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GENERALIZED WEYL'S THEOREM AND TENSOR PRODUCT

УЗАГАЛЬНЕНА ТЕОРЕМА ВЕЙЛЯ ТА ТЕНЗОРНИЙ ДОБУТОК

We give necessary and/or sufficient conditions ensuring the passage of generalized a-Weyl theorem and property (gw) from A and B to $A \otimes B$.

Наведено необхідні та/або достатні умови, що гарантують поширення узагальненої a-теореми Вейля та властивості (gw) із A та B на $A \otimes B$.

1. Introduction. Given Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, let $\mathbb{X}\otimes\mathbb{Y}$ denote the completion (in some reasonable uniform cross norm) of the tensor product of $\mathbb{X}$ and $\mathbb{Y}$. For Banach space operators $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$, let $A \otimes B \in \mathcal{L}(\mathbb{X} \otimes \mathbb{Y})$ denote the tensor product of A and B. Recall that for an operator S, the Browder spectrum $\sigma_b(S)$ and the Weyl spectrum $\sigma_w(S)$ of S are the sets

$$\sigma_b(S) = \{\lambda \in \mathbb{C}\colon S - \lambda \text{ is not Fredholm or } \operatorname{asc}(S-\lambda) \neq \operatorname{dsc}(S-\lambda)\},$$

$$\sigma_w(S) = \{\lambda \in \mathbb{C}\colon S - \lambda \text{ is not Fredholm or } \operatorname{ind}(S-\lambda) \neq 0\}.$$

In the case in which $\mathbb{X}$ and $\mathbb{Y}$ are Hilbert spaces, Kubrusly and Duggal [15] proved that

$$\text{if } \sigma_b(A) = \sigma_w(A) \quad \text{and} \quad \sigma_b(B) = \sigma_w(B), \quad \text{then} \quad \sigma_b(A\otimes B) = \sigma_w(A \otimes B)$$

$$\text{if and only if} \quad \sigma_w(A\otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B).$$

In other words, if A and B satisfy Browder's theorem, then their tensor product satisfies Browder's theorem if and only if the Weyl spectrum identity holds true. The same proof still holds in a Banach space setting.

For a bounded linear operator $S \in \mathcal{L}(\mathbb{X})$, let $\sigma(S), \sigma_p(S)$ and $\sigma_a(S)$ denote, respectively, the spectrum, the point spectrum and the approximate point spectrum of S and if $G \subseteq \mathbb{C}$, then G^{iso} denote the isolated points of G. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of S, defined by $\alpha(S) = \dim\ker(S)$ and $\beta(S) = \operatorname{codim}\Re(S)$.

If the range $\Re(S)$ of S is closed and $\alpha(S) < \infty$ (respectively $\beta(S) < \infty$), then S is called an upper semi-Fredholm (respectively a lower semi-Fredholm) operator. If $S \in \mathcal{L}(\mathbb{X})$ is either upper or lower semi-Fredholm, then S is called a semi-Fredholm operator, and $\operatorname{ind}(S)$, the index of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a Fredholm operator. The ascent, denoted $\operatorname{asc}(S)$, and the descent, denoted $\operatorname{dsc}(S)$, of S are given by $\operatorname{asc}(S) = \inf\left\{n \in \mathbb{N}\colon \ker(S^n) = \ker(S^{n+1}\right\}$, $\operatorname{dsc}(S) = \inf\left\{n \in \mathbb{N}\colon \Re(S^n) = \Re(S^{n+1}\right\}$ (where the infimum is taken over the set of non-negative integers); if no such integer n exists, then $\operatorname{asc}(S) = \infty$, respectively $\operatorname{dsc}(S) = \infty$.)

For $S \in \mathcal{L}(\mathbb{X})$ and a nonnegative integer n define $S_{[n]}$ to be the restriction of S to $\Re(S^n)$ viewed as a map from $\Re(S^n)$ into $\Re(S^n)$ (in particular, $S_{[0]} = S$). If for some integer n the range space $\Re(S^n)$ is closed and $S_{[n]}$ is an upper (a lower) semi-Fredholm operator, then S is called an upper (a lower) semi-B-Fredholm operator. In this case the index of S is defined as the index of the semi-B-Fredholm operator $S_{[n]}$, see [8]. Moreover, if $S_{[n]}$ is a Fredholm operator, then S is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator S is said to be a B-Weyl operator [9] (Definition 1.1) if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(S)$ of S is defined by $\sigma_{BW}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda I$ is not a B-Weyl operator$\}$.

An operator $S \in \mathcal{L}(\mathbb{X})$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_D(S)$ of an operator S is defined by $\sigma_D(S) = \{\lambda \in \mathbb{C} \colon S - \lambda I$ is not Drazin invertible$\}$. Define also the set $LD(\mathbb{X})$ by $LD(\mathbb{X}) = \big\{S \in \mathcal{L}(\mathbb{X}) \colon a(S) < \infty$ and $\Re(T^{a(S)+1})$ is closed$\big\}$ and $\sigma_{LD}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \notin LD(\mathbb{X})\}$. Following [10], an operator $S \in \mathcal{L}(\mathbb{X})$ is said to be left Drazin invertible if $S \in LD(\mathbb{X})$. We say that $\lambda \in \sigma_a(T)$ is a left pole of S if $S - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(S)$ is a left pole of S of finite rank if λ is a left pole of T and $\alpha(S - \lambda I) < \infty$. Let $\pi_a(S)$ denotes the set of all left poles of S and let $\pi_a^0(S)$ denotes the set of all left poles of S of finite rank. From [10] (Theorem 2.8) it follows that if $S \in \mathcal{L}(\mathbb{X})$ is left Drazin invertible, then S is an upper semi-B-Fredholm operator of index less than or equal to 0. Note that $\pi_a(S) = \sigma_a(S) \setminus \sigma_{LD}(S)$ and hence $\lambda \in \pi_a(S)$ if and only if $\lambda \notin \sigma_{LD}(S)$.

Following [9], we say that generalized Weyl's theorem holds for $S \in \mathcal{L}(\mathbb{X})$ (in symbol $S \in \mathrm{g}\mathcal{W}$) if $\Delta^g(S) = \sigma(S) \setminus \sigma_{BW}(S) = E(S)$, where $E(S) = \big\{\lambda \in \sigma^{\mathrm{iso}}(S) \colon 0 < \alpha(S - \lambda I)\big\}$ is the set of all isolated eigenvalues of S, and that generalized Browder's theorem holds for $S \in \mathcal{L}(\mathbb{X})$ (in symbol $S \in \mathrm{g}\mathcal{B}$) if $\Delta^g(S) = \pi(S)$, where $\pi(T)$ is the set of poles of the resolvent of T. It is proved in [5] (Theorem 2.1) that generalized Browder's theorem is equivalent to Browder's theorem. In [10] (Theorem 3.9), it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(S) = \pi(S)$, it is proved in [11] (Theorem 2.9) that generalized Weyl's theorem is equivalent to Weyl's theorem. Let $\Psi_+(\mathbb{X})$ be the class of all upper semi-B-Fredholm operators, $\Psi_+^-(\mathbb{X}) = \{S \in \Psi_+(\mathbb{X}) \colon \mathrm{ind}(S) \le 0\}$. The upper B-Weyl spectrum of S is defined by $\sigma_{SBF_+^-}(S) = \big\{\lambda \in \mathbb{C} \colon T - \lambda I \notin \Psi_+^-(\mathbb{X})\big\}$. We say that generalized a-Weyl's theorem holds for $S \in \mathcal{L}(\mathbb{X})$ (in symbol $S \in \mathrm{ga}\mathcal{W}$) if $\Delta_a^g(S) = \sigma_a(S) \setminus \sigma_{SBF_+^-}(S) = E_a(S)$, where $E_a(S) = \big\{\lambda \in \sigma_a^{\mathrm{iso}}(S) \colon \alpha(S - \lambda) > 0\big\}$ is the set of all eigenvalues of S which are isolated in $\sigma_a(S)$ and that $S \in \mathcal{L}(\mathbb{X})$ obeys generalized a-Browder's theorem ($S \in \mathrm{ga}\mathcal{B}$) if $\Delta_a^g(S) = \pi_a(S)$. It is proved in [5] (Theorem 2.2) that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [10] (Theorem 3.11) that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(S) = \pi_a(S)$ it is proved in [11] (Theorem 2.10) that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

The operator $T \in \mathcal{L}(\mathbb{X})$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ centred at λ_0, the only analytic function $f \colon \mathbb{D} \to$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in \mathcal{L}(\mathbb{X})$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Obviously, every $T \in \mathcal{L}(\mathbb{X})$ has SVEP at the points of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function, it easily follows that $T \in \mathcal{L}(\mathbb{X})$, as well as its dual

T^*, has SVEP at every point of the boundary $\partial\sigma(T) = \partial\sigma(T^*)$ of the spectrum $\sigma(T)$. In particular, both T and T^* have SVEP at every isolated point of the spectrum, see [1, 4, 2, 3].

Let

$$\Psi_+(S) = \{\lambda \in \mathbb{C}\colon S - \lambda \text{ is upper semi-B-Fredholm}\},$$

$$\Psi(S) = \{\lambda \in \mathbb{C}\colon S - \lambda \text{ is B-Fredholm}\},$$

$$\sigma_{SBF_+}(S) = \{\lambda \in \sigma_a(S)\colon \lambda \notin \Psi_+(S)\},$$

$$\sigma_{SBF_+^-}(S) = \left\{\lambda \in \sigma_a(S)\colon \lambda \in \sigma_{SBF_+}(S) \text{ or } \operatorname{ind}(S-\lambda) > 0\right\},$$

$$H_0(S) = \left\{x \in \mathbb{X}\colon \lim_{n\longrightarrow\infty} \|S^n x\|^{1/n} = 0\right\}.$$

2. Main results. Let $\sigma_s(S) = \{\lambda \in \sigma(S) : S - \lambda \text{ is not surjective}\}$ denote, the surjectivity spectrum. Let $\Psi_-(\mathbb{X})$ be the class of all lower semi-B-Fredholm operators, $\Psi_-^+(\mathbb{X}) = \{S \in \Psi_-(\mathbb{X})\colon \operatorname{ind}(S-\lambda) \geq 0\}$. The lower semi-B-Weyl spectrum of S is defined by $\sigma_{SBF_-^+}(S) = \{\lambda \in \mathbb{C}\colon S-\lambda \notin \Psi_-^+(\mathbb{X})\}$. Define $RD(\mathbb{X}) = \left\{S \in \mathcal{L}(\mathbb{X})\colon dsc(S) = d < \infty \text{ and } \Re(S^{d+1}) \text{ is closed}\right\}$. The right Drazin invertible is defined by $\sigma_{RD}(S) = \{\lambda \in \mathbb{C}\colon S - \lambda \notin RD(\mathbb{X})\}$. It is not difficult to see that $\sigma_D(S) = \sigma_{LD}(S) \cup \sigma_{RD}(S)$. Moreover, $\sigma_{LD}(S) = \sigma_{RD}(S^*)$ [7]. Then S satisfies generalized s-Browder's theorem if $\sigma_{SBF_-^+}(S) = \sigma_{RD}(S)$. Apparently, S satisfies generalized s-Browder's theorem if and only if S^* satisfies generalized a-Browder's theorem. A necessary and sufficient condition for S to satisfy generalized a-Browder's theorem is that S has SVEP at every $\lambda \in \Delta_a^g(S)$ [12] (Theorem 3.1); by duality, S satisfies generalized s-Browder's theorem if and only if S^* has SVEP at every $\lambda \in \sigma_s(S) \setminus \sigma_{SBF_-^+}(S)$. More generally, if either of S and S^* has SVEP, then S and S^* satisfy both generalized a-Browder's theorem and generalized s-Browder's theorem. Either of generalized a-Browder's theorem and generalized s-Browder's theorem implies generalized Browder's theorem, but the converse is false. generalized a-Browder's theorem fails to transfer from A and B to $A \otimes B$ [13] (Example 1).

Lemma 2.1. *Let $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$. Then $0 \notin \sigma_a(A \otimes B) \setminus \sigma_{SBF_+}(A \otimes B)$.*

Proof. Suppose $0 \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+}(A \otimes B)$. Then $0 \in \sigma_a(A \otimes B) \cap \Psi_+(A \otimes B)$. So, there exists an integer n_0 such that for any $n \geq n_0$, $A \otimes B - \frac{1}{n}I$ has closed range and $0 < \alpha\left(A \otimes B - \frac{1}{n}I\right) < \infty$. Since $A \otimes B - \frac{1}{n}I$ is injective if and only if A and B are injective, we have $\alpha(A) > 0$ or $\alpha(B) > 0$. But then $\alpha\left(A \otimes B - \frac{1}{n}I\right) = \infty$, and we have a contradiction.

Lemma 2.2. *Let $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$. Then*

$$\sigma_{SBF_+^-}(A \otimes B) \subseteq \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B) \subseteq$$

$$\subseteq \sigma_a(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_a(B) = \sigma_{LD}(A \otimes B).$$

Proof. Since $\sigma_{SBF_+^-}(S) \subseteq \sigma_{LD}(S)$ for every operator S, it follows that the inclusion $\sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B) \subseteq \sigma_a(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_a(B)$ is evident. To prove the inclusion $\sigma_{SBF_+^-}(A\otimes B)\subseteq\sigma_a(A)\sigma_{SBF_+^-}(B)\cup\sigma_{SBF_+^-}(A)\sigma_a(B)$, take $\lambda\notin\sigma_a(A)\sigma_{SBF_+^-}(B)\cup$ $\cup\,\sigma_{SBF_+^-}(A)\sigma_a(B)$. Since

$$\sigma_{SBF_+}(A\otimes B)\subseteq\sigma_a(A)\sigma_{SBF_+^-}(B)\cup\sigma_{SBF_+^-}(A)\sigma_a(B),$$

Lemma 2.1 implies that $\lambda\neq 0$. For every factorization $\lambda=\mu\nu$ such that $\mu\in\sigma_a(A)$ and $\nu\in\sigma_a(B)$ we have that $\mu\in\sigma_a\backslash\sigma_{SBF_+^-}(A)$ and $\nu\in\sigma_a(B)\backslash\sigma_{SBF_+^-}(B)$, i.e., $\mu\in\Psi_+(A),\nu\in\Psi_+(B),\operatorname{ind}(A-$ $-\,\mu)\leq 0$ and $\operatorname{ind}(B-\nu)\leq 0$. In particular, $\lambda\notin\sigma_{SBF_+}(A\otimes B)$.

We prove next that $\operatorname{ind}(A\otimes B-\lambda)\leq 0$. Suppose $\operatorname{ind}(A\otimes B-\lambda)>0$. Then there exists an integer n_0 such that for any $n\geq n_0$ we have $\alpha\left(A\otimes B-\lambda I-\dfrac{1}{n}I\right)<\infty$. But this implies that $\beta\left(A\otimes B-\lambda I-\dfrac{1}{n}I\right)<\infty$, so that $A\otimes B-\lambda$ is B-Weyl. Let

$$F=\left\{(\mu_i,\nu_i)_{i=1}^k\in\sigma(A)\sigma(B)\colon\,\mu_i\nu_i=\lambda\right\}.$$

Then F is a finite set. Furthermore

(i) if $m>1$, then $\mu_i\in\sigma^{\mathrm{iso}}(A)$ for $1\leq i\leq m$;

(ii) if $k>m$, then $\nu_i\in\sigma^{\mathrm{iso}}(B)$ for $m+1\leq i\leq k$;

(iii) $\operatorname{ind}(A\otimes B-\lambda)=\sum_{j=m+1}^k\operatorname{ind}(A-\mu_i)\dim H_0(B-\nu_i)+\sum_{j=1}^m\operatorname{ind}(B-\nu_i)\dim H_0(A-$ $-\,\mu_i)$.

Since $\operatorname{ind}(A-\mu_i)$ and $\operatorname{ind}(B-\nu_i)$ are non-positive, we have a contradiction. Hence, $\operatorname{ind}(A\otimes B-\lambda)\leq$ ≤ 0, and consequently, $\lambda\notin\sigma_{SBF_+^-}(A\otimes B)$. This leaves us to prove the equality $\sigma_{LD}(A\otimes B)=$ $=\sigma_a(A)\sigma_{LD}(B)\cup\sigma_{LD}(A)\sigma_a(B)$.

Suppose that $\lambda\notin\sigma_{LD}(A\otimes B)$. Then $\lambda\neq 0$, $\lambda\in LD(A\otimes B)$, $a=\operatorname{asc}(A\otimes B-\lambda)<\infty$ and $\Re(A\otimes B-\lambda)^{a+1}$ is closed and hence $\lambda\in\pi_a(A\otimes B)$. Observe that $\lambda\in\sigma_a^{\mathrm{iso}}(A\otimes B)$. Let $\lambda=\mu\nu$ be any factorization of λ such that $\mu\in\sigma_a(A)$ and $\nu\in\sigma_a(B)$; then $\mu\in LD(A)$ and $\nu\in LD(B)$. Furthermore, since $\sigma_a^{\mathrm{iso}}(A\otimes B)\subseteq\sigma_a^{\mathrm{iso}}(A)\cup\sigma_a^{\mathrm{iso}}(B)\cup\{0\}$, A has SVEP at μ and B has SVEP at ν. Consequently, $\mu\in\pi_a(A)$, $\nu\in\pi_a(B)$, that is, $\mu\notin\sigma_{LD}(A)$ and $\nu\notin\sigma_{LD}(B)$. But then $\lambda\notin\sigma_a(A)\sigma_{LD}(B)\cup\sigma_{LD}(A)\sigma_a(B)$. Hence $\sigma_a(A)\sigma_{LD}(B)\cup\sigma_{LD}(A)\sigma_a(B)\subseteq\sigma_{LD}(A\otimes B)$.

To prove the reverse inclusion we start by recalling the fact that if $\mu\in\sigma_a^{\mathrm{iso}}(A)$ and $\mu\in\sigma_a^{\mathrm{iso}}(B)$ for every factorization $\lambda=\mu\nu$ of $\lambda\neq 0$, then $\lambda=\mu\nu\in\sigma_a^{\mathrm{iso}}(A\otimes B)$. Let $\lambda\in\sigma_a(A)\sigma_{LD}(B)\cup$ $\cup\,\sigma_{LD}(A)\sigma_a(B)$. Then $\lambda\neq 0$. Furthermore, if $\lambda=\mu\nu$ is any factorization of λ such that $\mu\in\sigma_a(A)$ and $\nu\in\sigma_a(B)$, then the following implications hold:

$$\mu\notin\sigma_{LD}(A)\quad\text{and}\quad\nu\notin\sigma_{LD}(B)\Rightarrow\mu\in\pi_a(A)\quad\text{and}\quad\nu\in\pi_a(B)\Rightarrow$$

$$\Rightarrow\lambda\in\pi_a(A\otimes B),\mu\in\sigma_a^{\mathrm{iso}}(A)\quad\text{and}\quad\nu\in\sigma_a^{\mathrm{iso}}(B)\Rightarrow$$

$$\Rightarrow\lambda\in\pi_a(A\otimes B)\quad\text{and}\quad\lambda\in\sigma_a^{\mathrm{iso}}(A\otimes B)\Rightarrow$$

$$\Rightarrow\lambda\notin\sigma_{LD}(A\otimes B).$$

Hence $\sigma_{LD}(A\otimes B)\subseteq\sigma_a(A)\sigma_{LD}(B)\cup\sigma_{LD}(A)\sigma_a(B)$.

Lemma 2.2 is proved.

Lemma 2.3. *Let $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$. If $A\otimes B$ satisfies generalized a-Browder's theorem, then*

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B).$$

Proof. $A\otimes B$ satisfies generalized a-Browder's theorem if and only if $\sigma_{SBF_+^-}(A\otimes B) = \sigma_{LD}(A\otimes \otimes B)$. Thus the stated result is an immediate consequence of Lemma 2.2.

The next theorem, our main result, proves that A and B satisfy generalized a-Browder's theorem implies $A \otimes B$ satisfies generalized a-Browder's theorem if and only if $\sigma_{SBF_+^-}(A \otimes B) = = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$.

Theorem 2.1. *Let $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$. If A and B satisfy generalized a-Browder's theorem, then the following are equivalent*:

(i) *$A \otimes B$ satisfies generalized a-Browder's theorem*;

(ii) $\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$;

(iii) *A has SVEP at every $\mu \in \Psi_+(A)$ and B has SVEP at every $\nu \in \Psi_+(B)$ such that* $(0 \neq \neq \lambda) = \mu\nu \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$.

Proof. If A and B satisfy generalized a-Browder's theorem, then $\sigma_{LD}(A) = \sigma_{SBF_+^-}(A)$ and $\sigma_{LD}(B) = \sigma_{SBF_+^-}(B)$.

(i) $\Rightarrow$ (ii). By Lemma 2.3 we have, without any extra conditions.

(ii) $\Rightarrow$ (i). If (ii) is satisfied, then

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B) =$$

$$= \sigma_a(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_a(B) =$$

$$= \sigma_{LD}(A \otimes B) \quad \text{(by Lemma 2.2)}.$$

Hence $A \otimes B$ satisfies generalized a-Browder's theorem.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$. Then $\lambda \neq 0$ and for every factorization $\lambda = \mu\nu$ such that $\mu \in \sigma_a(A) \cap \Psi_+(A)$ and $\nu \in \sigma_a(B) \cap \Psi_+(B)$. Hence $\mu \in \pi_a(A)$ and $\nu \in \pi_a(B)$.So it follows from [10] (Remark 2.7) that $\mu \in \sigma_a^{\text{iso}}(A)$ and $\nu \in \sigma_a^{\text{iso}}(B)$. Therefore, A and B have SVEP at (all such) μ and ν, respectively.

(iii) $\Rightarrow$ (ii). In view of Lemma 2.2, we have to prove that $\sigma_{LD}(A \otimes B) \subseteq \sigma_{SBF_+^-}(A \otimes B)$. Suppose that (ii) is satisfied. Take a $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$. Then $(0 \neq)\lambda \in \Psi_+(A \otimes B)$ and $\operatorname{ind}(A\otimes B-\lambda) \leq 0$. The equality $\sigma_{SBF_+}(A\otimes B) = \sigma_a(A)\sigma_{SBF_+}(B)\cup\sigma_{SBF_+}(A)\sigma_a(B)$ implies that for any factorization $\lambda = \mu\nu$ (such that $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B)$) we have that $\mu \in \Psi_+(A)$ and $\nu \in \Psi_+(B)$. The SVEP hypotheses on A and B implies that $\operatorname{asc}(A - \mu I)$ and $\operatorname{asc}(B - \lambda)$ are finite. Hence, $\mu \in \sigma_a^{\text{iso}}(A)$ and $\mu \in \sigma_a^{\text{iso}}(B)$. So, it follows from Theorem 2.8 of [10] that $\mu \in \pi_a(A)$ and $\nu \in \pi_a(B)$. Therefore, $\mu \notin \sigma_{LD}(A)$ and $\nu \notin \sigma_{LD}(B)$. But then $\lambda \notin \sigma_{LD}(A \otimes B)$. Hence $\sigma_{LD}(A \otimes B) \subseteq \sigma_{SBF_+^-}(A \otimes B)$.

Theorem 2.1 is proved.

The next theorem gives a sufficient condition for $A \otimes B$ to satisfy generalized a-Weyl theorem, given that A and B satisfy generalized a-Weyl theorem. But before that a couple of technical lemmas. Recall that an operator S is said to be a-isoloid if $\lambda \in \sigma_a^{\text{iso}}(S)$ implies $\lambda \in \sigma_p(S)$.

Lemma 2.4. *Suppose that A, B and $A \otimes B$ satisfy generalized a-Browder's theorem. If $\mu \in \in \pi_a(A)$ and $\nu \in \pi_a(B)$, then $\lambda = \mu\nu \in \pi_a(A \otimes B)$.*

Proof. Since $\mu \in \sigma_a(A) \setminus \sigma_{SBF_+^-}(A)$, $\nu \in \sigma_a(B) \setminus \sigma_{SBF_+^-}(B)$ and $\sigma_{SBF_+^-}(A \otimes B) = = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$. Hence, $\lambda = \mu\nu \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B) = \pi_a(A \otimes B)$.

Theorem 2.2. *Suppose that $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$ are a-isoloid which satisfy generalized a-Weyl theorem. If $\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$, then $A \otimes B$ satisfies generalized a-Weyl theorem.*

Proof. The hypotheses imply that $A \otimes B$ satisfies generalized a-Browder's theorem, that is, $\sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B) = \pi_a(A \otimes B)$. Since $\pi_a(A \otimes B) \subseteq E_a(A \otimes B)$, we have to prove that $E_a(A \otimes B) \subseteq \pi_a(A \otimes B)$. Let $\lambda \in E_a(A \otimes B)$. Then $0 \neq \lambda = \mu\nu$ for some $\mu \in \sigma_a^{\text{iso}}(A)$ and $\nu \in \sigma_a^{\text{iso}}(B)$. The operators A and B being a-isoloid, it follows from $\lambda = \mu\nu \in E_a(A \otimes B)$ that $\mu \in E_a(A) = \pi_a(A)$ and $\nu \in E_a(B) = \pi_a(B)$. By Lemma 2.4, $\lambda \in \pi_a(A \otimes B)$.

Theorem 2.2 is proved.

Following [16], we say that $S \in \mathcal{L}(\mathbb{X})$ satisfies property (w) if $\sigma_a(S) \setminus \sigma_{aw}(S) = E^0(S)$. The property (w) has been studied in [2, 3, 4, 16]. In [3] (Theorem 2.8), it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. An operator $S \in \mathcal{L}(\mathbb{X})$ is said to be satisfies property (gw) if $\sigma_a(S) \setminus \sigma_{SBF_+^-}(S) = E(S)$. Property (gw) has been introduced and studied in [6]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [6] that an operator satisfying property (gw) satisfies property (w) and generalized Weyl's theorem but the converse is not true in general.

The following theorem gives a necessary and sufficient condition for the transference of property (gw) from isoloid A and B to $A \otimes B$ But before that a lemma and some observations, which will often be used in the sequel. Let $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$. Then $\sigma^{\text{iso}}(A \otimes B) \subseteq \sigma^{\text{iso}}(A).\sigma^{\text{iso}} \cup \{0\}$. If 0 is in the point spectrum of either of A and B, then $\alpha(A \otimes B) = 0$; in particular, $0 \notin E(A \otimes B)$). It is easily seen, see the argument of the proof of [15] (Proposition 2), that $E(A \otimes B) \subseteq E(A)E(B)$.

Theorem 2.3. *If $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$ are isoloid operators which satisfy property (gw), then the following conditions are equivalent*:

(i) *$A \otimes B$ satisfies property (gw)*;

(ii) *the generalized a-Weyl spectrum equality $\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$ is satisfied*;

(iii) *$A \otimes B$ satisfies generalized a-Browder's theorem.*

Proof. Since property (gw) implies generalized a-Browder's theorem, the equivalence (ii) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (iii) follows from Theorem 2.2. We prove (iii) $\Rightarrow$ (i). The hypothesis A and B satisfy property (gw) implies

$$\sigma_a(A) \setminus \sigma_{SBF_+^-}(A) = E(A), \qquad \sigma_a(B) \setminus \sigma_{SBF_+^-}(B) = E(B).$$

Observe that (iii) implies generalized a-Browder's theorem transfers from A and B to $A \otimes B$: hence $\sigma_{SBF_+^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B)$. Let $\lambda \in E(A \otimes B)$; then $\lambda \neq 0$ and hence there exist $\mu \in \sigma^{\text{iso}}(A)$ and $\nu \in \sigma^{\text{iso}}(B)$ such that $\lambda = \mu\nu$. By hypotheses A and B are isoloid; hence μ is an eigenvalue of A and ν is an eigenvalue of B. Hence $\mu \in E(A) = \sigma_a(A) \setminus \sigma_{SBF_+^-}(A)$ and $\nu \in E(B) = \sigma_a(B) \setminus \sigma_{SBF_+^-}(B)$. Consequently, $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$; hence

$E(A \otimes B) \subseteq \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$. Conversely, if $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$, then $\lambda \neq 0$. So, there exist $\mu \in \sigma_a(A) \setminus \sigma_{SBF_+^-}(A) = E(A)$ and $\nu \in \sigma_a(B) \setminus \sigma_{SBF_+^-}(B)$ such that $\lambda = \mu\nu$. But then $\lambda \in E(A \otimes B)$. Hence $\sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B) \subseteq E(A \otimes B)$. Therefore, the proof is achieved.

An operator $S \in \mathcal{L}(\mathbb{X})$ is said to be polaroid (respectively, a-polaroid) if $\sigma^{\text{iso}}(S)$ (respectively, $\sigma_a^{\text{iso}}(S)$) is empty or every isolated point of $\sigma(S)$ (respectively, $\sigma_a(S)$) is a pole of the resolvent. $S \in \mathcal{L}(\mathbb{X})$ is polaroid implies S^* polaroid. It is well known that if S orS^* has SVEP and S is polaroid, then S and S^* satisfy generalized Weyl's theorem. Not as well known is the fact [6] (Theorem 2.10), that if S is polaroid and S^* (respectively, S) has SVEP, then S (respectively, S^*) satisfies property (gw). Here the SVEP hypotheses on S and S^* can not be exchanged. The following theorem is the tensor product analogue of this result.

Theorem 2.4. *Suppose that the operators $A \in \mathcal{L}(\mathbb{X})$ and $B \in \mathcal{L}(\mathbb{Y})$ are polaroid.*

(i) *If A^* and B^* have SVEP, then $A \otimes B$ satisfies property (gw).*

(ii) *If A and B have SVEP, then $A^* \otimes B^*$ satisfies property (gw).*

Proof. (i) The hypothesis A^* and B^* have SVEP implies

$$\sigma(A) = \sigma_a(A), \qquad \sigma(B) = \sigma_a(B), \qquad \sigma_{SBF_+^-}(A) = \sigma_{BW}(A), \qquad \sigma_{SBF_+^-}(B) = \sigma_{BW}(B)$$

and

$$A^*,\ B^* \quad \text{and} \quad A^* \otimes B^* \quad \text{satisfy generalized s-Browder's theorem.}$$

Thus generalized s-Browder's theorem and generalized Browder's theorem transform from A^* and B^* to $A^* \otimes B^*$. Hence

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_-^+}(A^* \otimes B^*) = \sigma_s(A^*)\sigma_{SBF_-^+}(B^*) \cup \sigma_{SBF_-^+}(A^*)\sigma_s(B^*) =$$

$$= \sigma_a(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_a(B) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B),$$

and

$$\sigma_{BW}(A \otimes B) = \sigma_{BW}(A^* \otimes B^*) = \sigma(A^*)\sigma_{BW}(B^*) \cup \sigma_{BW}(A^*)\sigma(B^*) =$$

$$= \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B).$$

Consequently,

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{BW}(A \otimes B).$$

Already,

$$\sigma_a(A \otimes B) = \sigma_a(A)\sigma_a(B) = \sigma(A)\sigma(B) = \sigma(A \otimes B).$$

Evidently, $A \otimes B$ is polaroid by Lemma 2 of [14]; combining this with $A \otimes B$ satisfies generalized Browder's theorem, it follows that $A \otimes B$ satisfies generalized Weyl's theorem, i.e., $\sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E(A \otimes B)$. It follows then

$$\sigma_a(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E(A \otimes B),$$

that is, $A \otimes B$ satisfies property (gw).

(ii) In this case $\sigma(A) = \sigma_a(A^*)$, $\sigma(B) = \sigma_a(B^*)$, $\sigma_{BW}(A^*) = \sigma_{SBF_+^-}(A^*)$, $\sigma_{BW}(B^*) = \sigma_{SBF_+^-}(B^*$, $\sigma(A^* \otimes B^*) = \sigma_a(A^* \otimes B^*)$, both generalized Browder's theorem and generalized s-Browder's theorem transfer from A and B to $A \otimes B$. Hence

$$\sigma_{SBF_+^-}(A^* \otimes B^*) = \sigma_{SBF_-^+}(A \otimes B) = \sigma_s(A)\sigma_{SBF_-^+}(B) \cup \sigma_{SBF_-^+}(A)\sigma_s(B) =$$

$$= \sigma_a(A^*)\sigma_{SBF_+^-}(B^*) \cup \sigma_{SBF_+^-}(A^*)\sigma_a(B^*) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) =$$

$$= \sigma_{BW}(A \otimes B) = \sigma_{BW}(A^* \otimes B^*).$$

Thus, since $A^* \otimes B^*$ polaroid and $A \otimes B)$ satisfies generalized Browder's theorem imply $A^* \otimes B^*$ satisfy generalized Weyl's theorem,

$$\sigma_a(A^* \otimes B^*) \setminus \sigma_{SBF_+^-}(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_{BW}(A^* \otimes B^*) = E(A^* \otimes B^*),$$

that is, $A^* \otimes B^*$ satisfies property (gw).

Theorem 2.4 is proved.

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