma(x)$ *for* $(x,y)\in E_0$, *where* $\gamma\in C([-h,0],R_+)$;

3) $\omega(\cdot,\gamma)$ *is the maximum solution of* (4) *and*

$$|\psi(x,y)-\tilde{\psi}(x,y)|\le\omega(x,y),\qquad (x,y)\in Z_0;$$

4) $\tilde{g}:\tilde{\Gamma}\to R$ *and, on* $\tilde{\Gamma}$,

$$|g(x,y,p,z)-\tilde{g}(x,y,\overline{p},\overline{z})|\le\tilde{\sigma}(x,|p-\overline{p}|,T(z-\overline{z}));$$

5) *functions* u, $v\in C_{\text{imp}}(E^*,R)$ *satisfy problems* (1) – (3) *and* (5) *respectively.*

Then

$$|u(x,y)-v(x,y)|\le\omega(x,\gamma),\qquad (x,y)\in E^*. \tag{6}$$

Proof. Let us define $\overline{z}(x,y)=u(x,y)-v(x,y)$, $(x,y)\in E^*$. We have $|\overline{z}(x,y)|\le\gamma(x)$ for $(x,y)\in E_0$ and $|\overline{z}(x,y)|\le\omega(x,\gamma)$, $(x,y)\in Z_0$. Since

$$D_x\overline{z}(x,y)=\sum_{i=1}^{n}D_{y_i}\overline{z}(x,y)\int_0^1 D_{q_i}f(W(\mu,x,y))\,d\mu+$$
$$+\;f(x,y,u(x,y),u,D_yv(x,y))-\tilde{f}(x,y,v(x,y),v,D_yv(x,y)),$$

where

$$W(\mu,x,y)=(x,y,u(x,y),u,D_yv(x,y))+\mu\,(D_yu(x,y)-D_yv(x,y))), \tag{7}$$

we get

$$\left|D_x\overline{z}(x,y)-\sum_{i=1}^{n}D_{y_i}\overline{z}(x,y)\int_0^1 D_{q_i}f(W(\mu,x,y))\,d\mu\right|\le$$
$$\le\sigma(x,|\overline{z}(x,y)|,T\overline{z}).$$

Moreover, $|\Delta\overline{z}(x,y)|\le\tilde{\sigma}(x,|\overline{z}(x^-,y)|,T\overline{z})$, $(x,y)\in E_{\text{imp}}$. By Theorem 1, we have (6).

In order to simplify the formulation of the subsequent theorem, we introduce the following assumption:

Assumption H_4. Suppose that Assumption H_2 is satisfied and the function $w(t)\equiv 0$, $t\in[-h,a)$, is the unique solution of the problem (4) with $\gamma(t)\equiv 0$, $t\in[-h,0]$.

The following uniqueness theorem is a consequence of Theorem 2:

Theorem 3. *If Assumptions* H_0, H_3, *and* H_4 *are satisfied and*

1) *the inequality*

$$|f(x,y,p,z,q)-f(x,y,\overline{p},\overline{z},q)|\le\sigma(x,|p-\overline{p}|,T(z-\overline{z}))$$

holds on Γ;

2) *the inequality*

$$|g(x,y,p,z)-g(x,y,\overline{p},\overline{z})|\le\tilde{\sigma}(x,|p-\overline{p}|,T(z-\overline{z}))$$

holds on $\tilde{\Gamma}$,

then solution of (1) – (3) *is unique in the class* $C^*_{\text{imp}}(E^*, R)$.

Example. Assume that $\Omega = (-h, a) \times (-b, b)$, where $h \geq 0$, $a > 0$, $b = (b_1, \ldots, b_n)$, and $b_i > 0$ for $i = 1, \ldots, n$. Then Ω satisfies Assumption H_0.

Let $f\colon \Gamma \to R$ and $g\colon \tilde{\Gamma} \to R$ with Ω given above satisfy Assumption H_3 and let the function

$$\operatorname{sign} D_y f = (\operatorname{sign} D_{q_1} f, \ldots, \operatorname{sign} D_{q_n} f)$$

be constant on Γ.

We define the sets I_+ and I_-, $I_+ \cap I_- = \varnothing$, in the following way:

$$I_+ = \{ i \in \{1, \ldots, n\} \colon D_{q_i} f \geq 0 \text{ on } \Gamma \},$$

$$I_- = \{ i \in \{1, \ldots, n\} \colon D_{q_i} f \leq 0 \text{ on } \Gamma \}.$$

Let $Z_0 = \{ (x, y) \in (0, a) \times [-b, b] \colon x \neq x_j,\ j = 1, \ldots, k,$ and there exists $i \in I_+$ such that $y_i = b_i$ or there exists $i \in I_-$ such that $y_i = -b_i \}$.

By using Ω introduced above and Z_0, we define the sets E_0, Z, E, E_{imp}, and E^*. A solution of problem (1) – (3) in this case is unique in the class $C^*_{\text{imp}}(E^*, R)$.

The next theorem shows continuous dependence of solutions on the initial data and on the right-hand side of the equation.

Theorem 4. *Suppose that the conditions of Theorem* 3 *are satisfied and the functions* $u, v \in C^*_{\text{imp}}(E^*, R)$ *are solutions of* (1) – (3) *and* (5), *respectively.*

Then for every $\varepsilon > 0$, *one can find* $\delta > 0$ *such that if, on* Γ,

$$|f(x, y, p, z, q) - \tilde{f}(x, y, p, z, q)| < \delta$$

and

$$|\varphi(x, y) - \tilde{\varphi}(x, y)| \leq \delta, \quad (x, y) \in E_0,$$

$$|\psi(x, y) - \tilde{\psi}(x, y)| \leq \delta, \quad (x, y) \in Z_0,$$

$$|g(x, y, p, z) - \tilde{g}(x, y, p, z)| \leq \delta, \quad (x, y, p, z) \in \tilde{\Gamma},$$

then

$$|u(x, y) - v(x, y)| < \varepsilon, \quad (x, y) \in E^*.$$

Proof. For $\varepsilon > 0$, we can choose $\delta > 0$ such that the maximum solution $\omega(\cdot, \delta)$ of the problem

$$\alpha'(x) = \sigma(x, \alpha(x), \alpha) + \delta, \quad x \in (0, a) \setminus I_{\text{imp}},$$

$$\Delta\alpha(x) = \tilde{\sigma}(x, \alpha(x^-), \alpha) + \delta, \quad x \in I_{\text{imp}},$$

$$\alpha(x) = \delta, \quad x \in [-h, 0],$$

is defined in the interval $[-h, a)$ and $\omega(x, \delta) < \varepsilon$ for $x \in [-h, a)$. Denote $\bar{z}(x, y) = u(x, y) - v(x, y)$. By Theorem 1, we have

$$|\bar{z}(x, y)| \leq \omega(x, \delta) < \varepsilon, \quad (x, y) \in E^*,$$

which completes the proof.

4. Differential-functional inequalities. In this part of our paper, we give theorems on impulsive differential-functional inequalities.

Theorem 5. *Suppose that Assumptions* H_0 *and* H_3 *are satisfied and*

1) *the function* $f\colon \Gamma \to R$ *is nondecreasing with respect to the functional argument and the function* $g\colon \tilde{\Gamma} \to R$ *is nondecreasing with respect to the last two arguments*;

2) *the functions* $u, v \in C^*_{\mathrm{imp}}(E^*, R)$ *satisfy the conditions* $u(x,y) < v(x,y)$, $(x,y) \in E_0 \bigcup Z_0$, *and the inequalities*

$$\begin{aligned} D_x u(x,y) &\le f(x,y,u(x,y),u,D_y u(x,y)), \\ D_x v(x,y) &> f(x,y,v(x,y),v,D_y v(x,y)), \end{aligned} \tag{8}$$

hold for $(x,y) \in S \setminus E_{\mathrm{imp}}$, *where*

$$S = \{(\xi,\eta) \in E \colon u(\tilde{\xi},\tilde{\eta}) < v(\tilde{\xi},\tilde{\eta}),\ \tilde{\xi} < \xi,\ (\tilde{\xi},\tilde{\eta}) \in E,\ u(\xi,\eta) = v(\xi,\eta)\}$$

and

$$\Delta u(x,y) \le g(x,y,u(x^-,y),u), \quad (x,y) \in E_{\mathrm{imp}} \bigcap S,$$

$$\Delta v(x,y) > g(x,y,v(x^-,y),v), \quad (x,y) \in E_{\mathrm{imp}} \bigcap S.$$

Then $u(x,y) < v(x,\gamma)$, $(x,y) \in E^*$.

Proof. It is sufficient to prove that $S = \varnothing$. Let $(x^*, y^*) \in S$. We have $u(x,y) < v(x,\gamma)$ for $x < x^*$, $(x,y) \in E^*$, and $u(x^*,y^*) = v(x^*,y^*)$. It follows from (8) and condition 1 that

$$D_x u(x^*,y^*) - D_x v(x^*,y^*) <$$

$$< \sum_{i=1}^{n} (D_{y_i} u(x^*,y^*) - D_{y_i} v(x^*,y^*)) \int_0^1 D_{q_i} f(W(\mu, x^*, y^*))\, d\mu$$

where W is defined by (7).

Assume that $(x^*,y^*) \in \operatorname{Int} E \setminus E_{\mathrm{imp}}$. Then $D_y u(x^*,y^*) - D_y v(x^*,y^*) = 0$ and $D_x u(x^*,y^*) - D_x v(x^*,y^*) \ge 0$.

Assume that $(x^*,y^*) \in Z$. Let $\varepsilon_0 > 0$ be such that

$$P(\tau) = \left(x^* - \tau,\ y^* + \tau \int_0^1 D_y f(W(\mu, x^*, y^*))\, d\mu \right) \in \overline{\Omega}, \quad \tau \in [0, \varepsilon_0].$$

Denote $\tilde{u}(\tau) = u(P(\tau)) - v(P(\tau))$. We have $\tilde{u}(0) = 0$ and $\tilde{u}(\tau) < 0$, $\tau \in (0, \varepsilon_0]$. Hence,

$$0 \le \frac{d}{d\tau}\tilde{u}(0) = -(D_x u(x^*,y^*) - D_x v(x^*,y^*)) +$$

$$+ \sum_{i=1}^{n} (D_{y_i} u(x^*,y^*) - D_{y_i} v(x^*,y^*)) \int_0^1 D_{q_i} f(W(\mu, x^*, y^*))\, d\mu$$

or

$$D_x u(x^*,y^*) - D_x v(x^*,y^*) \ge$$

$$\ge \sum_{i=1}^{n} (D_{y_i} u(x^*,y^*) - D_{y_i} v(x^*,y^*)) \int_0^1 D_{q_i} f(W(\mu, x^*, y^*))\, d\mu .$$

Assume that $x^* = x_i$ for some $i = 1, \ldots, k$. Then

$$u(x_i, y^*) - v(x_i, y^*) \le$$

$$\le u(x_i^-, y^*) - v(x_i^-, y^*) + g(x_i, y^*, u(x_i^-, y^*), u) - g(x_i, y^*, v(x_i^-, y^*), v) \le 0.$$

In each of these cases, we arrive at a contradiction, which proves that $u(x, y) < v(x, y)$ for $(x, y) \in E^*$.

We now formulate a theorem on weak inequalities.

Theorem 6. *Suppose that Assumptions H_0, H_3, and H_4 are satisfied and*

1) *$f\colon \Gamma \to R$ is nondecreasing with respect to the functional argument and $g\colon \tilde{\Gamma} \to R$ is nondecreasing with respect to the last two arguments;*

2) *the following estimates hold on Γ and $\tilde{\Gamma}$, respectively:*

$$|f(x, y, p, z, q) - \tilde{f}(x, y, \overline{p}, \overline{z}, q)| \le \sigma(x, |p - \overline{p}|, T(z - \overline{z})),$$

$$|g(x, y, p, z) - \tilde{g}(x, y, \overline{p}, \overline{z})| \le \tilde{\sigma}(x, |p - \overline{p}|, T(z - \overline{z}));$$

3) *the functions $u, v \in C^*_{\text{imp}}(E^*, R)$ satisfy the initial-boundary inequalities*

$$u(x, y) \le v(x, y), \qquad (x, y) \in E_0 \cup Z_0,$$

$$D_x u(x, y) \le f(x, y, u(x, y), u, D_y u(x, y)), \qquad (x, y) \in E \setminus E_{\text{imp}},$$

$$D_x v(x, y) \ge f(x, y, v(x, y), v, D_y v(x, y)), \qquad (x, y) \in E \setminus E_{\text{imp}},$$

and

$$\Delta u(x, y) \le g(x, y, u(x^-, y), u), \qquad (x, y) \in E_{\text{imp}},$$

$$\Delta v(x, y) \ge g(x, y, v(x^-, y), v), \qquad (x, y) \in E_{\text{imp}}.$$

Then $u(x, y) \le v(x, \gamma)$, $(x, y) \in E^*$.

Proof. Let $\varepsilon_1 > 0$ be such that, for each $0 < \varepsilon < \varepsilon_1$, the maximum solution $\omega(\cdot, \varepsilon)$ of the problem

$$\alpha'(x) = \sigma(x, \alpha(x), \alpha) + \varepsilon, \qquad x \in (0, a) \setminus I_{\text{imp}},$$

$$\Delta\alpha(x) = \tilde{\sigma}(x, \alpha(x^-), \alpha) + \varepsilon, \qquad x \in I_{\text{imp}},$$

$$\alpha(x) = \varepsilon, \qquad x \in [-h, 0],$$

exists on $[-h, a)$ and $\lim\limits_{\varepsilon \to 0} \omega(x, \varepsilon) = 0$ uniformly on $[-h, a)$.

We define $\tilde{v}(x, y) = v(x, y) + \omega(x, \varepsilon)$, $(x, y) \in E^*$, $0 < \varepsilon < \varepsilon_1$. For $(x, y) \in E_0 \cup Z_0$, we have $u(x, y) \le v(x, y) < \tilde{v}(x, y)$.

Let $(x, y) \in E \setminus E_{\text{imp}}$. Then

$$D_x \tilde{v}(x, y) = D_x v(x, y) + \omega'(x, \varepsilon) \ge$$

$$\ge f(x, y, v(x, y), v, D_y v(x, y)) + \sigma(x, \omega(x, \varepsilon), \omega(\cdot, \varepsilon)) + \varepsilon =$$

$$= f(x, y, v(x, y), v, D_y v(x, y)) - f(x, y, \tilde{v}(x, y), \tilde{v}, D_y \tilde{v}(x, y)) +$$

$$+ f(x, y, \tilde{v}(x, y), \tilde{v}, D_y \tilde{v}(x, y)) + \sigma(x, \omega(x, \varepsilon), \omega(\cdot, \varepsilon)) + \varepsilon \ge$$

$$\ge -\sigma(x, |v(x, y) - \tilde{v}(x, y)|, T(v - \tilde{v})) +$$

$$+ f(x, y, \tilde{v}(x, y), \tilde{v}, D_y \tilde{v}(x, y)) + \sigma(x, \omega(x, \varepsilon), \omega(\cdot, \varepsilon)) + \varepsilon =$$

$$= f(x, y, \tilde{v}(x, y), \tilde{v}, D_y \tilde{v}(x, y)) + \varepsilon >$$

$$> f(x, y, \tilde{v}(x, y), \tilde{v}, D_y \tilde{v}(x, y)).$$

Moreover, for $(x, y) \in E_{\text{imp}}$, we have

$$\Delta \tilde{v}(x, y) = \Delta v(x, y) + \Delta\omega(x, \varepsilon) \ge$$

$$\ge g(x, y, v(x^-, y), v) - g(x, y, v(x^-, y), \tilde{v}) +$$

$$+ \; g(x, y, v(x^-, y), \tilde{v}) + \tilde{\sigma}(x, \omega(x^-, \varepsilon), \omega(\cdot, \varepsilon)) + \varepsilon \geq$$

$$\geq -\tilde{\sigma}(x, |v(x^-, y) - \tilde{v}(x^-, y)|, T(v - \tilde{v})) +$$

$$+ \; g(x, y, v(x^-, y), \tilde{v}) + \tilde{\sigma}(x, \omega(x^-, \varepsilon), \omega(\cdot, \varepsilon)) + \varepsilon >$$

$$> g(x, y, \tilde{v}(x^-, y), \tilde{v}).$$

Hence, by Theorem 5, the inequality $u(x, y) < \tilde{v}(x, y)$ holds for $(x, y) \in E^*$. Passing to the limit as $\varepsilon \to 0$, we get

$$u(x, y) \leq v(x, y), \quad (x, y) \in E^*.$$

Theorem 6 has the following consequence:

Remark. If a function $f\colon \Gamma \to R$ satisfies Assumptions H_0, H_3, and H_4 and conditions 1 and 2 of Theorem 6, then a solution of the problem (1) – (3) is unique in the class $C^*_{\text{imp}}(E^*, R)$.

1. *Lakshmikantham V., Leela S.* Differential and integral inequalities. – New York; London, 1969.
2. *Szarski J.* Differential inequalities. – Warszawa, 1967. – 256 p.
3. *Kamont Z.* On the Cauchy problem for system of first order partial differential-functional equations // Serdica Bulg. Math. Publ. – 1979. – **5**. – P. 327–339.
4. *Kamont Z.* On the Chaplygin method for partial differential-functional equations of the first order // Ann. Polon. Math. – 1980. – **38**, № 1. – P. 27–46.
5. *Brandi P., Kamont Z., Salvadori A.* Differential and differential-difference inequalities related to mixed problems for first order partial differential-functional equations // Atti Semin. e mat. fis. Univ. Modena. – 1991. – **39**. – P. 255–276.
6. *Jaruszewska-Walczak D.* Differential-functional inequalities related to initial-boundary problems for first order partial differential-functional equations // Periodica math. hung. – 1993. – **26**, № 2. – P. 163–174.
7. *Bainov D., Lakshmikantham V., Simeonov P.* Theory of impulsive differential equations. – Singapore: World Sci., 1989.
8. *Samoilenko A. M., Perestyuk N. A.* Impulsive differential equations. – Kiev, 1987 (in Russian); World Sci. Ser. Nonlinear Sci. Ser. A. – 1995. – **14**.
9. *Rogovchenko S.* Periodic solutions of hyperbolic systems with fixed moments of impulse effect. – Kiev, 1988. – 38 p. – (Preprint / Acad. Sci. Ukraine. Inst. Math.; 88.8) (in Russian).
10. *Rogovcenko Yu. V., Trofimchuk S. I.* Periodical solutions of weakly nonlinear impulsive partial differential equations of parabolic type and their stability. – Kiev, 1986. – 44 p. – (Preprint / Acad. Sci. Ukraine. Inst. Math.; 86-65) (in Russian).
11. *Rogovchenko Yu. V., Trofimchuk S. I.* Restricted and periodical solutions of weakly non-linear impulse evolutionary systems // Ukr. Mat. Zh. – 1987. – **39**, № 2. – P. 260–264 (in Russian).
12. *Erbe L. H., Freedman H. I., Liu X. Z., Wu J. H.* Comparison principles for impulsive parabolic equations with applications to models of single species growth // J. Austral. Math. Soc. Ser. B. – 1991. – **32**. – P. 382–400.
13. *Bainov D., Kamont Z., Minchev E.* On first order impulsive partial differential inequalities // Appl. Math. Comp. – 1994. – **61**. – P. 207–230.

Received 25.03.96