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INTEGRATION OF THE MODIFIED DOUBLE-INFINITE TODA LATTICE WITH THE HELP OF INVERSE SPECTRAL PROBLEM*

ІНТЕГРУВАННЯ ЗМІНЕНОГО ДВОСТОРОННЬО НЕСКІНЧЕННОГО ЛАНЦЮЖКА ТОДИ ЗА ДОПОМОГОЮ ОБЕРНЕНОЇ СПЕКТРАЛЬНОЇ ЗАДАЧІ

The approach to finding the solution of the Cauchy problem for the indicated Toda lattice by means of inverse spectral problem is given.

Викладено підхід до знаходження розв'язку задачі Коші для вказаного ланцюжка Тоди за допомогою оберненої спектральної задачі.

1. Introduction. The classical method of investigation of the Cauchy problem for the KdV equation via an application of inverse spectral problem for Sturm – Liouville equation (I. Gelfand, B. Levitan, V. Marchenko, M. Krein; account see in the book [1]) can be adjusted for the Toda semi-infinite lattice

$$\dot{\alpha}_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)),$$
$$\dot{\beta}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n = 0, 1, \ldots, \quad t \in [0, T]; \quad \alpha_{-1} = 0.$$

Using some results for a finite Toda lattice [2, 3], this approach was proposed in the articles [4, 5] of author. There the role of the Sturm – Liouville equation was played by the simpler spectral theory of Jacobi matrices.

But finding solutions of the double-infinite Toda lattice (when $n = \ldots, -1, 0, 1, \ldots$) is a more difficult problem. In the periodic case, this equation was integrated in terms of theta-functions in [6] (see also [7]); the inverse scattering problem method (for difference equations) was applied in [8 – 10]; the method similar to that of [4, 5] was used in [11 – 13] in the case when initial data tend to zero when $|n| \to \infty$; new classes of solutions were found in [14]; see also [15 – 17].

In [18], Ch. 7, the author proposed to investigate spectral problems for double-infinite Jacobi matrices by doubling such a matrix and reducing the problem to the case of block one-sided Jacobi matrices with (2×2)-matrix blocks (the spectral theory of such block Jacobi matrices was proposed in [19] and developed in [18], Ch. 7; in this theory, instead of ordinary scalar-valued spectral measure the matrix-valued spectral measure appeared). In [20] author and M. Gekhtman tried to apply the inverse spectral theory of (2×2)-block one-sided Jacobi matrices for integration of double-infinite Toda lattice, but the corresponding differential equation for (2×2)-matrix-spectral measure was impossible to solve and therefore this approach to solving the Cauchy problem was ineffective.

In this paper, the author applies an analogue of the approach in [20], but instead of (2×2)-block Jacobi matrices here the dimensions of blocks are changed: the first

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diagonal block is 1×1, i.e., scalar, and therefore, instead of (2×2)-matrix-spectral measure we have an ordinary scalar spectral measure (such idea has appeared in the works [21 – 23]). As a result, we integrate the double-infinite Toda chain as in [4, 5], but with an additional condition:

$$\alpha_0(t)\alpha_{-1}(t) = 0, \quad t \in [0, T].$$

Thus, we can integrate an appropriately modified double-infinite Toda chain when functions $\alpha_0(t), \alpha_{-1}(t), t \in [0, T]$, are given.

2. The spaces and the corresponding block Jacobi matrix. We will investigate an operator on the complex Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots, \qquad \mathcal{H}_0 = \mathbb{C}^1 := \mathbb{C}, \qquad \mathcal{H}_1 = \mathcal{H}_2 = \ldots = \mathbb{C}^2. \tag{1}$$

Vectors f from $\mathbf{l}_2$ have a form $f = (f_n)_{n=0}^{\infty}$ where $f_n \in \mathcal{H}_n$; so $f_0 = f_0 e_0$, $f_n = f_{n;0}e_{n;0} + f_{n;1}e_{n;1} =: (f_{n;0}, f_{n;1})$ where $e_0 = 1$ and $e_n = (e_{n;0}, e_{n;1})$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, form the standard basis in $\mathbb{C}^1$ and $\mathcal{H}_n = \mathbb{C}^2$ respectively.

By $\mathbf{l}_{fin}$ we denote the linear space of finite vectors from $\mathbf{l}_2$ and by $\mathbf{l}_2(p)$ we denote the corresponding weighted space of vectors for which

$$\|f\|^2_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} p_n < \infty, \qquad (f, g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n} p_n. \tag{2}$$

Here $p = (p_n)_{n=0}^{\infty}$, $p_n > 0$, is a given sequence of weights. In what follows, $p_n \geq 1$ and $\sum_{n=0}^{\infty} p_n^{-1} < \infty$, therefore the imbedding of the positive space $\mathbf{l}_2(p) \subset \mathbf{l}(p)$ is quasinuclear. The corresponding negative space is $\mathbf{l}_2(p^{-1})$, $p^{-1} := (p_n^{-1})_{n=0}^{\infty}$. As a result, we construct the quasinuclear rigging (see, e.g., [24], Ch. 15)

$$\mathbf{l} = (\mathbf{l}_{\text{fin}})' \supset (\mathbf{l}_2(p^{-1})) \supset \mathbf{l}_2 \supset (\mathbf{l}_2(p)) \supset \mathbf{l}_{\text{fin}} \tag{3}$$

($\mathbf{l}$ denotes the space of all sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$ are arbitrary).

In the space (1), consider a Hermitian matrix $J = (J_{j,k})_{j,k=0}^{\infty}$ with operator (real matrix) — valued elements $J_{j,k} \colon \mathcal{H}_k \to \mathcal{H}_j$, $J_{j,k} = (J_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{1}$, of the following block Jacobi structure:

$$b_0 \colon \mathbb{R}^1 \to \mathbb{R}^1,$$

$$J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \ldots \\ a_0^* & b_1 & a_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & a_2 & 0 & \ldots \\ 0 & 0 & a_2 & b_3 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$a_0 = [a_{0;0,0} a_{0;0,1}] \colon \mathbb{R}^2 \to \mathbb{R}^1, \qquad a_0^* = \begin{bmatrix} a_{0;0,0} \\ a_{0;0,1} \end{bmatrix} \colon \mathbb{R}^1 \to \mathbb{R}^2, \tag{4}$$

$$a_n = a_n^* = \begin{bmatrix} a_{n;0,0} & a_{n;0,1} \\ a_{n;1,0} & a_{n;1,1} \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$b_n = b_n^* = \begin{bmatrix} b_{n;0,0} & b_{n;0,1} \\ b_{n;1,0} & b_{n;1,1} \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2, \quad n \in \mathbb{N}.$$

By assumption, all elements of matrix J (4) are real and uniformly bounded. Therefore the operator $\boldsymbol{J}$ constructed in a usual way from the matrix J is a bounded self-adjoint operator acting in the space $\mathbf{l}_2$. It is connected with the chain (3) in a standard way.

3. The spectral theory of the operator *J*. Basic constructions. Now we will use the result from [18], Ch. 5, and [24], Ch. 15, on the generalized eigenvector expansion for a bounded self-adjoint operator standardly connected with the chain (3). For our operator $\boldsymbol{J}$ we have the representation

$$\boldsymbol{J}f = \int\limits_{\mathbb{R}} \lambda\Phi(\lambda)d\sigma(\lambda)f, \quad f \in \mathbf{l}_2, \tag{5}$$

where $\Phi(\lambda)\colon \mathbf{l}_2(p) \to \mathbf{l}_2(p^{-1})$ is a generalized projection operator and $d\sigma(\lambda)$ is a spectral measure (with a bounded support). For all $f,\ g \in \mathbf{l}_{\mathrm{fin}}$ we have the Parseval equality

$$(f,g)_{\mathbf{l}_2} = \int\limits_{\mathbb{R}} (\Phi(\lambda)f, g)_{\mathbf{l}_2} d\sigma(\lambda); \tag{6}$$

and, after extending by continuity, the equality (6) takes place for all $f,\ g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection in $\mathbf{l}_2$ on $\mathcal{H}_n$, $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. Hence for all $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously in the space $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$ but possibly with norm which is not equal to one.

Let us consider the operator matrix $(\Phi_{j,k}(\lambda))_{j,k=0}^{\infty}$ where

$$\Phi_{j,k}(\lambda) = \pi_j \Phi(\lambda)\pi_k \colon \mathbf{l}_2 \to \mathcal{H}_j \quad (\text{or} \quad \mathcal{H}_k \to \mathcal{H}_j). \tag{7}$$

The Parseval equality (6) can be rewritten as follows: $\forall f,\ g \in \mathbf{l}_2$

$$(f,g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} \int\limits_{\mathbb{R}} (\Phi(\lambda)\pi_k f, \pi_j g)_{\mathbf{l}_2} d\sigma(\lambda) =$$

$$= \sum_{j,k=0}^{\infty} \int\limits_{\mathbb{R}} (\pi_j\Phi(\lambda)\pi_k f, g)_{\mathbf{l}_2} d\sigma(\lambda) = \sum_{j,k=0}^{\infty} \int\limits_{\mathbb{R}} (\Phi_{j,k}(\lambda) f_k, g_j)_{\mathbf{l}_2} d\sigma(\lambda). \tag{8}$$

In what follows we will assume that *all matrices* a_n, $n \in \mathbb{N}$, *are invertible and* $a_0 \neq 0$. The difference equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$, $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty} \in (\mathbf{l}_{\mathrm{fin}})' = \mathbf{l}$ has the following form: let $\varphi_0(\lambda) = \varphi_0$ be independent of λ:

$$b_0\varphi_0 + a_0\varphi_1(\lambda) = \lambda\varphi_0,$$

i.e.,

$$
\begin{aligned}
&b_0\varphi_0 + (a_{0;0,0}\varphi_{1;0}(\lambda) + a_{0;0,1}\varphi_{1;1}(\lambda)) = \lambda\varphi_0, \quad \varphi_{-1} = 0,\\
&a_0\varphi_0 + b_1\varphi_1(\lambda) + a_1\varphi_2(\lambda) = \lambda\varphi_1(\lambda),\\
&a_1\varphi_1 + b_2\varphi_2(\lambda) + a_2\varphi_3(\lambda) = \lambda\varphi_2(\lambda),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{n-1}\varphi_{n-1} + b_n\varphi_n(\lambda) + a_n\varphi_{n+1}(\lambda) = \lambda\varphi_n(\lambda),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\qquad n \in \mathbb{N}.
\end{aligned} \tag{9}
$$

Let us explain that above we have assumed that, as in the classical theory of Jacobi matrices, the initial value $\varphi_0(\lambda)$ of an eigenvector $\varphi(\lambda) = \big(\varphi_0(\lambda), \varphi_1(\lambda), \varphi_2(\lambda), \ldots\big) \in \in (\mathbf{l}_{\text{fin}})' = \mathbf{l}$ is $\varphi_0 \in \mathbb{R}$ and does not depend on λ (we will often assume $\varphi_0 = 1$).

The expression $P_m(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, $\lambda \in \mathbb{R}$, where A_j are linear operators acting: $\mathbb{R}^2 \to \mathbb{R}^2$ and A_m is invertible, we will call an operator polynomial of degree $m \in \mathbb{N}_0$ w.r.t. λ. For fixed λ it is an operator, acting: $\mathbb{R}^2 \to \mathbb{R}^2$. If the coefficients $A_0, \ldots, A_m \colon \mathbb{C}^2 \to \mathbb{C}^2$, such operator polynomial will be called a complex operator polynomial.

The system (9) is a system of recurrence relations: starting from some $\varphi_1(\lambda) \in \mathbb{R}^2$ we can find, step by step, $\varphi_2(\lambda)$, $\varphi_3(\lambda), \ldots$; since the matrices a_1, $a_2, \ldots$ are invertible we have (below $\mathbf{1}$ is the identity operator in $\mathbb{R}^2$)

$$
\begin{aligned}
&\varphi_2(\lambda) = a_1^{-1}\big((\lambda\mathbf{1} - b_1)\varphi_1(\lambda) - a_0\varphi_0\big) =: Q_2(\lambda)(\varphi_0, 0) =: Q_2(\lambda)\varphi_0,\\
&\varphi_3(\lambda) = a_2^{-1}\big((\lambda\mathbf{1} - b_2)\varphi_2(\lambda) - a_1\varphi_1(\lambda)\big) =: Q_3(\lambda)(\varphi_0, 0) =: Q_3(\lambda)\varphi_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\varphi_{n+1}(\lambda) = a_n^{-1}\big((\lambda\mathbf{1} - b_n)\varphi_n(\lambda) - a_{n-1}\varphi_{n-1}(\lambda)\big) =\\
&\qquad = a_n^{-1}\big((\lambda\mathbf{1} - b_n)Q_n(\lambda) - a_{n-1}Q_{n-1}(\lambda)\big) =:\\
&\qquad =: Q_{n+1}(\lambda)(\varphi_0, 0) =: Q_{n+1}(\lambda)\varphi_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\qquad n \in \mathbb{N}.
\end{aligned} \tag{10}
$$

The first vector $\varphi_1(\lambda) \in \mathbb{R}^2$ cannot be found uniquely from the first equation in (9). Our way around this is as follows: we fix some θ_1, $\omega_1 \neq 0$, then after easy calculations we get

$$
\varphi_1(\lambda) = (\lambda - b_0)\begin{bmatrix} \theta_1^{-1} & 0 \\ 0 & \omega_1^{-1} \end{bmatrix}(\varphi_0, 0) =: Q_1(\lambda)(\varphi_0, 0) =: Q_1(\lambda)\varphi_0, \tag{11}
$$

where θ_1, ω_1 are some solutions of one equation with two unknowns

$$
a_{0;0,0}\theta_1^{-1} + a_{0;0,1}\omega_1^{-1} = 1 \tag{12}
$$

(these solutions exist because the matrix $a_0 \colon \mathbb{R}^2 \to \mathbb{R}$ is nonzero). *These* θ_1, ω_1 *will be fixed in Sections* 3 – 5 *of the paper.*

Thus, we start from $Q_1(\lambda)\varphi_0$ of the form (11) with condition (12) and then find $Q_2(\lambda)\varphi_0, Q_3(\lambda)\varphi_0, \ldots$. It is easy to understand that every $Q_n(\lambda)$ is an operator polynomial of degree $n \in \mathbb{N}$. The solution of the difference equation (9) has a form

$$\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}, \qquad \varphi_n(\lambda) = Q_n(\lambda)\varphi_0, \quad n \in \mathbb{N}_0;$$
$$Q_0(\lambda) := \mathbf{1}, \qquad Q_0(\lambda)\varphi_0 := \varphi_0. \tag{13}$$

It is possible to find the representation of elements (7) of matrix $(\Phi_{j,k}(\lambda))_{j,k=0}^{\infty}$ by operator polynomials $Q_0(\lambda)$, $Q_1(\lambda), \ldots$ (compare with [18], Ch. 7, and [21, 22]).

Lemma 1. *For every fixed* j, $k \in \mathbb{N}_0$ *the operator* (7) $\Phi_{j,k}(\lambda)$ *has the following representation*:

$$\Phi_{j,k}(\lambda) = Q_j(\lambda)\Phi_{0,0}(\lambda)(Q_k(\lambda))^* \tag{14}$$

where $\Phi_{0,0}(\lambda) \geq 0$ *is understood as an operator of multiplication by the scalar* $\Phi_{0,0}(\lambda)$.

Proof. For a fixed $k \in \mathbb{N}_0$, the vector (with a fixed $\lambda \in \mathbb{R}$ and $f \in \mathbf{l}_{\text{fin}}$) $\varphi(\lambda) = (\varphi_j(\lambda)_{j=0}^{\infty})$, where

$$\varphi_j(\lambda) = \Phi_{j,k}(\lambda)f = \pi_j\Phi(\lambda)\pi_k f \in \mathcal{H}_j, \quad \lambda \in \mathbb{R}, \tag{15}$$

is a generalized solution, in $\mathbf{l} = (\mathbf{l}_{\text{fin}})'$, of the equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$, since $\Phi(\lambda)$ is a projector onto a generalized eigenvector of the self-adjoint operator $\boldsymbol{J}$ with the corresponding generalized eigenvalue λ. Therefore for all $g \in \mathbf{l}_{\text{fin}}$ we have $(\varphi(\lambda), Jg)_{\mathbf{l}_2} = \lambda(\varphi(\lambda), g)_{\mathbf{l}_2}$. Transferring the finite difference Hermitian expression J to $\varphi(\lambda)$ we get $(J\varphi(\lambda), g)_{\mathbf{l}_2} = \lambda(\varphi(\lambda), g)_{\mathbf{l}_2}$. Hence, it follows that $\varphi(\lambda) \in \mathbf{l}_2(p^{-1})$ exists as a usual solution of the difference equation $J\varphi = \lambda\varphi$ with the initial condition $\varphi_0(\lambda) = \pi_0\Phi(\lambda)\pi_k f \in \mathcal{H}_0$.

The uniqueness of the solution of the Cauchy problem for the difference equation (9) (with a condition $\varphi_{-1}(\lambda) = 0$ and finding θ_1, ω_1, according to (12)), i.e., for $J\varphi = \lambda\varphi$, ensures that the solution (15) $\big(\varphi_0(\lambda), \varphi_1(\lambda), \ldots\big)$ and the solution (13) with the initial condition $\varphi_0(\lambda) = \pi_0\Phi(\lambda)\pi_k f$ are the same. Vector $f \in \mathbf{l}_{\text{fin}}$ is arbitrary, therefore we obtain

$$\Phi_{j,k}(\lambda) = Q_j(\lambda)(\pi_0\Phi(\lambda)\pi_k) = Q_j(\lambda)\Phi_{0,k}(\lambda), \quad j \in \mathbb{N}_0. \tag{16}$$

The operator $\Phi(\lambda)\colon \mathbf{l}_2(p) \to \mathbf{l}_2(p^{-1})$ is formally self-adjoint on $\mathbf{l}_2$ (since it is equal to the derivative of the resolution of identity of the operator $\boldsymbol{J}$ in $\mathbf{l}_2$ with respect to the spectral measure). Hence, according to (7), we get

$$\big(\Phi_{j,k}(\lambda)\big)^* = (\pi_j\Phi(\lambda)\pi_k)^* = \pi_k\Phi(\lambda)\pi_j = \Phi_{k,j}(\lambda), \quad j,\ k \in \mathbb{N}_0. \tag{17}$$

For a fixed $j \in \mathbb{N}_0$, it follows from (16) and the previous discussion that the vector

$$\psi(\lambda) = (\psi_k(\lambda))_{k=0}^{\infty}, \psi_k(\lambda) = \Phi_{k,j}(\lambda)f, \quad k \in \mathbb{N}_0, \quad f \in \mathbf{l}_{\text{fin}},$$

is the usual solution of the difference equation $J\psi = \lambda\psi$ with the initial condition $\psi_0(\lambda) = \Phi_{0,j}(\lambda)f = (\Phi_{j,0}(\lambda))^*f$.

Again as above, we obtain the representation of the type (16)

$$\Phi_{j,k}(\lambda) = Q_k(\lambda)(\pi_0\Phi(\lambda)\pi_j) = Q_k(\lambda)\Phi_{0,j}(\lambda), \quad k \in \mathbb{N}_0. \tag{18}$$

Taking into an account (16) with j (k) replaced by 0 (k), we get

$$\Phi_{0,k}(\lambda) = (\Phi_{k,0}(\lambda))^* = (Q_k(\lambda)(\pi_0\Phi(\lambda)\pi_0))^* = \Phi_{0,0}(\lambda)(Q_k(\lambda))^* \tag{19}$$

(here we used the fact that the scalar $\Phi_{0,0}(\lambda) \geq 0$; this inequality follows from (6) and (8)). Substituting (19) into (16), we get for all $j,\ k \in \mathbb{N}_0$: $\Phi_{j,k}(\lambda) = Q_j(\lambda)\Phi_{0,k}(\lambda) = Q_j(\lambda)\Phi_{0,0}(\lambda)(Q_k(\lambda))^*$.

The lemma is proved.

It will be essential for us to rewrite the Parseval equality (6), (8) in the form that involves the operator polynomials $Q_0(\lambda), Q_1(\lambda), \ldots$ introduced above.

Using Parseval equality (8) and representation (14) we get: $\forall f,\ g \in \mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2} = \sum_{j,k=}^{\infty} \int_{\mathbb{R}} (\Phi_{j,k}(\lambda)f_k, g_j)_{\mathbf{l}_2} d\sigma(\lambda) =$$

$$= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (Q_j(\lambda)\Phi_{0,0}(\lambda)Q_k^*(\lambda)f_k, g_j)_{\mathbf{l}_2} d\sigma(\lambda) =$$

$$= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (Q_k^*(\lambda)f_k, Q_j^*(\lambda)g_j)_{\mathbf{l}_2} d\rho(\lambda) =$$

$$= \int_{\mathbb{R}} \left(\left(\sum_{k=0}^{\infty} Q_k^*(\lambda)f_k \right), \left(\sum_{j=0}^{\infty} Q_j^*(\lambda)g_j \right) \right)_{\mathbb{C}^2} d\rho(\lambda). \tag{20}$$

Here $d\rho(\lambda) = \Phi_{0,0}(\lambda)d\sigma(\lambda)$ is the spectral measure of our operator $\boldsymbol{J}$, it is a probability Borel measure on $\mathbb{R}$ with a bounded support.

***Remark* 1.** The operator polynomials $Q_1(\lambda), Q_2(\lambda), \ldots$ form a solution of equations (9)–(11) with real coefficients. Therefore they are real and the star * means transposed matrix.

Introduce the Fourier transform $\widehat{\ }$ induced by self-adjoint bounded operators $\boldsymbol{J}$ in the space $\mathbf{l}_{\mathrm{fin}}$:

$$\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \mapsto \widehat{f}(\lambda) = f_0 + \sum_{n=1}^{\infty} Q_n^*(\lambda)f_n =$$

$$= \sum_{n=0}^{\infty} Q_n^*(\lambda)f_n \in L^2\big(\mathbb{R}, d\rho(\lambda)\big). \tag{21}$$

Hence, (20) gives the Parseval equality in the final form: $\forall f,\ g \in \mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)}d\rho(\lambda). \tag{22}$$

Extending (22) by continuity, it becomes valid for all $f,\ g \in \mathbf{l}_2$.

We find now the orthogonal properties of operator polynomials $Q_n^*(\lambda)$, $n \in \mathbb{N}_0$.

Remind, that the zeroth polynomial $Q_0^*(\lambda)$ is equal to $\mathbf{1}$. The orthonormal properties of these polynomials (which are some operators: $\mathbb{R}^2 \to \mathbb{R}^2$, polynomially depending on $\lambda \in \mathbb{R}$) are easy to find. Namely, recall that every $\mathbb{C}^2 \ni f_k$, $k \in \mathbb{N}$, has a 2-vector form: $f_k = (f_{k;0}e_{k;0}, f_{k;1}e_{k;1})$ where $e_k = (e_{k;0}, e_{k;1})$ is a fixed standard basis in the space $\mathbb{C}^2$. In the case $k = 0$ we can view the value $f_0 \in \mathbb{C}$ as vector $(f_{0;0}) \in \mathbb{C}^2$. Therefore, taking in (21) the vectors f, g of the form $f = (0, \ldots, 0, f_{k;\beta}, 0, \ldots)$, $g = (0, \ldots, 0, g_{j;\alpha}, 0, \ldots)$, we conclude that $\widehat{f}(\lambda) = \big(Q_k^*(\lambda) f_{k;\beta} e_{k;\beta}\big)_\alpha = (Q_k^*(\lambda))_{\alpha,\beta} f_{k;\beta} = (Q_k(\lambda))_{\beta,\alpha} f_{k;\beta}$, $\widehat{g}(\lambda) = (Q_j^*(\lambda) g_{j;\alpha} e_{j;\alpha})_\beta = (Q_j^*(\lambda))_{\alpha,\beta} g_{j;\alpha} = (Q_j(\lambda))_{\beta,\alpha} g_{j;\alpha}$ and than the Parseval equality (22) gives

$$\int_{\mathbb{R}} (Q_k(\lambda))_{\beta,\alpha} f_{k;\beta} (Q_j(\lambda))_{\beta,\alpha} \overline{g}_{j;\alpha} d\rho(\alpha) = f_{k;\beta} \overline{g}_{j;\alpha}, \quad j,\ k \in \mathbb{N}_0, \quad \alpha,\ \beta = 0,\ 1,$$

or in a more symmetric form

$$\int_{\mathbb{R}} (Q_j(\lambda))_{\alpha,\beta} (Q_k(\lambda))^*_{\beta,\alpha} d\rho(\alpha) = \delta_{j,k} (e_\alpha, e_\beta)_{\mathbb{C}^2}, \quad j,\ k \in \mathbb{N}_0, \quad \alpha,\ \beta = 0, 1, \tag{23}$$

where $d\rho(\lambda)$ is a probability Borel measure on $\mathbb{R}$.

It is easy enough to find the elements of matrix J in terms of operator polynomials $Q_0(\lambda)$, $Q_1(\lambda), \ldots$. For this, we take f, $g \in \mathbf{l}_{\text{fin}}$. The representation (5) and (8), (20) give

$$(Jf, g)_{\mathbf{l}_2} = \big(\boldsymbol{J} f, g\big)_{\mathbf{l}_2} = \int_{\mathbb{R}} \lambda (\Phi(\lambda) f, g)_{\mathbf{l}_2} d\sigma(\lambda) =$$

$$= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} \lambda (\Phi_{j,k}(\lambda) f_k, g_j)_{\mathbf{l}_2} d\sigma(\lambda) = \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} \lambda (Q_k^*(\lambda) f_k, Q_j^*(\lambda) g_j)_{\mathbf{l}_2} d\rho(\lambda).$$

Setting in this equality $f_{(k)} = (f_n)_{n=0}^\infty \in \mathbf{l}_{\text{fin}}$ for which $f_n = 0$ except for $f_k = e_k$ and, similarly, $g_{(j)} = (g_n)_{n-0}^\infty \in \mathbf{l}_{\text{fin}}$ composed from zeros except for $g_j = e_j$ $(j,\ k \in \mathbb{N}_0$ are fixed) we find: $\forall j,\ k \in \mathbb{N}_0$

$$J_{j,k} = (J f_{(k)}, g_{(j)})_{\mathbf{l}_2} = \int_{\mathbb{R}} \lambda (Q_k^*(\lambda) e_k, Q_j^*(\lambda) e_j)_{\mathbb{C}^2} d\rho(\lambda) =$$

$$= \int_{\mathbb{R}} \lambda (Q_j(\lambda) Q_k^*(\lambda) e_k, e_j)_{\mathbb{C}^2} d\rho(\lambda). \tag{24}$$

Taking into an account that $J_{j,k}$ is a matrix $(J_{j,k;\alpha,\beta})_{\alpha,\beta=0}^1$, we can rewrite (24) in the form

$$(J_{j,k;\alpha,\beta})_{\alpha,\beta=0}^1 = \int_{\mathbb{R}} \lambda (Q_j(\lambda) Q_k^*(\lambda))_{\alpha,\beta=0}^1 d\rho(\lambda), \quad j,\ k \in \mathbb{N}_0 \tag{25}$$

$\left(\text{in particular, } b_0 = \int_{\mathbb{R}} \lambda d\rho(\lambda)\right)$.

4. The inverse spectral problem for operator *J*. Basic constructions. Our goal now is to prove that the spectral measure $d\rho(\lambda)$ of operator ***J*** (a probability Borel measure on $\mathbb{R}$, with a bounded support $\operatorname{supp} d\rho(\lambda)$), allows one to reconstruct the matrix J (4).

We first recall some constructions (more general results can be found in [18], Ch. 7). Consider the space $\mathcal{F}$ of all matrix-valued (i.e., operator valued of the type: $\mathbb{C}^2 \to \mathbb{C}^2$) continuous functions $\mathbb{R} \ni \lambda \mapsto F(\lambda)\colon \mathbb{C}^2 \to \mathbb{C}^2$ and introduce for such functions the operator-valued scalar product $\{\cdot,\cdot\}$ putting for all $F,\, G \in \mathcal{F}$

$$\{F, G\} = \int_{\mathbb{R}} (F(\lambda))^* G(\lambda) d\rho(\lambda)\colon \mathbb{C}^2 \to \mathbb{C}^2. \tag{26}$$

Some examples of functions $F(\lambda)$ were introduced earlier: the operator polynomials $Q_m(\lambda)$. Of course, it is possible to perform some procedure of completion, but for our purposes it is not essential. We only note that for an arbitrary operator "scalar" $C\colon \mathbb{C}^2 \to \mathbb{C}^2$ we have

$$\{FC, G\} = C^*\{F, G\}, \qquad \{CF, G\} = \{F, C^*G\}, \qquad \{F, GC\} = \{F, G\}C,$$
$$\{F, G\}^* = \{G^*, F\}, \qquad \{F, F\} \geq 0, \quad F,\ G \in \mathcal{F}. \tag{27}$$

Let $x,\ y \in \mathbb{C}^2$, than for all $F,\ G \in \mathcal{F}$

$$\big(\{F, G\}x, y\big)_{\mathbb{C}^2} = \int_{\mathbb{R}} \big((F(\lambda))^* G(\lambda)x, y\big)_{\mathbb{C}^2} d\rho(\lambda) = \int_{\mathbb{R}} \big(G(\lambda)x, F(\lambda)y\big)_{\mathbb{C}^2} d\rho(\lambda). \tag{28}$$

For us will be essential the procedure of normalization of vectors F from $\mathcal{F}$. If for $F \in \mathcal{F}$ the operator $\{F, F\}^{-1}\colon \mathbb{C}^2 \to \mathbb{C}^2$ exists, than we have from (27)

$$\{FC, FC\} = C^*\{F, F\}C = 1 \quad \text{for} \quad C = \{F, F\}^{-1/2}, \tag{29}$$

i.e., FC is "normalized" F.

Lemma 2. *Assume additionally, that* $\operatorname{supp} d\rho(\lambda)$ *contains infinitely many different points. Then every complex operator polynomial* $P_m(\lambda) \in \mathcal{F}$, $m \in \mathbb{N}_0$, *can be normalized.*

Proof. According to (29), it is only necessary to prove that $\big\{P_m(\lambda), P_m(\lambda)\big\}^{-1}\colon \mathbb{C}^2 \to \mathbb{C}^2$ exists. Taking into an account that $\big\{P_m(\lambda), P_m(\lambda)\big\}$ is an operator on finite dimensional space $\mathbb{C}^2$, the existence of $\big\{P_m(\lambda), P_m(\lambda)\big\}^{-1}$ is equivalent to the following assertion: if for some $x \in \mathbb{C}^2$ $\big(\big\{P_m(\lambda), P_m(\lambda)\big\}x, x\big)_{\mathbb{C}^2} = 0$, then $x = 0$. Using (28), we assume that

$$0 = \big(\big\{P_m(\lambda), P_m(\lambda)\big\}x, x\big)_{\mathbb{C}^2} = \int_{\mathbb{R}} \|P_m(\lambda)x\|^2_{\mathbb{C}^2}\, d\rho(\lambda).$$

The expression $\|P_m(\lambda)x\|^2_{\mathbb{C}^2}$ is an ordinary polynomial of degree $2m$, therefore the last equality gives $\|P_m(\lambda)x\|^2_{\mathbb{C}^2} = 0$. But this equality shows that $x = 0$ because the higher coefficient of $P_m(\lambda)$ is an invertible operator.

The lemma is proved.

In what follows, the following special generalization of the usual Schmidt orthogonalization procedure will be very essential (see, e.g., [18], Ch. 7).

Lemma 3. *Let* $\operatorname{supp} d\rho(\lambda)$ *contains infinitely many different points. Denote by* $P_m(\lambda)$ *some complex operator polynomial of degree* $m \in \mathbb{N}_0$. *Assume that we have a system* $P_1(\lambda), \ldots, P_n(\lambda)$, $n \in \mathbb{N}$, *of such polynomials with properties*:

$$\begin{gathered} \{P_1(\lambda), P_1(\lambda)\} = \ldots = \{P_n(\lambda), P_n(\lambda)\} = \mathbf{1}, \\ \{P_j(\lambda), P_k(\lambda)\} = 0, \quad j \neq k, \quad j,\ k = 1, \ldots, n. \end{gathered} \tag{30}$$

Then one can be construct a complex operator polynomial $P_{n+1}(\lambda)$ *of degree* $n+1$ *for which*

$$\{P_{n+1}(\lambda), P_{n+1}(\lambda)\} = \mathbf{1}, \qquad \{P_{n+1}(\lambda), P_j(\lambda)\} = 0, \quad j = 1, \ldots, n. \tag{31}$$

Proof. Let $D_{n+1}(\lambda)$ be some fixed arbitrary complex operator polynomial of degree $n+1$. We will find polynomial $P_{n+1}(\lambda)$ at first in a non-normalized form (denote it by $Q_{m+1}(\lambda)$). Put

$$Q_{n+1}(\lambda) = D_{n+1}(\lambda) - \sum_{j=1}^{n} P_j(\lambda) C_j \tag{32}$$

where $C_j : \mathbb{C}^2 \to \mathbb{C}^2$ are some unknown operator coefficients.

Multiply (32) on $P_k(\lambda)$, $k = 1, \ldots, n$, and using (27), (30) obtain

$$0 = \{Q_{n+1}(\lambda), P_k(\lambda)\} = \{D_{n+1}(\lambda), P_k(\lambda)\} - \sum_{j=1}^{n} \{P_j(\lambda) C_j, P_k(\lambda)\} =$$

$$= \{D_{n+1}(\lambda), P_k(\lambda)\} - \sum_{j=1}^{n} C_j^* \{P_j(\lambda), P_k(\lambda)\} = \{D_{n+1}(\lambda), P_k(\lambda)\} - C_k^*, \tag{33}$$

i.e., $C_k = \{(P_k(\lambda))^*, D_{n+1}(\lambda)\}$, $k = 1, \ldots, n$.

Last equalities and (32) show that the following operator polynomial of degree $n+1$:

$$Q_{n+1}(\lambda) = D_{n+1}(\lambda) - \sum_{j=1}^{n} P_j(\lambda) \{(P_j(\lambda))^*, D_{n+1}(\lambda)\} \tag{34}$$

is orthogonal (w.r.t. $\{\cdot, \cdot\}$) to all $P_1(\lambda), \ldots, P_n(\lambda)$.

According to Lemma 2, this polynomial can be normalized. We get as a result the normalized polynomial $P_{n+1}(\lambda)$ which is equal to $Q_{n+1}(\lambda) C$ where $C = \{Q_{n+1}(\lambda), Q_{n+1}(\lambda)\}^{-1/2}$. The conditions (33) of orthogonality are not violated because for all $k = 1, \ldots, n$ $\{P_{n+1}(\lambda), P_k(\lambda)\} = \{Q_{n+1}(\lambda) C, P_k(\lambda)\} = C^* \{Q_{n+1}(\lambda), P_k(\lambda)\} = 0$.

The lemma is proved.

A few simple remarks are in order.

***Remark* 2.** The results of Lemma 3 are true for (real) operator polynomials.

***Remark* 3.** Introduce a vector-valued polynomial of degree $m \in \mathbb{N}_0$ as a function on $\mathbb{R}$ such that:

$$\mathbb{R} \ni \lambda \mapsto \sum_{j=0}^{m} \lambda^j a_j,$$

where $a_j \in \mathbb{R}^2$ are real coefficients; $a_m \neq 0$. It is obvious that the set $\{P_m(\lambda)x \mid x \in \in \mathbb{R}^2\}$, where $P_m(\lambda)$ is an operator polynomial of degree m, is a subset of a set of such polynomials with the highest coefficient $a_m = A_m x$.

***Remark* 4.** Consider a fixed system $\{P_1(\lambda)\}, \ldots, \{P_m(\lambda)\}$, $m \in \mathbb{N}$, of operator polynomials with orthogonality properties (30) for which every polynomial $P_n(\lambda)$, $n = = 2, \ldots, m-1$, is constructed according to the rules of Lemma 3; where polynomial $D_{n+1}(\lambda)$ from (32) has a form $D_{n+1}(\lambda) = \lambda^{n+1} A_{n+1}$, $\lambda \in \mathbb{R}$, $A_{n+1} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a fixed invertible operator. Operator polynomial $P_1(\lambda)$ is fixed of the form $P_1(\lambda) = = \lambda A_1 + A_0$, where A_1, $A_0 \colon \mathbb{R}^2 \to \mathbb{R}^2$ are invertible; $P_0(\lambda) = 1$.

Introduce a set of vector-valued polynomials

$$\mathbb{R} \ni \lambda \mapsto \sum_{j=0}^{m} P_j(\lambda) x_j, \tag{35}$$

where $x_j \in \mathbb{R}^2$ are arbitrary. Then this set coincides with the set of all vector-valued polynomials $\mathbb{R} \ni \lambda \mapsto P(\lambda)$ of degree $\leq m$.

This assertion immediately follows from the proof of Lemma 3 and Remark 2.

***Remark* 5.** The union of all the sets (35) $(m = 0, 1, \ldots)$ is equal to the set $\mathcal{P}(\mathbb{R})$ of all vector-valued polynomials on $\mathbb{R}$.

***Remark* 6.** In Section 7, in the study of Toda lattices, it will be essential to use in Remarks 3 – 5 the invertible operators $A_2, A_3, \ldots \colon \mathbb{R}^2 \to \mathbb{R}^2$ with matrices

$$\begin{bmatrix} \theta_n & 0 \\ 0 & \omega_n \end{bmatrix}, \quad \theta_n,\ \omega_n > 0, \quad n = 2, 3, \ldots. \tag{36}$$

Let $d\rho(\lambda)$ be the spectral measure of our operator $\boldsymbol{J}$; assume that its $\operatorname{supp} d\rho(\lambda)$ contains infinitely many different points. Consider the linear space $\mathcal{P}(\mathbb{R})$ of all vector-valued (real) polynomials on $\mathbb{R}$. Every vector $p(\lambda)$ from $\mathcal{P}(\mathbb{R})$ can be viewed as a function of $\lambda \in \mathbb{R}$ with values in $\mathbb{R}^2$ for which $\int_{\mathbb{R}} \|p(\lambda)\|^2_{\mathbb{R}^2}\, d\rho(\lambda) < \infty$. If we introduce the real Hilbert space $L^2\big(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\big)$ of functions $\mathbb{R} \ni \lambda \mapsto f(\lambda) \in \mathbb{R}^2$ with a scalar product $\big(f(\lambda), g(\lambda)\big)_{L^2(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda))} = \int_{\mathbb{R}} \big(f(\lambda), g(\lambda)\big)_{\mathbb{R}^2} d\rho(\lambda)$, then we can say that $\mathcal{P}(\mathbb{R}) \subset L^2\big(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\big)$ is dense in the latter space. Similarly, it is possible to introduce the complex Hilbert space $L^2\big(\mathbb{R}, \mathbb{C}^2; d\rho(\lambda)\big)$.

We will fix some normalized operator polynomial $P_1(\lambda) \in L^2\big(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\big)$ of degree one and after this we apply the Lemma 3 (see Remark 2) and construct by orthogonalization the normalized operator polynomials $P_2(\lambda)$, $P_3(\lambda), \ldots$; $\{P_j(\lambda), P_k(\lambda)\} = \mathbf{1}\delta_{j,k}$, j, $k = 1, 2, \ldots$. The form of this polynomials depend on initial operator polynomials $D_2(\lambda), D_3(\lambda), \ldots$. We will take, in agreement with Remark 6:

$$D_n(\lambda) = \lambda^n \begin{bmatrix} \theta_n & 0 \\ 0 & \omega_n \end{bmatrix}, \quad \theta_n,\ \omega_n > 0, \quad n = 2, 3, \ldots. \tag{37}$$

Lemma 4. *Assume that matrices a_n in* (4) *have a form*

$$a_n = \begin{bmatrix} \theta_n^{-1} & 0 \\ 0 & \omega_n^{-1} \end{bmatrix}, \quad \theta_n,\ \omega_n > 0, \quad n \in \mathbb{N}. \tag{38}$$

For $j = 2, 3, \ldots$ *the polynomials* $\bigl(Q_j(\lambda)\bigr)^*$, *constructed as solutions of equations* (9), (10) (*see* (13)), *are equal to operator polynomials* $P_j(\lambda)$ *constructed via the procedure of orthogonalization with fixed* $P_1(\lambda)$ *equal to* $Q_1(\lambda)$ *from* (13), (11), *and* $D_n(\lambda)$ *of the form* (37).

Thus, these $\bigl(Q_j(\lambda)\bigr)^*$ *can be found via a procedure of orthogonalization, described in Lemma* 3.

Proof. According to Remark 5, the set of all vectors (35) coincides with $\mathcal{P}(\mathbb{R})$ and is dense in the space $L^2\bigl(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\bigr)$ (we first fixed an operator polynomial $P_1(\lambda) = \lambda A_1 + A_0$, where $A_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ is invertible, $P_0(\lambda) = \mathbf{1}$).

Thus, we have, in the real Hilbert space $L^2\bigl(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\bigr) \ominus \bigl\{$ set F_1 of all vector-valued polynomials of degree $\leq 1\bigr\}$, the orthogonal systems $\bigl\{P_n(\lambda)x_n \in \mathbb{R}^2\bigr\}$, $n \in 2, 3, \ldots$, constructed by means of polynomials (37).

On the other hand from formulas (10) and (38) we see, that the operator polynomials $Q_2(\lambda)$, $Q_3(\lambda), \ldots$ (and, therefore the polynomials $Q_2^*(\lambda)$, $Q_3^*(\lambda), \ldots$) have the leading coefficients, i.e., the coefficients near λ^2, $\lambda^3, \ldots$, that coincide with those for $P_2(\lambda)$, $P_3(\lambda), \ldots$ (their corresponding leading terms are (37)). As we proved in Section 3 (see (23)) the system of vectors $Q_2^*(\lambda)\varphi_0$, $Q_3^*(\lambda)\varphi_0, \ldots$ from $L^2\bigl(\mathbb{R}, \mathbb{R}^2; d\rho(\lambda)\bigr) \ominus \{F_1\}$ is also orthogonal and gives the basis in the latter space. Since the coefficients of highest degree λ of these two systems are equal, these systems must coincide: $Q_2^*(\lambda) = P_2(\lambda)$, $Q_3^*(\lambda) = P_3(\lambda), \ldots$.

The lemma is proved.

5. The direct and inverse spectral problem for the Jacobi matrix considered. Formulation of results. We can now collect constructions of last two sections and formulate the main results. Thus, we consider on the Hilbert space $\mathbf{l}_2$ (1) the operator $\boldsymbol{J}$ generated by Jacobi matrix J of the form (4) with the following additional condition: the matrices a_n, $n \in \mathbb{N}$, are diagonal of the form (38):

$$a_n = \begin{bmatrix} \theta_n^{-1} & 0 \\ 0 & \omega_n^{-1} \end{bmatrix}, \quad \theta_n,\ \omega_n > 0, \quad n \in \mathbb{N}. \tag{39}$$

Recall that all elements of matrix J are real and uniformly bounded. The matrices a_0 and a_0^* will be denoted now by

$$a_0 = \begin{bmatrix} \alpha_0 & \alpha_{-1} \end{bmatrix} \neq 0, \qquad a_0^* = \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \end{bmatrix}, \quad \alpha_0,\ \alpha_{-1} \in \mathbb{R}. \tag{40}$$

This matrix J generates (starting from its action on finite sequence $\mathbf{l}_{\mathrm{fin}}$) a bounded self-adjoint operator $\boldsymbol{J}$ in the space $\mathbf{l}_2$. Its spectral measure $d\rho(\lambda)$ is a bounded Borel measure with a bounded $\operatorname{supp} d\rho(\lambda)$, that, we assume in addition, contains infinitely many different points.

Theorem 1. *The full system of generalized eigenvectors* $\varphi(\lambda)$ *of the operator* $\boldsymbol{J}$ *has the following form*: *for all* λ *from the generalized spectrum of* $\boldsymbol{J}$

$$\varphi(\lambda) = \bigl(\varphi_0 = 1, \varphi_1(\lambda), \varphi_2(\lambda), \ldots\bigr) \in \mathbf{l}_2(p^{-1}), \tag{41}$$

where $\varphi_n(\lambda), n \in \mathbb{N}$, *are the solutions* (10), (11) *of difference equations* (9).

The corresponding Fourier transform $\widehat{f}(\lambda)$ *of a vector* $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_{\mathrm{fin}}$ *is defined by the formula*

$$\mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \mapsto \widehat{f}(\lambda) = \sum_{n=0}^{\infty} Q_n^*(\lambda) f_n \in L^2(\mathbb{R}, \mathbb{C}^2; d\rho(\lambda)) \tag{42}$$

and maps $\mathbf{l}_{fin}$ *onto right-hand side in* (42) *isometrically. The closure of* (42) *by continuity gives the unitary operator between* $\mathbf{l}_2$ *and the right-hand side of* (42).

The inverse spectral problem consist in the following. Assume that we know an element $a_0 = \begin{bmatrix}\alpha_0 & \alpha_{-1}\end{bmatrix} \neq 0$ *of the matrix* J *and the spectral measure* $d\rho(\lambda)$ *of the corresponding operator* $\boldsymbol{J}$. *Then we can find all elements of the matrix* J *in the following manner. Construct by* (11) *the normalized operator polynomial* $Q_1(\lambda)$ *with fixed* $\theta_1, \omega_1 \neq 0$ *and* $b_0 = \int_{\mathbb{R}} \lambda \rho(\lambda)$. *Then apply the procedure of orthogonalization described in Lemma* 3 *and Remark* 2 *with initial operator polynomials* $D_n(\lambda)$, $n = 2, 3, \ldots$, *given by* (37). *As a result, we obtain the orthonormal sequence of operator polynomials* $Q_1(\lambda)$, $Q_2(\lambda), \ldots$. *The elements of the matrix* J (*with* a_n, $n \in \mathbb{N}$, *of the form* (39)) *are reconstructed by formulas* (25).

***Remark* 7.** The conditions (39) on the blocks a_n are not essential: using in the Lemma 3 instead $D_n(\lambda)$ of the form (37) more complicated expressions (connected with matrices a_n from (4)), we can treat the case of general matrix J of type (4).

6. The Lax equation corresponding to Jacobi matrices of our type. Assume that elements $a_{n;\alpha,\beta}$, $b_{n;\alpha,\beta}$, $n \in \mathbb{N}_0$, α, $\beta = 0, 1$, of matrix (4) are once continuously differentiable uniformly bounded real functions of $t \in [0, T]$, $T < \infty$. Denote this matrix by $J(t)$. Let $A(t)$, $t \in [0, T]$, be some other matrix of the same type as $J(t)$. The Lax equation connected with these two matrices $J(t)$ and $A(t)$, has a form

$$\left(\frac{dJ}{dt}\right)(t) =: \dot{J}(t) = \bigl[J(t), A(t)\bigr] := J(t)A(t) - A(t)J(t), \quad t \in [0, T]. \tag{43}$$

When elements of matrix $A(t)$ do not depend on elements of $J(t)$ the equation

$$\dot{J}(t) = \bigl[J(t), A(t)\bigr] = 0, \quad t \in [0, T], \tag{44}$$

is a linear differential equation w.r.t. matrix $J(t)$. But if these elements depend on elements of $J(t)$, the system (44) is a system of nonlinear differential-difference equation for the elements of matrix $J(t)$. As in [1, 25, 4] and in an extensive list of other works, we will discuss precisely this situation.

We will use for elements of matrix $A(t)$ the same notations as for elements of $J(t)$, but with tildes: $\widetilde{a_n}(t)$, $\widetilde{b_n}(t)$, $n \in \mathbb{N}_0$.

For every $t \in [0, T]$, we construct a bounded self-adjoint operator $\boldsymbol{J}(t)$ using the matrix $J(t)$. The spectrum of $\boldsymbol{J}(t)$, $t \in [0, T]$, is located in a bounded segment $[a, b] \subset \mathbb{R}$ and therefore we can apply to our case the scheme of [23], § 3. Thus, in our case the essential role will be played by the Weyl function

$$m(z;t) = \int\limits_a^b \frac{d\rho(\lambda;t)}{\lambda - z}, \quad t \in [0,T], \quad z \in \mathbb{C}\backslash[a,b], \tag{45}$$

where $d\rho(\lambda;t)$ is the spectral probability measure of the bounded self-adjoint operator $\boldsymbol{J}(t)$. This operator is weekly continuously differentiable w.r.t. $t \in [0,T]$.

We will use a matrix $A(t)$ similar to the matrix used in the article [23], § 2[1]. Namely, we assume that analogically to (29), [23],

$$(\widetilde{a_0})^*(t) = 0 \qquad \text{and} \qquad \widetilde{a_0}(t) = -a_0(t), \quad t \in [0,T]. \tag{46}$$

Then mimicking calculations from [23], § 3, we can prove an analogue of Theorem 3 from [23], which gives the following differential equation for the Weyl function (45):

$$\dot{m}(z;t) = (z - b_0(t))m(z;t) + 1, \quad z \in \mathbb{C}\backslash[a,b], \tag{47}$$

and the representation of spectral measure $d\rho(\lambda;t)$ of the following form:

$$d\rho(\lambda;t) = C(t)e^{\lambda t}d\rho(\lambda;0), \quad \lambda \in [a,b], \quad t \in [0,T]. \tag{48}$$

Here $d\rho(\lambda;0)$ is the spectral measure of initial operator $\boldsymbol{J}(0)$ and $C(t) > 0$ is a factor normalizing the measure (48) to a probability measure, i.e., the equality $\rho([a,b];t) = 1$, $t \in [0,T]$ must hold.

For the particular case of conditions (46), namely, if

$$\widetilde{a_n^*}(t) = 0, \qquad \widetilde{b_n}(t) = -\frac{1}{2}b_n(t), \qquad \widetilde{a_n}(t) = -a_n(t), \quad n \in \mathbb{N}_0, \quad t \in [0,T], \tag{49}$$

it is easy to write analogously to § $2,3$ from [23] (see [23], (33)) the following form of Lax equations: $\forall t \in [0,T]$

$$\begin{aligned} &\dot{a}_n(t) = \frac{1}{2}(a_n(t)b_{n+1}(t) - b_n(t)a_n(t)), \quad n \in \mathbb{N}_0, \\ &\dot{b}_n(t) = a_n^2(t) - a_{n-1}^2(t), \quad n = 2,3,\ldots, \\ &\dot{b}_1(t) = a_1^2(t) - (a_0(t))^*a_0(t), \qquad \dot{b}_0(t) = a_0(t)(a_0(t))^*. \end{aligned} \tag{50}$$

Thus, we have the system (50) of nonlinear differential-difference equation with respect to matrix unknowns $a_0(t),\ a_1(t),\ldots;\ b_0(t),\ b_1(t),\ldots,\ t \in [0,T]$. For this system we formulate the Cauchy problem: for given $a_0(0),\ a_1(0),\ldots$; $b_0(0),\ b_1(0),\ldots,$ find the solution $a_0(t),\ a_1(t),\ldots$; $b_0(t),\ b_1(t),\ldots$ for an arbitrary $t \in [0,T]$.

By using the inverse spectral problem we can formulate the following result:

Theorem 2. *Assume that the matrices $a_n(t)$ from* (4) *have the following form*: $\forall t \in [0,t]$

[1] When comparing these formulas with those of [23], it is necessary take into an account that the role of matrices $a_0,\ a_1,\ldots$ was played by matrices $c_0,\ c_1,\ldots$ in [23].

$$a_n(t) = \begin{bmatrix} \theta_n^{-1}(t) & 0 \\ 0 & \omega_n^{-1}(t) \end{bmatrix}, \quad \theta_n(t), \quad \omega_n(t) > 0, \quad n \in \mathbb{N},$$
$$a_0(t) = \begin{bmatrix} \alpha_0(t) & \alpha_{-1}(t) \end{bmatrix} \neq 0. \tag{51}$$

Consider the Cauchy problem for (50) *formulated above.*

Suppose that for this problem, we know the part of its solutions: $\alpha_0(t)$, $\alpha_{-1}(t)$ for every $t \in [0, t]$. Then the full solution can be reconstructed through following procedure.

Find the spectral measure $d\rho(\lambda; 0)$ of the initial operator $\boldsymbol{J}(0)$ and then construct, by formula (48), *the spectral measure $d\rho(\lambda; t)$ for $\boldsymbol{J}(t)$, $t \in (0, T]$. Using the second part of Theorem* 1, *find the sequence of orthonormal operator polynomials $Q_1(\lambda; t)$, $Q_2(\lambda; t), \ldots; Q_1(\lambda; t)$ is constructed by rules* (11), (12), *where $a_0(t)$ has the form* (51) *and $\theta_1(t), \omega_1(t) \neq 0$ depend on $t \in [0, T]$ in a continuously differentiable manner. Using these polynomial find, according to formula* (25), *all the elements of the Jacobi matrix $J(t), t \in [0, T]$, whose elements constitute the full solution of our Cauchy problem.*

Proof. It is only necessary to check that the formulas (25) give the matrices $a_n(t)$, $b_n(t)$, $t \in [0, T]$, which are the solution of our Cauchy problem for the system (50). But this assertion is a partial case of a more general Theorem 4 from [23].

The theorem is proved.

7. The integration of the modified double-infinite Toda lattice. The double-infinite Toda lattice has the form

$$\dot{\alpha}_n = \frac{1}{2}\alpha_n(\beta_{n+1} - \beta_n), \qquad \dot{\beta}_n = \alpha_n^2 - \alpha_{n-1}^2, \quad n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}, \tag{52}$$

where $\alpha_n = \alpha_n(t), \beta_n = \beta_n(t)$ are real once continuously differentiable functions of $t \in [0, T]$. (52) is a differential-difference nonlinear equation and for (52) it is possible to consider the Cauchy problem: we know initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{Z}$, and it is necessary to find the solution $\alpha_n(t)$, $\beta_n(t)$, $n \in \mathbb{Z}$, for $t > 0$.

The equation (52) is connected with the special case of the equation (50). Namely, we take in (50) matrices

$$a_n(t) = \begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix}, \qquad b_n(t) = \begin{bmatrix} \beta_n(t) & 0 \\ 0 & -\beta_{-n}(t) \end{bmatrix}, \quad n \in \mathbb{N},$$
$$a_0(t) = \begin{bmatrix} \alpha_0(t) & \alpha_{-1}(t) \end{bmatrix}, \qquad b_0(t) = \begin{bmatrix} \beta_0(t) \end{bmatrix}, \quad t \in [0, T], \tag{53}$$

with positive uniformly bounded once continuously differentiable functions $\alpha_m(t)$, $\beta_m(t)$, $m \in \mathbb{Z}$. Then equations (50) transform into a system:

$$\begin{bmatrix} \dot{\alpha}_n(t) & 0 \\ 0 & \dot{\alpha}_{-n-1}(t) \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} \beta_{n+1}(t) & 0 \\ 0 & -\beta_{-n-1}(t) \end{bmatrix} \begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix} - \right.$$
$$\left. - \begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix} \begin{bmatrix} \beta_n(t) & 0 \\ 0 & -\beta_{-n}(t) \end{bmatrix} \right), \quad n \in \mathbb{N};$$

$$\begin{bmatrix}\dot\alpha_0(t) & \dot\alpha_{-1}(t)\end{bmatrix} =$$

$$= \frac{1}{2}\left(\begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\end{bmatrix}\begin{bmatrix}\beta_1(t) & 0\\ 0 & -\beta_{-1}(t)\end{bmatrix} - \begin{bmatrix}\beta_0(t)\end{bmatrix}\begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\end{bmatrix}\right), \quad n \in \mathbb{N};$$

$$\begin{bmatrix}\dot\beta_n(t) & 0\\ 0 & -\dot\beta_{-n}(t)\end{bmatrix} = \tag{54}$$

$$= \begin{bmatrix}\alpha_n(t) & 0\\ 0 & \alpha_{-n-1}(t)\end{bmatrix}^2 - \begin{bmatrix}\alpha_{n-1}(t) & 0\\ 0 & \alpha_{-n}(t)\end{bmatrix}^2, \quad n = 2, 3, \ldots;$$

$$\begin{bmatrix}\dot\beta_1(t) & 0\\ 0 & -\dot\beta_{-1}(t)\end{bmatrix} = \begin{bmatrix}\alpha_1(t) & 0\\ 0 & \alpha_{-2}(t)\end{bmatrix}^2 - \begin{bmatrix}\alpha_0(t)\\ \alpha_{-1}(t)\end{bmatrix}\begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\end{bmatrix} =$$

$$= \begin{bmatrix}\alpha_1(t) & 0\\ 0 & \alpha_{-2}(t)\end{bmatrix}^2 - \begin{bmatrix}\alpha_0^2(t) & \alpha_0(t)\alpha_{-1}(t)\\ \alpha_{-1}(t)\alpha_0(t) & \alpha_{-1}^2(t)\end{bmatrix};$$

$$\begin{bmatrix}\dot\beta_0(t)\end{bmatrix} = \begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\end{bmatrix}\begin{bmatrix}\alpha_0(t)\\ \alpha_{-1}(t)\end{bmatrix} = \begin{bmatrix}\alpha_0^2(t) + \alpha_{-1}^2(t)\end{bmatrix}.$$

From (54) we conclude: $\forall t \in [0, T]$

$$\begin{aligned}
&\dot\alpha_n(t) = \frac{1}{2}\alpha_n(t)\big(\beta_{n+1}(t) - \beta_n(t)\big),\\
&\dot\alpha_{-n-1}(t) = \frac{1}{2}\alpha_{-n-1}(t)(-\beta_{-n-1}(t) - \beta_{-n}(t)), \quad n \in \mathbb{N};\\
&\dot\alpha_0(t) = \frac{1}{2}\alpha_0(t)\big(\beta_1(t) - \beta_0(t)\big),\\
&\dot\alpha_{-1}(t) = \frac{1}{2}\alpha_{-1}(t)\big(-\beta_{-1}(t) - \beta_0(t)\big);\\
&\dot\beta_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t),\\
&-\dot\beta_{-n}(t) = \alpha_{-n-1}^2(t) - \alpha_{-n}^2(t), \quad n = 2, 3, \ldots;\\
&\dot\beta_1(t) = \alpha_1^2(t) - \alpha_0^2(t), \qquad \alpha_0(t)\alpha_{-1}(t) = 0,\\
&-\dot\beta_{-1}(t) = \alpha_{-2}^2(t) - \alpha_{-1}^2(t),\\
&\dot\beta_0(t) = \alpha_0^2(t) + \alpha_{-1}^2(t).
\end{aligned} \tag{55}$$

Above mentioned results give the possibility to present, instead of the Cauchy problem for (52), the solution of the following problem. Consider the "modified" double-infinite Toda lattice: we take the equation (55) and denote $\alpha_0 = \varphi, \alpha_{-1} = \psi$. Then for all $t \in [0, T]$

$$\dot\alpha_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)), \quad n \in \mathbb{Z} \setminus \{-1\};$$

$$\dot{\beta}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{Z} \setminus \{-1, 0, 1\};$$
$$\dot{\psi}(t) = \frac{1}{2}\psi(t)(-\beta_0(t) - \beta_{-1}(t)), \qquad \dot{\beta}_1(t) = \alpha_1^2(t) - \varphi^2(t), \tag{56}$$
$$\dot{\beta}_0(t) = \varphi^2(t) + \psi^2(t), \qquad \dot{\beta}_{-1}(t) = \psi^2(t) - \alpha_{-2}^2(t).$$

Here $\varphi(t)$, $\psi(t)$ are given real continuously differentiable functions for which

$$\varphi(t)\psi(t) = 0, \quad t \in [0, T], \qquad \text{and} \qquad \forall t \in [0, T] \quad \varphi(t) \neq 0 \quad \text{or} \quad \psi(t) \neq 0. \tag{57}$$

For system (56) with condition (57) and unknowns $\alpha_n(t)$, $\beta_n(t)$, $n \in \mathbb{Z}$, we state the Cauchy problem: for known initial data $\alpha_n(0)$, $n \in \mathbb{Z} \setminus \{-1\}$; $\beta_n(0)$, $n \in \mathbb{Z}$, find the solution of (56).

Theorem 3. *The Cauchy problem formulated above has the solution that can be found as follows.*

Find the initial spectral measure $d\rho(\lambda; 0)$ of the operator $\boldsymbol{J}(0)$ which is constructed from the initial matrix $J(0)$ in the space $\mathbf{l}_2$ (1), *where $a_n(0)$, $b_n(0)$, $n \in \mathbb{N}$, are given by* (53) *and $a_0(0) = [\varphi(0)\,\psi(0)]$, $b_0(0) = \beta_0(0)$. By formula* (48), *construct the spectral measure $d\rho(\lambda; t)$, $t \in (0, T]$. Using the second part of Theorem* 1 *and Theorem* 2, *find the sequence of orthonormal operator polynomials $Q_1(\lambda; t), Q_2(\lambda; t), \ldots; Q_1(\lambda; t)$ has the form* (11), (12) *where $a_0(t) = [\varphi(t)\,\psi(t)]$ and $\theta_1(t), \omega_1(t) \neq 0$ depend on $t \in [0, T]$ in a continuously differentiable manner. Then elements of matrix $J(t)$, i.e., the solution of our Cauchy problem, can be found according to formula* (25).

The proof of this theorem follows from Theorem 2.

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