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NONISOSPECTRAL FLOWS ON SEMIINFINITE UNITARY BLOCK JACOBI MATRICES

НЕІЗОСПЕКТРАЛЬНІ ПОТОКИ НА НАПІВНЕСКІНЧЕННИХ УНІТАРНИХ БЛОЧНИХ ЯКОБІЄВИХ МАТРИЦЯХ

It is proved that if the spectrum and spectral measure of a unitary operator generated by a semiinfinite block Jacobi matrix $J(t)$ vary appropriately, then the corresponding operator $\boldsymbol{J}(t)$ satisfies the generalized Lax equation $\dot{\boldsymbol{J}}(t) = \Phi(\boldsymbol{J}(t), t) + [\boldsymbol{J}(t), A(\boldsymbol{J}(t), t)]$, where $\Phi(\lambda, t)$ is a polynomial in λ and $\bar{\lambda}$ with t-dependent coefficients and $A(J(t), t) = \Omega + I + \frac{1}{2}\Psi$ is a skew-symmetric matrix.

The operator $\boldsymbol{J}(t)$ is analyzed in the space $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$. It is mapped into the unitary operator of multiplication $L(t)$ in the isomorphic space $L^2(\mathbb{T}, d\rho)$, where $\mathbb{T} = \{z\colon |z| = 1\}$. This fact enables one to construct an efficient algorithm for solving the block lattice of differential equations generated by the Lax equation. A procedure that allows one to solve the corresponding Cauchy problem by the Inverse-Spectral-Problem method is presented.

The article contains examples of block difference-differential lattices and the corresponding flows that are analogues of the Toda and van Moerbeke lattices (from self-adjoint case on $\mathbb{R}$) and some notes about applying this technique for Schur flow (unitary case on $\mathbb{T}$ and OPUC theory).

Доведено, що у випадку, коли спектр та спектральна міра унітарного оператора, породженого напівнескінченною блочною якобієвою матрицею $J(t)$, змінюються заданим чином, відповідний оператор $\boldsymbol{J}(t)$ задовольняє узагальнене рівняння Лакса $\dot{\boldsymbol{J}}(t) = \Phi(\boldsymbol{J}(t), t) + \big[\boldsymbol{J}(t), A(\boldsymbol{J}(t), t)\big]$, де $\Phi(\lambda, t)$ є поліномом по λ та $\bar{\lambda}$ з коефіцієнтами, що залежать від t, і $A(J(t), t) = \Omega + I + \frac{1}{2}\Psi$ – деяка кососиметрична матриця.

Оператор $\boldsymbol{J}(t)$ аналізується у просторі $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$. Він відображається в унітарний оператор множення $L(t)$ в ізоморфному просторі $L^2(\mathbb{T}, d\rho)$, де $\mathbb{T} = \{z\colon |z| = 1\}$. Це дає можливість побудувати ефективний алгоритм розв'язування блочного ланцюжка диференціальних рівнянь, що породжується рівнянням Лакса. У статті наведено процедуру, що дозволяє розв'язувати відповідну задачу Коші методом оберненої спектральної задачі.

Розглянуто приклади блочних диференціально-різницевих ланцюжків та відповідних їм потоків, що є аналогами ланцюжків Тоди та Ван Мербека (у самоспряженому випадку на $\mathbb{R}$), а також деякі зауваження стосовно застосування цієї техніки до потоку Шура (унітарний випадок на $\mathbb{T}$ та OPUC теорія).

1. Introduction. This article is the next logical step in developing the theory of difference-differential lattices of equations generated by various forms of Lax equation $\dot{\boldsymbol{J}}(t) = \Phi(\boldsymbol{J}(t), t) + [\boldsymbol{J}(t), A(\boldsymbol{J}(t), t)]$ of the following type. It is required that $\boldsymbol{J}(t)\colon \mathbf{l}_2 \to \mathbf{l}_2$ can be mapped into the operator $L(t)$ of multiplication by independent variable in separable Hilbert space $L^2(\mathbb{C}, d\rho)$. Probability measure $d\rho$ has an infinite compact support and is defined on the Borelean σ-algebra $\mathfrak{B}(\mathbb{C})$. In the whole article it is assumed that all operators are bounded. These restrictions define the class of difference-differential lattices of equations that can be integrated by the method presented here.

This work is based on numerous results by Yu. Berezansky, N. Dudkin, M. Shmoish, L. Golinskii. And it became possible because of advance in OPUC theory (see related article by M. J. Cantero, L. Moral, L. Velázquez [1] and Simon's works [2 – 4]) and CMV matrices theory (see [5, 6]).

In [7 – 9] Yu. M. Berezansky developed an approach to Cauchy problem for Toda lattice on semiaxis and other similar difference-differential lattices. The author used a number of results from spectral theory of classical Jacobi matrices. The main idea in these works is as follows. Solution $u(t),\ t \in [0, \infty)$, was attached in a very simple manner to a bounded self-adjoint Jacobi matrix $J(t)$. At some restrictions for initial difference-differential lattice the evolution of spectral measure $d\rho(\lambda; t)$ of the corresponding operator $\boldsymbol{J}(t)$ could be found for initial spectral measure $d\rho(\lambda; 0)$ of any pre-given $\boldsymbol{J}(0)$ that corresponds to initial condition $u(0)$. Measure $d\rho(\lambda; 0)$ was built using Direct-Spectral-Problem. Final result $u(t)$ was obtained as the set of entries $J(t)$ that was reconstructed from $d\rho(\lambda; t)$ using Inverse-Spectral-Problem for ordinary Jacobi matrices.

Later in [10 – 13] this method was extended for nonisospectral equations. In this case the spectrum of $\boldsymbol{J}(t)$ varies with time t (in the case of Toda lattice the spectrum is always the same — this is *isospectral* lattice). And in [14] the case of unbounded selfadjoint $\boldsymbol{J}(t)$ was investigated.

In all previously mentioned articles $\boldsymbol{J}(t)$ was a self-adjoint operator in ordinary ℓ_2. So the support of its spectral measure (spectrum $\sigma(\boldsymbol{J}(t))$) always laid on $\mathbb{R}$. Currently considerable attention is paid to the theory of Orthogonal Polynomials on the Unit Circle (OPUC) mostly because of the influence of Simon's books (see. [2, 3]). These polynomials are connected with five-diagonal operators in ordinary ℓ_2 (instead of three-diagonal ones in self-adjoint case, see [1]). At the assumption that $\boldsymbol{J}(t)$ is unitary operator one can build the analogous theory of difference-differential lattices with $\sigma(\boldsymbol{J}(t))$ concentrated on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. First results in this direction were obtained by L. Golinskii in [15]. Toda lattice is replaced with Schur flow here.

In mentioned above articles all operators were considered in ordinary $\ell_2 = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \oplus \ldots$. Recently Yu. Berezansky and M. Dudkin noticed that five-diagonal matrices in OPUC can be concidered as ordinary three-diagonal block Jacobi matrices (see [16]). Moreover this structure is absolutely natural (and arises in much simpler way as essential construction) from slightly more general point of view (see [17]). $J(t)$ must be considered as Jacobi matrix in $\mathbf{l}_2 = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \ldots$ if it is normal operator. In particular it can be unitary. In this case it should be considered in subspace $\mathbf{l}_{2,u} = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots \subset \mathbf{l}_2$. Then one has Jacobi matrix for which it is possible to apply a wide range of ideas developed in the last 150 years for ordinary Jacobi matrices. The origins of this fruitful idea can be found in Mark Krein's article [18].

The main aim of this article is to show that the described above approach gives results that are fully compatible with already obtained results (see [2, 3, 1, 15]). We restrict ourselves with the unitary case. However most of the results can be formally copied for more general case of normal $\boldsymbol{J}(t)$ (because all the proofs have algebraic taste and does not depend on space structure). Now we are about to give a mechanism of solving quite general lattice of block difference-differential equations and show that already known results can be easily obtained as particular samples.

It is worth stressing that the described approach (use of block three-diagonal Jacobi matrices in block spaces) allows to obtain in simple algebraic way a wide range of well-known and completely new results which otherwise (if considered in ordinary ℓ_2) would be technically complicated. This method reveals algebraic structure of spaces and operators and in particular gives the hope to make OPUC much simpler. A series of articles is planned by the author on this subject in the nearest future.

2. Common notes. We shall start with definitions of the main objects that are used in this article: spaces, equations and operators. In the next section we shall formulate the main result. Last sections are devoted to the explanation why the result is as it is shown here and why is it convenient just in this form. All the proofs are contained in Section 4.

The article is synchronized with [11]. So it is very simple to compare old and new results if one has the two articles at hand simultaneously. Remind that one of the goals of this work is to show that the corresponding well-known results (see [11, 1, 15]) naturally embed into the new theory and are really simple here. Thus the article is organized in such a way that the comparison of new and old results is as convenient as possible. The last section will contain samples that show how one can construct the embedding of theories.

Consider three-diagonal block Jacobi matrix

$$J(t) = \begin{pmatrix} b_0(t) & c_0(t) & & & \cdot \\ a_0(t) & b_1(t) & c_1(t) & & \cdot \\ & a_1(t) & b_2(t) & c_2(t) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad (1)$$

in the space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots, \qquad \mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_n = \mathbb{C}^2, \quad n \geq 1. \qquad (2)$$

$\mathbf{l}_2$ is Hilbert space with natural scalar product: for $f,\ g \in \mathbf{l}_2$ with coordinates in the *standard orthonormal basis*

$$e_0 = \left(1, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \ldots\right),$$

$$e_{n,1} = \left(0, \begin{pmatrix}0\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\end{pmatrix}, \underset{n}{\begin{pmatrix}1\\0\end{pmatrix}}, \begin{pmatrix}0\\0\end{pmatrix} \ldots\right), \qquad (3)$$

$$e_{n,2} = \left(0, \begin{pmatrix}0\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\end{pmatrix}, \underset{n}{\begin{pmatrix}0\\1\end{pmatrix}}, \begin{pmatrix}0\\0\end{pmatrix} \ldots\right),$$

the norm and scalar product are defined as follows:

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n}, \qquad (f,g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}.$$

Here $\|f_n\|^2_{\mathcal{H}_n} = \|f_n\|^2_{\mathbb{C}^2} = |f_{n,1}|^2 + |f_{n,2}|^2$,

$$(f_n, g_n)_{\mathcal{H}_n} = (f_n, g_n)_{\mathbb{C}^2} = f_{n,1} \cdot \overline{g_{n,1}} + f_{n,2} \cdot \overline{g_{n,2}}.$$

Let J in standard orthonormal basis have the following view:

$$J(t) = \left(\begin{array}{c|cc|cc|cc|cc} b_{0;22} & c_{0;21} & c_{0;22} & & & & & & \\ \hline a_{0;12} & b_{1;11} & b_{1;12} & 0 & 0 & & & & \\ 0 & b_{1;21} & b_{1;22} & c_{1;21} & c_{1;22} & & & & \\ \hline & a_{1;11} & a_{1;12} & b_{2;11} & b_{2;12} & 0 & 0 & & \\ & 0 & 0 & b_{2;21} & b_{2;22} & c_{2;21} & c_{2;22} & & \\ \hline & & & a_{2;11} & a_{2;12} & b_{3;11} & b_{3;12} & 0 & 0 \\ & & & 0 & 0 & b_{3;21} & b_{3;22} & c_{3;21} & c_{3;22} \end{array} \right) \tag{4}$$

All the entries are assumed to be continuously differentiable functions on the interval $[0, T]$. Matrix $J(t)$ generates the corresponding operator $\boldsymbol{J}(t)$. The next step we need is to apply [16] (Theorem 2). So we make assumptions for the entries of $J(t)$ that are described in this theorem. Assume that $\boldsymbol{J}(t)$ is unitary operator (in particular, it is bounded, defined on the whole $\mathbf{l}_2$ and operators $a_n(t), b_n(t), c_n(t)$ are uniformly bounded). Additionally we assume that the entries of $J(t)$ have the following properties (see [16], formula (42)):

$$\begin{gathered} a_{0;12} > 0, \quad a_{1;11} > 0, \quad a_{2;11} > 0, \quad a_{3;11} > 0, \ldots, \\ c_{0;22} > 0, \quad c_{1;22} > 0, \quad c_{2;22} > 0, \quad c_{3;22} > 0, \ldots. \end{gathered} \tag{5}$$

Now it is necessary to cite [16] (Lemma 5).

Denote by $\mathbf{l}_{\text{fin}}$ the linear subset of finite vectors in $\mathbf{l}_2$ (i.e., vectors that have only finite number of non-zero coordinates) and by $\mathbf{l}'_{\text{fin}}$ denote the corresponding conjugate space.

Lemma 1. *Let $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, $z \in \mathbb{T}$, be a fixed solution from $\mathbf{l}'_{\text{fin}}$ of the following system with boundary condition $\varphi_0(z) \equiv \varphi_0 \in \mathbb{C}$:*

$$\begin{gathered} (J\varphi(z))_n = a_{n-1}\varphi_{n-1}(z) + b_n\varphi_n(z) + c_n\varphi_{n+1}(z) = z\varphi_n(z), \\ (J^+\varphi(z))_n = c^*_{n-1}\varphi_{n-1}(z) + b^*_n\varphi_n(z) + a^*_n\varphi_{n+1}(z) = \bar{z}\varphi_n(z), \\ n \in \mathbb{N}_0, \quad \varphi_{-1}(z) \equiv 0. \end{gathered} \tag{6}$$

Then this solution exists for all φ_0 and has the form for all $n \in \mathbb{N}$

$$\varphi_n(z) = Q_n(z)\varphi_0 = \big(Q_{n;1}(z),\ Q_{n;2}(z)\big)^T \varphi_0,$$

where $Q_{n;1}$ and $Q_{n;2}$ are polynomials in z and $\bar{z}$ of the following form:

$$Q_{n;1}(z) = l_{n;1}\bar{z}^n + q_{n;1}(z), \qquad Q_{n;2}(z) = l_{n;2}z^n + q_{n;2}(z).$$

Here $l_{n;1} > 0$, $l_{n;2} > 0$, $Q_0(z) \equiv 1$ and $q_{n;1}(z)$, $q_{n;2}(z)$ are linear combinations of 1, z, $\bar{z}, \ldots,$ z^{n-1}, $\bar{z}^{n-1}$ for $q_{n;1}(z)$ and 1, z, $\bar{z}, \ldots,$ z^{n-1}, $\bar{z}^{n-1}$, $\bar{z}^n$ for $q_{n;2}(z)$.

This lemma will be very important for us a bit later: this is conceptual point that glues difference-differential lattices of equations with spectral theory technique. Now we follow [16] (Theorem 2). Take any complex number $\varphi_0 \in \mathbb{C}$ and build the solution $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$ of equations (6) using Lemma 1 [16] (Theorem 2) says that the mapping

$$\begin{aligned} \widehat{\ } : \mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} &\longrightarrow L^2(\mathbb{T}, d\rho(z)), \\ f = (f_n)_{n=0}^{\infty} &\longmapsto f_0 + \sum_{n=1}^{\infty} \left(\overline{Q_{n;1}(z)} f_{n;1} + \overline{Q_{n;2}(z)} f_{n;2} \right) \end{aligned}$$

after closure by continuity is unitary mapping between $\mathbf{l}_2$ and $L^2(\mathbb{T}, d\rho(z))$. Here ρ is probability spectral measure of $\boldsymbol{J}$. Moreover the following Parseval equality takes place: $\forall f,\, g \in \mathbf{l}_{\text{fin}}$

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{T}} \hat{f}(z) \cdot \overline{\hat{g}(z)} d\rho(z), \qquad (\boldsymbol{J} f, g)_{\mathbf{l}_2} = \int_{\mathbb{T}} z \cdot \hat{f}(z) \cdot \overline{\hat{g}(z)} d\rho(z).$$

Explicit substitution of the elements of *standard orthonormal basis* (3) reveals that Fourier transform $\widehat{\ }$ maps them to $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$ and $Q_{0,\alpha}(z) \equiv 1$. The last statement of [16] (Theorem 2) claims that $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$ and $Q_{0,\alpha}(z) \equiv 1$ constitute orthonormal basis of $L^2(\mathbb{T}, d\rho)$. Thus we can jump to two conclusions: ***first*** under Fourier transform operator $\boldsymbol{J}$ maps to operator L of *multiplication by independent variable* in $L^2(\mathbb{T}, d\rho)$:

$$\begin{array}{ccc} \mathbf{l}_2 & \xrightarrow{\ \boldsymbol{J}\ } & \mathbf{l}_2 \\ {\scriptstyle \widehat{}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{}} \\ L^2(\mathbb{T}, d\rho) & \xrightarrow[\ L\ ]{} & L^2(\mathbb{T}, d\rho) \end{array}$$

and ***second***: matrices of $\boldsymbol{J}$ (in *standard orthonormal basis*) and L (in $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$ and $Q_{0,\alpha}(z) \equiv 1$) coincide: they both are equal to J. Second conclusion is of great importance for us. It says that there will be no need to make any changes to the coefficients of difference-differential equations while passing from initial task formulation (that is being performed in $\mathbf{l}_2$) to the space $L^2(\mathbb{T}, d\rho)$ where it can be solved by using the explicit sense of L (see first conclusion).

Mentioned above [16] (Theorem 2) solves the Direct-Spectral-Problem. The corollary of [16] (Theorem 2) (see the same page: corollary is unnumbered) solves Inverse-Spectral-Problem. It says that if we apply [16] (Theorem 1) to measure $d\rho$ then we reconstruct the original matrix J.

Lemma 1 ([16], Lemma 5) gives an interesting result (see remarks in the proof of ISP mentioned above in [16], Theorems 1, 2). Polynomials $\overline{Q_{n;\alpha}(z)}, n \in \mathbb{N}, \alpha = 1, 2$ and $Q_{0,\alpha}(z) \equiv 1$ can be constructed in the same manner as orthonormal basis was built in the same situation in [1]. This is particular case of more common construction (see [17]) and that's the way how [16] (Theorem 1) is being proved. We give only the necessary brief sketch of this construction.

Denote by $\mathfrak{M}$ the set of probability Borel measures on the unit circle $\mathbb{T}$ with infinite support. Take a measure $\rho \in \mathfrak{M}$. Functions

$$1, \quad z, \quad \bar{z} = \frac{1}{z}, \quad z^2, \quad \bar{z}^2 = \frac{1}{z^2}, \ \dots \tag{7}$$

are linearly independent in the space $L^2(\mathbb{T}, d\rho)$. Denote moments of ρ by

$$t_n = \int\limits_{\mathbb{T}} z^n d\rho(z), \quad n \in \mathbb{Z}. \tag{8}$$

By using standard Gramm – Schmidt orthogonalization procedure construct the following orthonormal basis of the space $L^2(\mathbb{T}, d\rho)$:

$$P_0(z) \equiv 1,$$

$$P_{1,1}(z) = \frac{z - (z, P_0(z))_{L^2} P_0(z)}{\big\|z - (z, P_0(z))_{L^2} P_0(z)\big\|_{L^2}} = \frac{z - \int_{\mathbb{T}} z d\rho(z)}{1/k_{1,1}} = k_{1,1}(z - t_1), \quad k_{1,1} > 0,$$

$$P_{1,2}(z) = \frac{\frac{1}{z} - \left(\frac{1}{z}, P_0(z)\right)_{L^2} P_0(z) - \left(\frac{1}{z}, P_{1,1}(z)\right)_{L^2} P_{1,1}(z)}{\left\|\frac{1}{z} - \left(\frac{1}{z}, P_0(z)\right)_{L^2} P_0(z) - \left(\frac{1}{z}, P_{1,1}(z)\right)_{L^2} P_{1,1}(z)\right\|_{L^2}} = \tag{9}$$

$$= k_{1,2}\frac{1}{z} + \big(-t_{-1}k_{1,2} + k_{1,1}^2 k_{1,2} t_1 \big(t_{-2} - \overline{t_1} t_{-1}\big)\big) - k_{1,1}^2 k_{1,2} \left(t_{-2} - \overline{t_1} t_{-1}\right) z;$$

$$P_{n,1}(z) = k_{n,1} z^n + \dots, \quad k_{n,1} > 0,$$

$$P_{n,2}(z) = k_{n,2} z^{-n} + \dots, \quad k_{n,2} > 0.$$

The fact is that in this way we actually obtain basis elements $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$, and $Q_{0,\alpha}(z) \equiv 1$. According to [16] (proof of Corollary from Theorem 2) the following equality holds:

$$Q_0(z) \equiv 1 = P_0(z), \qquad \overline{Q_{n;1}(z)} = P_{n;1}(z), \qquad \overline{Q_{n;2}(z)} = P_{n;2}(z).$$

It is worth noting that P_0, $P_{n;\alpha}$, $n \in \mathbb{N}_0$, $\alpha = 1, 2$, are the same polynomials as χ_n, $n \in \mathbb{N}_0$, in [3, p. 442].

Finally from (6) we have the following system of equations:

$$\begin{aligned} a_{n-1}\overline{P_{n-1}(z)} + b_n \overline{P_n(z)} + c_n \overline{P_{n+1}(z)} &= z\overline{P_n(z)}, \\ c^*_{n-1}\overline{P_{n-1}(z)} + b^*_n \overline{P_n(z)} + a^*_n \overline{P_{n+1}(z)} &= \bar{z}\overline{P_n(z)}. \end{aligned} \tag{10}$$

In coordinate form it has the following view:

$$\begin{pmatrix} \overline{a_{n-1,11}} & \overline{a_{n-1,12}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_{n-1,1} \\ P_{n-1,2} \end{pmatrix} + \begin{pmatrix} \overline{b_{n,11}} & \overline{b_{n,12}} \\ \overline{b_{n,21}} & \overline{b_{n,22}} \end{pmatrix} \begin{pmatrix} P_{n,1} \\ P_{n,2} \end{pmatrix} +$$

$$+\begin{pmatrix} 0 & 0 \\ \overline{c_{n,21}} & \end{pmatrix}\overline{c_{n,22}}\begin{pmatrix} P_{n+1,1} \\ P_{n+1,2} \end{pmatrix} = \overline{z}\begin{pmatrix} P_{n,1} \\ P_{n,2} \end{pmatrix},$$

$$\begin{pmatrix} 0 & c_{n-1,21} \\ 0 & c_{n-1,22} \end{pmatrix}\begin{pmatrix} P_{n-1,1} \\ P_{n-1,2} \end{pmatrix} + \begin{pmatrix} b_{n,11} & b_{n,21} \\ b_{n,12} & b_{n,22} \end{pmatrix}\begin{pmatrix} P_{n,1} \\ P_{n,2} \end{pmatrix} +$$

$$+\begin{pmatrix} a_{n,11} & 0 \\ a_{n,12} & 0 \end{pmatrix}\begin{pmatrix} P_{n+1,1} \\ P_{n+1,2} \end{pmatrix} = z\begin{pmatrix} P_{n,1} \\ P_{n,2} \end{pmatrix}.$$

This result will play significant role for us. It gives explicit formulae for entries of multiplication operators by z (operator L) and by $\bar{z} = \dfrac{1}{z}$ (operator $L^* = L^{-1}$). It is worth noting that these equations are equivalent to Szegő recursion (this question is not considered in this article). For now it is sufficient to note that described above ideas give the possibility to establish a connection with OPUC theory. To make the first step towards OPUC it is necessary to rewrite entries of J in terms of Verblunsky coefficients using [1].

The last remark touches the block structure of the spaces and operators used in this article. Space $\mathbf{l}_2$ is built as block space from the most start. Its image under Fourier transform $L^2(\mathbb{T}; d\rho)$ does not have any block structure. To be accurate it is necessary to show the image of each cell of $\mathbf{l}_2$. This can be done fairly easily.

Introduce the spaces $\mathcal{P}_0 = \operatorname{span}\{1\} = \mathbb{C}$, $\mathcal{P}_{n,1} = \operatorname{span}\big\{1, z, \bar{z}, \ldots, z^{(n-1)}, z^{-(n-1)}, z^n\big\}$, $\mathcal{P}_{n,2} = \operatorname{span}\big\{1, z, \bar{z}, \ldots, z^{(n-1)}, z^{-(n-1)}, z^n, z^{-n}\big\}$. It is obvious by construction of elements of orthonormal basis that

$$\mathcal{P}_{n,1} = \{P_0\} \oplus \{P_{1,1}\} \oplus \{P_{1,2}\} \oplus \ldots \oplus \{P_{n-1,1}\} \oplus \{P_{n-1,2}\} \oplus \{P_{n,1}\},$$
$$\mathcal{P}_{n,2} = \mathcal{P}_{n,1} \oplus \{P_{n,2}\}. \tag{11}$$

It is quite natural to combine pairs of one-dimensional subspaces in (11). Unite each pair $P_{n,1}, P_{n,2}$ and construct the vector $P_n(z) = \begin{pmatrix} P_{n,1}(z) \\ P_{n,2}(z) \end{pmatrix}$.

Denote $P(z) = \big(P_n(z)\big)_{n=0}^{\infty}$. Final answer is as follows:

$$\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty} \mapsto \hat{f}(z) = \sum_{n=0}^{\infty} (f_n, P_n(z))_{\mathcal{H}_n} \in L^2(\mathbb{T}, d\rho).$$

3. Main result. Now let us pass to the central result of the article. Consider the following polynomials in λ:

$$\Phi(\lambda, t) = \sum_{j=-l}^{l} \varphi_j(t)\lambda^j, \qquad \varphi_j(t) \in C^1_{[0,\infty)\to\mathbb{C}}, \quad \lambda \in \mathbb{T}, \tag{12}$$

$$\Psi(\lambda, t) = \sum_{j=-m}^{m} \psi_j(t)\lambda^j, \qquad \psi_j(t) \in C^1_{[0,\infty)\to\mathbb{C}}, \quad \lambda \in \mathbb{T}. \tag{13}$$

Denote by D — differential operator $\dfrac{\partial}{\partial\lambda}$ and consider operators:

$$\Omega = \Phi(L(t),t)D, \quad \widehat{\Omega} = \Phi(L^*(t),t)D, \quad \Psi = \Psi(L(t),t), \quad \Xi = -\Omega - \widehat{\Omega}^* - \Psi,$$

$$I = \left(\begin{array}{c|cc|cc} \frac{1}{2}\Xi_{0,1;0,1} & \Xi_{0,1;1,1} & \Xi_{0,1;1,2} & \Xi_{0,1;2,1} & \Xi_{0,1;2,2} \\ \hline 0 & \frac{1}{2}\Xi_{1,1;1,1} & \Xi_{1,1;1,2} & \Xi_{1,1;2,1} & \Xi_{1,1;2,2} \\ 0 & 0 & \frac{1}{2}\Xi_{1,2;1,2} & \Xi_{1,2;2,1} & \Xi_{1,2;2,2} \\ \hline 0 & 0 & 0 & \frac{1}{2}\Xi_{2,1;2,1} & \Xi_{2,1;2,2} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Xi_{2,2;2,2} \end{array}\right). \tag{14}$$

Consider the following differential equation:

$$\frac{d}{dt}L(t) = \Phi(L(t),t) + \left[L(t), \Omega + I + \frac{1}{2}\Psi\right]. \tag{15}$$

Here $[A,B] = AB - BA$. This Lax equation is equivalent to the following differential-difference chain of equations in matrix-variables a_n, b_n, c_n:

$$\dot{a}_n(t) = (a_n \cdot \Omega_{n,n} + b_{n+1} \cdot \Omega_{n+1,n}) +$$

$$+(\widehat{\Omega}^*_{n-1,n+1} \cdot c_{n-1} + \widehat{\Omega}^*_{n,n+1} \cdot b_n + \widehat{\Omega}^*_{n+1,n+1} \cdot a_n) +$$

$$+ \left[a_n \cdot \begin{pmatrix} \frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2} \\ 0 & \frac{1}{2}\Xi_{n,2;n,2} \end{pmatrix}\right] +$$

$$+ \Bigg[\left(\Xi^*_{n-1,n+1} \cdot c_{n-1} + \Xi^*_{n,n+1} \cdot b_n + \Xi^*_{n+1,n+1} \cdot a_n\right) -$$

$$- \frac{1}{2}\begin{pmatrix} \Xi_{n+1,1;n+1,1} & 0 \\ 0 & \Xi_{n+1,2;n+1,2} \end{pmatrix} \cdot a_n\Bigg] +$$

$$+ \left(a_n \cdot \Psi_{n,n} + b_{n+1} \cdot \Psi_{n+1,n}\right) + \Phi_{n+1,n},$$

$$\dot{b}_n(t) = (a_{n-1} \cdot \Omega_{n-1,n} + b_n \cdot \Omega_{n,n} + c_n \cdot \Omega_{n+1,n}) +$$

$$+(\widehat{\Omega}^*_{n-1,n} \cdot c_{n-1} + \widehat{\Omega}^*_{n,n} \cdot b_n + \widehat{\Omega}^*_{n+1,n} \cdot a_n) +$$

$$+ \left[a_{n-1} \cdot \{-\Omega - \widehat{\Omega}^* - \Psi\}_{n-1,n} + b_n \cdot \begin{pmatrix} \frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2} \\ 0 & \frac{1}{2}\Xi_{n,2;n,2} \end{pmatrix}\right] =$$

$$+ \left[\{-\Omega^* - \widehat{\Omega} - \Psi^*\}_{n-1,n} \cdot c_{n-1} + \begin{pmatrix} \frac{1}{2}\Xi_{n,1;n,1} & 0 \\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2} \end{pmatrix} \cdot b_n\right] +$$

$$+(a_{n-1}\cdot\Psi_{n-1,n}+b_n\cdot\Psi_{n,n}+c_n\cdot\Psi_{n+1,n})+\Phi_{n,n}, \tag{16}$$

$$\dot c_n(t)=(a_{n-1}\cdot\Omega_{n-1,n+1}+b_n\cdot\Omega_{n,n+1}+c_n\cdot\Omega_{n+1,n+1})+$$
$$+(\widehat{\Omega}^*_{n,n}\cdot c_n+\widehat{\Omega}^*_{n+1,n}\cdot b_{n+1}+\widehat{\Omega}^*_{n+2,n}\cdot a_{n+1})+$$
$$+\Bigg[(a_{n-1}\cdot\Xi_{n-1,n+1}+b_n\cdot\Xi_{n,n+1}+c_n\cdot\Xi_{n+1,n+1})-$$
$$-\frac{1}{2}c_n\cdot\begin{pmatrix}\Xi_{n+1,1;n+1,1} & 0\\ 0 & \Xi_{n+1,2;n+1,2}\end{pmatrix}\Bigg]+$$
$$+\left[\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot c_n\right]+$$
$$+\ (a_{n-1}\cdot\Psi_{n-1,n+1}+b_n\cdot\Psi_{n,n+1}+c_n\cdot\Psi_{n+1,n+1})+\Phi_{n,n+1}.$$

The Cauchy problem for the differential equation (15) can be stated as follows.

Suppose we have bounded unitary block Jacobi matrix L_0 *with entries* $a_{n;11}>0$, $c_{n,22}>0$. *Find* $L(t)$, $t\in[0;T]$, *with continuously differentiable entries such that:* $L(t)$ *is a solution of* (15) *for* $t\in[0,T]$ *where* T *depends only on initial condition* L_0 *and functions* Φ, Ψ (*see* (12), (13)) *and*

$$L(0)=L_0. \tag{17}$$

Here we introduce the following algorithm that solves the described above Cauchy problem.

Algorithm. Let $\rho(\cdot,0)$ be the spectral measure of the Jacobi matrix L_0. It is built using Direct-Spectral-Problem discussed in Section 2. Denote by $M=\operatorname{supp}\rho(\cdot,0)\subset\mathbb{T}$. Consider the Cauchy problem

$$\frac{d\lambda(t)}{dt}=\Phi(\lambda(t),t),\quad \lambda(0)=\mu,\quad \mu\in M,\quad t\ge 0. \tag{18}$$

From the standard theory of differential equations it is well known that one can choose $T>0$ such that for every $\mu\in M$ there exists unique solution $\lambda(\cdot,\mu)$ of the Cauchy problem (18) defined on the interval $[0,T]$. We suppose that polynomial $\Phi(\lambda,t)$ is such that $|\lambda(t)|=1\ \forall t\in[0;T]$.

For every fixed $t\in[0,T]$ consider the mapping

$$\begin{aligned}\omega_t: M &\longrightarrow \mathbb{T},\\ \mu &\longmapsto \lambda(t,\mu)\end{aligned} \tag{19}$$

and construct the following measure (mapping step):

$$\tilde{\rho}(\Delta, t) = \rho(\omega_t^{-1}(\Delta), 0), \quad \Delta \in \mathfrak{B}(\mathbb{T}). \tag{20}$$

Here $\omega_t^{-1}(\Delta)$ is full preimage of the set Δ under the mapping ω_t. Let us consider the following partial differential equation:

$$\frac{\partial s(\lambda,t)}{\partial \lambda}\Phi(\lambda,t) + \frac{\partial s(\lambda,t)}{\partial t} = \Psi(\lambda,t)s(\lambda,t), \qquad s(\lambda,0) = 1, \quad \lambda \in \mathbb{T}, \quad t \geq 0. \tag{21}$$

Let $s(\lambda, t)$ be its nonnegative solution. Build the final measure tranformation (multiplication step):

$$\rho(\Delta, t) = \int_{\Delta} s(\lambda, t) d\tilde{\rho}(\lambda, t), \quad \Delta \in \mathfrak{B}(\mathbb{T}). \tag{22}$$

The last step is to reconstruct $L(t)$ from its spectral measure $\rho(\cdot, t), t \in [0, T]$ by solving the Inverse-Spectral-Problem discussed in Section 2. Briefly recall the corresponding algorithm.

Consider the following family of functions:

$$1, \quad z, \quad \bar{z} = \frac{1}{z}, \quad z^2, \quad \bar{z}^2 = \frac{1}{z^2}, \; \ldots . \tag{23}$$

Build orthonormal basis $P_0(z,t)$, $P_{1,1}(z,t)$, $P_{1,2}(z,t)$, $P_{2,1}(z,t)$, $P_{2,2}(z,t), \ldots$ of the space $L^2(\mathbb{T}, d\rho(\cdot, t))$ (using standard Schmidt orthogonalization procedure). $L(t)$ is operator of multiplication by independent variable in the space $L^2(\mathbb{T}, d\rho(\cdot, t))$. Thus its entries (that are the desired solution) can be found as:

$$L_{j,\alpha;k,\beta}(t) = \int_{\mathbb{T}} \lambda P_{k,\beta}(\lambda, t)\overline{P_{j,\alpha}(\lambda, t)} d\rho(\lambda, t).$$

Theorem 1. *A solution of the Cauchy problem* (15), (17) *exists and can be found using the described above algorithm.*

4. Proof. Here we restrict ourselves with existence theorem only. Uniqueness theorem can be found in [11]. The aim now is to prove that if we take the described above measure transformation then operator of multiplication by independent variable satisfies differential equation (15) with initial condition (17).

Let $F(\lambda, t) \in C^1(\mathbb{T} \times [0, T] \to \mathbb{C})$ and consider the following function:

$$f(t) = \int_{\mathbb{T}} F(\lambda, t) d\rho(\lambda, t) = \int_{\mathbb{T}} F(\lambda, t) s(\lambda, t) d\tilde{\rho}(\lambda, t) =$$

$$= \int_{\omega_t^{-1}(\mathbb{T})} F(\omega_t(\mu), t) s(\omega_t(\mu), t) d\rho(\mu, 0) =$$

$$= \int_{\mathbb{T}} F(\lambda(t, \mu), t) s(\lambda(t, \mu), t) d\rho(\mu, 0). \tag{24}$$

Using (18) and (21) obtain the formula for df/dt:

$$\frac{df}{dt} = \int\limits_{\mathbb{T}} \left\{ \left(\frac{\partial F(\lambda(t,\mu),t)}{\partial \lambda} \frac{\partial \lambda(t,\mu)}{\partial t} + \frac{\partial F(\lambda(t,\mu),t)}{\partial t} \right) s(\lambda(t,\mu),t) + \right.$$

$$\left. + \left(\frac{\partial s(\lambda(t,\mu),t)}{\partial \lambda} \frac{\partial \lambda(t,\mu)}{\partial t} + \frac{\partial s(\lambda(t,\mu),t)}{\partial t} \right) F(\lambda(t,\mu),t) \right\} d\rho(\mu,0) =$$

$$= \int\limits_{\mathbb{T}} \left\{ \left(\frac{\partial F(\lambda(t,\mu),t)}{\partial \lambda} \cdot \Phi(\lambda(t,\mu),t) + \frac{\partial F(\lambda(t,\mu),t)}{\partial t} \right) s(\lambda(t,\mu),t) + \right.$$

$$\left. + \left(\frac{\partial s(\lambda(t,\mu),t)}{\partial \lambda} \cdot \Phi(\lambda(t,\mu),t) + \frac{\partial s(\lambda(t,\mu),t)}{\partial t} \right) F(\lambda(t,\mu),t) \right\} d\rho(\mu,0) =$$

$$= \int\limits_{\mathbb{T}} \left\{ \left(\frac{\partial F(\lambda(t,\mu),t)}{\partial \lambda} \cdot \Phi(\lambda(t,\mu),t) + \frac{\partial F(\lambda(t,\mu),t)}{\partial t} \right) s(\lambda(t,\mu),t) + \right.$$

$$\left. + \Psi(\lambda(t,\mu),t) s(\lambda(t,\mu),t) F(\lambda(t,\mu),t) \right\} d\rho(\mu,0) =$$

$$= \int\limits_{\mathbb{T}} \left\{ \frac{\partial F(\lambda,t)}{\partial \lambda} \cdot \Phi(\lambda,t) + \frac{\partial F(\lambda,t)}{\partial t} + \Psi(\lambda,t) F(\lambda,t) \right\} d\rho(\lambda,t).$$

Final result:

$$\frac{d}{dt} \int\limits_{\mathbb{T}} F(\lambda,t) d\rho(\lambda,t) =$$

$$= \int\limits_{\mathbb{T}} \left\{ \frac{\partial F(\lambda,t)}{\partial \lambda} \cdot \Phi(\lambda,t) + \frac{\partial F(\lambda,t)}{\partial t} + \Psi(\lambda,t) F(\lambda,t) \right\} d\rho(\lambda,t). \tag{25}$$

The next step is to take left-hand side of (15) and obtain its right-hand side. So it is necessary to compute $\frac{d}{dt} L_{j,\alpha;k,\beta}(t)$.

Let $P_0(\cdot,t), P_{1,1}(\cdot,t), P_{1,2}(\cdot,t), \ldots$ be the elements of orthonormal basis of the space $L^2(\mathbb{T}, d\rho(\cdot,t))$ according to (9). Consider two operators: operator of multiplication and differentiation operator in $L^2\big(\mathbb{T}, d\rho(\cdot,t)\big)$:

$$\begin{aligned} L(t) : L^2(\mathbb{T}, \rho(\cdot,t)) &\longrightarrow L^2(\mathbb{T}, \rho(\cdot,t)), \\ f(z) &\longmapsto z \cdot f(z), \end{aligned} \tag{26}$$

$$\begin{aligned} D(t) : C^\infty &\longrightarrow L^2(\mathbb{T}, \rho(\cdot,t)), \\ f(z,t) &\longmapsto \frac{df(z,t)}{dz}. \end{aligned} \tag{27}$$

By applying (25) with $F(\lambda,t) = \lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}$ one can find the expression for derivative of the coordinate $L_{j,\alpha;k,\beta}$:

$$\frac{d}{dt}L_{j,\alpha;k,\beta}(t) =$$

$$= \int\limits_{\mathbb{T}} \left\{ \frac{\partial}{\partial\lambda} \left(\lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) \cdot \Phi(\lambda,t) + \frac{\partial}{\partial t} \left(\lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) + \right.$$

$$\left. + \Psi(\lambda,t) \left(\lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) \right\} d\rho(\lambda,t) \stackrel{\text{df}}{=} J_1 + J_2 + J_3. \tag{28}$$

Simplify each components $J_1,\ J_2,\ J_3$ one-by-one:

$$J_3 = \big\{L(t)\Psi(L(t),t)\big\}_{j,\alpha;k,\beta} = \{L\Psi\}_{j,\alpha;k,\beta}, \tag{29}$$

$$J_1 = \int\limits_{\mathbb{T}} \frac{\partial}{\partial\lambda} \left(\lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) \cdot \Phi(\lambda,t) d\rho(\lambda,t) =$$

$$= \int\limits_{\mathbb{T}} P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\Phi(\lambda,t)d\rho(\lambda,t) +$$

$$+ \int\limits_{\mathbb{T}} \lambda \frac{\partial P_{k,\beta}(\lambda,t)}{\partial\lambda}\overline{P_{j,\alpha}(\lambda,t)}\Phi(\lambda,t)d\rho(\lambda,t) +$$

$$+ \int\limits_{\mathbb{T}} \lambda P_{k,\beta}(\lambda,t)\overline{\frac{\partial P_{j,\alpha}(\lambda,t)}{\partial\lambda}} \cdot \Phi(\lambda,t)d\rho(\lambda,t) =$$

$$= \Big\{\Phi(L(t),t) + L(t)\Phi(L(t),t)D(t) + D^*(t)\Phi(L(t),t)L(t)\Big\}_{j,\alpha;k,\beta} =$$

$$= \big\{\Phi + L\Omega + \widehat{\Omega}^* L\big\}_{j,\alpha;k,\beta}. \tag{30}$$

Here $\Omega = \Phi(L(t),t)D(t)$, $\widehat{\Omega} = \Phi(L^*(t),t)D(t)$.

We used the following facts:

$$\overline{\Phi(\lambda,t)} = \overline{\sum_{j=-l}^{l} \varphi_j(t)\lambda^j} = \sum_{j=-l}^{l} \varphi_j(t)\bar{\lambda}^j = \sum_{j=-l}^{l} \varphi_j(t)\frac{1}{\lambda^j} = \Phi\left(\frac{1}{\lambda},t\right).$$

The second fact is that $L(t)$ is unitary operator in $L(\mathbb{T}, d\rho(\cdot,t))$, so $L^*(t) = L^{-1}(t)$. And inverse operator $L^{-1}(t)$ is obviously the operator of multiplication by $1/\lambda$:

$$J_2 = \int\limits_{\mathbb{T}} \frac{\partial}{\partial t} \left(\lambda P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) d\rho(\lambda,t) =$$

$$= \int\limits_{\mathbb{T}} \lambda \frac{\partial P_{k,\beta}(\lambda,t)}{\partial t}\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) + \int\limits_{\mathbb{T}} \lambda P_{k,\beta}(\lambda,t)\overline{\frac{\partial P_{j,\alpha}(\lambda,t)}{\partial t}}d\rho(\lambda,t) =$$

$$= \left(\frac{\partial P_{k,\beta}}{\partial t}; \frac{1}{\lambda} P_{j,\alpha} \right)_{L^2} + \left(\lambda P_{k,\beta}; \frac{\partial P_{j,\alpha}}{\partial t} \right)_{L^2}.$$

Denote by

$$\begin{aligned} I : C^\infty &\longrightarrow L^2(\mathbb{T}, \rho(\cdot, t)), \\ f(z,t) &\longmapsto \frac{df(z,t)}{dt}. \end{aligned} \tag{31}$$

Operator J_2 can be represented as:

$$J_2 = (LIP_{k,\beta}, P_{j,\alpha})_{L^2} + (I^* L P_{k,\beta}, P_{j,\alpha})_{L_2}. \tag{32}$$

Thus

$$\{J_2\}_{j,\alpha;k,\beta} = \{LI + I^* L\}_{j,\alpha;k,\beta}. \tag{33}$$

Finally we have the following expression:

$$\frac{d}{dt} L = \Phi + L\Omega + \widehat{\Omega}^* L + LI + I^* L + L\Psi. \tag{34}$$

To obtain from this formula (15) it is necessary to express I through Ω and Ψ. We shall synchronize our notations and proof with the one from [11] (Lemma 2). Note that in this article operator $\boldsymbol{J}$ is self-adjoint and the main space is ordinary ℓ_2. We shall use block-matrix ideology in our case to make the reasonings as close to [11] as possible.

Lemma 2 [analogue of [11], Lemma 2]**.** *Denote by*:

$$I_{j,\alpha;k,\beta} = \left(\frac{\partial P_{k,\beta}}{\partial t}; P_{j,\alpha} \right)_{L^2} \tag{35}$$

and $\Xi = -\Omega - \widehat{\Omega}^* - \Psi$.

The following matrix equalities hold:

1) $I_{j;k} = 0$ *for* $j > k$;

2) $I_{k;k} = \begin{pmatrix} \frac{1}{2}\Xi_{k,1;k,1} & \Xi_{k,1;k,2} \\ 0 & \frac{1}{2}\Xi_{k,2;k,2} \end{pmatrix}$, $k > 0$, $I_{0;0} = \frac{1}{2}\Xi_{0;0}$;

3) $I_{j;k} = \Xi_{j;k}$ *for* $j < k$.

Note that in [11] $a_k = c_k$, $\Theta = \Psi$, $\Omega^* = \widehat{\Omega}$.

Proof. Since $\dfrac{\partial P_{k,\beta}}{\partial t} \in \mathcal{P}_{k,\beta}$ it is obvious that

$$I_{j,\alpha;k,\beta} = 0 \quad \text{for} \quad j > k \qquad \text{and for} \quad j = k,\ \alpha = 2,\ \beta = 1. \tag{36}$$

All the other cases must be examined in the following specific way. We shall differentiate the equality

$$\int_{\mathbb{T}} P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) = \delta_{k,j}\delta_{\alpha,\beta}.$$

To do this we use formula (25) where $F(\lambda,t) = P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}$:

$$0 = \frac{d}{dt}\int_{\mathbb{T}} P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) =$$

$$= \int_{\mathbb{T}} \left\{ \frac{\partial}{\partial\lambda}\left(P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right)\cdot\Phi(\lambda,t) + \frac{\partial}{\partial t}\left(P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right) +\right.$$

$$\left. +\Psi(\lambda,t)\left(P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}\right)\right\}d\rho(\lambda,t) =$$

$$= \int_{\mathbb{T}} \Phi(\lambda,t)\frac{\partial}{\partial\lambda}P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) +$$

$$+ \int_{\mathbb{T}} \Phi(\lambda,t)P_{k,\beta}(\lambda,t)\overline{\frac{\partial}{\partial\lambda}P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) +$$

$$+ \int_{\mathbb{T}} \frac{\partial}{\partial t}P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) + \int_{\mathbb{T}} P_{k,\beta}(\lambda,t)\overline{\frac{\partial}{\partial t}P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) +$$

$$+ \int_{\mathbb{T}} \Psi(\lambda,t)P_{k,\beta}(\lambda,t)\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) =$$

$$= \big\{\Phi(L(t),t)D(t)\big\}_{j,\alpha;k,\beta} + \overline{\big\{\Phi(L^*(t),t)D(t)\big\}_{k,\beta;j,\alpha}} +$$

$$+ I_{j,\alpha;k,\beta} + \overline{I_{k,\beta;j,\alpha}} + \big\{\Psi(L(t),t)\big\}_{j,\alpha;k,\beta}.$$

Thus the following equality takes place:

$$I_{j,\alpha;k,\beta} + \overline{I_{k,\beta;j,\alpha}} =$$

$$= -\left[\{\Phi(L(t),t)D(t)\}_{j,\alpha;k,\beta} + \overline{\{\Phi(L^*(t),t)D(t)\}_{k,\beta;j,\alpha}} + \{\Psi(L(t),t)\}_{j,\alpha;k,\beta}\right].$$

Thus we have

$$I + I^* = -\Omega - \widehat{\Omega}^* - \Psi. \tag{37}$$

If $j < k$ or $(j = k,\ \beta = 2,\ \alpha = 1)$ then $I_{k,\beta;j,\alpha} = 0$ and we obtain

$$I_{j,\alpha;k,\beta} = -\left[\{\Phi(L(t),t)D(t)\}_{j,\alpha;k,\beta} + \overline{\{\Phi(L^*(t),t)D(t)\}_{k,\beta;j,\alpha}} + \{\Psi(L(t),t)\}_{j,\alpha;k,\beta}\right].$$

The last option left is $j = k$; $\alpha = \beta$. First let us show that $I_{j,\alpha;j,\alpha} \in \mathbb{R}$.

Let $\alpha = 1$ (option $\alpha = 2$ is analyzed in the same manner). As we saw in (9) $P_{j,1} = k_{j,1}\lambda^j + \ldots$ where $k_{j,1} > 0$. It is obvious that $\dfrac{\partial}{\partial t}P_{j,1}(\lambda,t) = c_0P_0 + c_{1,1}P_{1,1} + c_{1,2}P_{1,2} + \ldots + c_{j,1}P_{j,1}$. That's why $\left(\dfrac{\partial}{\partial t}P_{j,1}(\cdot,t),\ P_{j,1}(\cdot,t)\right)_{L^2} = c_{j,1}$. Compare coefficients at λ^j. We have $\dfrac{dk_{j,1}(t)}{dt} = c_{j,1}(t)k_{j,1}(t)$. Here $k_{j,1}(t)$ is real-valued non-zero function, so $c_{j,1}(t)$ is real-valued too. So $I_{j,\alpha;k,\beta} + \overline{I_{k,\beta;j,\alpha}} = 2 \cdot I_{j,\alpha;k,\beta}$, $j = k, \alpha = \beta$.

Finally we obtain the following expression for $I_{j,\alpha;j,\alpha}$, $j \in \mathbb{N}_0$, $\alpha = 1, 2$:

$$I_{j,\alpha;j,\alpha} =$$

$$= -\frac{1}{2}\left[\left\{\Phi(L(t),t)D(t)\right\}_{j,\alpha;j,\alpha} + \overline{\left\{\Phi(L^*(t),t)D(t)\right\}_{j,\alpha;j,\alpha}} + \left\{\Psi(L(t),t)\right\}_{j,\alpha;j,\alpha}\right].$$

Thus we have the following view of I:

$$I = \left(\begin{array}{c|cc|cc} \frac{1}{2}\Xi_{0,1;0,1} & \Xi_{0,1;1,1} & \Xi_{0,1;1,2} & \Xi_{0,1;2,1} & \Xi_{0,1;2,2} \\ \hline 0 & \frac{1}{2}\Xi_{1,1;1,1} & \Xi_{1,1;1,2} & \Xi_{1,1;2,1} & \Xi_{1,1;2,2} \\ 0 & 0 & \frac{1}{2}\Xi_{1,2;1,2} & \Xi_{1,2;2,1} & \Xi_{1,2;2,2} \\ \hline 0 & 0 & 0 & \frac{1}{2}\Xi_{2,1;2,1} & \Xi_{2,1;2,2} \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Xi_{2,2;2,2} \end{array}\right). \qquad (38)$$

The lemma is proved.

Actually in this lemma the key role for us has formula (37). It allows us to finish the proof of (15). The following lemma makes it obvious.

Lemma 3. *The following equality takes place*:

$$\frac{d}{dt}L(t) = \Phi(L(t),t) + \left[L(t), \Omega + I + \frac{1}{2}\Psi\right]. \qquad (39)$$

Proof. Taking into account that $\lambda \cdot \Psi(\lambda,t) = \Psi(\lambda,t) \cdot \lambda \Rightarrow L\Psi = \Psi L$ we obtain

$$\frac{d}{dt}L = \Phi + L\Omega + \widehat{\Omega}^*L + LI + I^*L + L\Psi =$$

$$= \left(\Phi + L\Omega + \widehat{\Omega}^*L\right) + (LI + I^*L) + \left(\frac{1}{2}(L\Psi) + \frac{1}{2}(\Psi L)\right) =$$

$$= \Phi + L\left(\Omega + I + \frac{1}{2}\Psi\right) + \left(\widehat{\Omega}^* + I^* + \frac{1}{2}\Psi\right)L =$$

$$= \Phi + L\left(\Omega + I + \frac{1}{2}\Psi\right) + \left(\widehat{\Omega}^* + (-\Omega - \widehat{\Omega}^* - \Psi - I) + \frac{1}{2}\Psi\right)L =$$

$$= \Phi + L\left(\Omega + I + \frac{1}{2}\Psi\right) - \left(\Omega + I + \frac{1}{2}\Psi\right)L = \Phi + \left[L, \Omega + I + \frac{1}{2}\Psi\right].$$

For algebraic purposes this result is the most convenient one. Note that in [11] the same one-dimensional result was obtained in much more complicated and obscure way.

5. Difference-differential lattices. We follow [11] and give coordinate-wise interpretation of (15). This section is devoted to the proof of (16). It is important for numerical applications and also gives the possibility (by choosing appropriate Φ and Ψ) to obtain different matrix flows. This is obviously interesting in comparison with e.g. Schur flow (see [15]).

The idea is to establish a connection between difference-differential lattices and Lax equations (this section), Lax equations and spectral measures (previous section), spectral measures and block Jacobi matrices (see [16, 17]), block Jacobi matrices and OPUC theory (in particular with Verblunsky coefficients and their flows, Szegő recursion etc. — see further papers). First we need another lemma.

Lemma 4 [analogue of [11], Lemma 3]**.** *Denote by*

$$E_{j,\alpha;k,\beta} = \int\limits_{\mathbb{T}} \lambda P_{k,\beta}(\lambda,t)\overline{\frac{\partial P_{j,\alpha}(\lambda,t)}{\partial t}}d\rho(\lambda,t) = \Big(LP_{k,\beta}; IP_{j,\alpha}\Big)_{L_2},$$

$$\widehat{E}_{j,\alpha;k,\beta} = \int\limits_{\mathbb{T}} \lambda \frac{\partial P_{k,\beta}(\lambda,t)}{\partial t}\overline{P_{j,\alpha}(\lambda,t)}d\rho(\lambda,t) = \Big(LIP_{k,\beta}; P_{j,\alpha}\Big)_{L_2}.$$

The following equalities take place:

(a) $E_{j,k} = 0, \;\; \widehat{E}_{j,k} = \{L\Xi\}_{j,k}, \;\; j < k-1;$

(b) $E_{k-1,k} = \begin{pmatrix} \frac{1}{2}\Xi_{k-1,1;k-1,1} & 0 \\ \overline{\Xi_{k-1,1;k-1,2}} & \frac{1}{2}\Xi_{k-1,2;k-1,2} \end{pmatrix} \cdot c_{k-1},$

$\widehat{E}_{k-1,k} = \{L\cdot\Xi\}_{k-1,k} - \frac{1}{2}c_{k-1}\cdot \begin{pmatrix} \Xi_{k,1;k,1} & 0 \\ 0 & \Xi_{k,2;k,2} \end{pmatrix};$

(c) $E_{k,k} = \{\Xi^*\}_{k-1,k}\cdot c_{k-1} + \begin{pmatrix} \frac{1}{2}\Xi_{k,1;k,1} & 0 \\ \overline{\Xi_{k,1;k,2}} & \frac{1}{2}\Xi_{k,2;k,2} \end{pmatrix}\cdot b_k,$

$\widehat{E}_{k,k} = a_{k-1}\cdot\Xi_{k-1,k} + b_k \cdot \begin{pmatrix} \frac{1}{2}\Xi_{k,1;k,1} & \Xi_{k,1;k,2} \\ 0 & \frac{1}{2}\Xi_{k,2;k,2} \end{pmatrix};$

(d) $E_{k+1,k} = \{\Xi^*\cdot L\}_{k+1,k} - \frac{1}{2}\begin{pmatrix} \Xi_{k+1,1;k+1,1} & 0 \\ 0 & \Xi_{k+1,2;k+1,2} \end{pmatrix}\cdot a_k,$

$\widehat{E}_{k+1,k} = a_k \cdot \begin{pmatrix} \frac{1}{2}\Xi_{k,1;k,1} & \Xi_{k,1;k,2} \\ 0 & \frac{1}{2}\Xi_{k,2;k,2} \end{pmatrix};$

(e) $E_{j,k} = \{\Xi^*\cdot L\}_{j,k}, \;\; \widehat{E}_{j,k} = 0, \;\; j > k+1.$

Proof. J is the matrix of multiplication operator L. So the following holds:

$$E^*_{j;k} = \overline{\begin{pmatrix} zP_{k,1} \\ zP_{k,2} \end{pmatrix} \cdot_{L^2} \left(\frac{\partial P_{j,1}}{\partial t}; \frac{\partial P_{j,2}}{\partial t} \right)} =$$

$$= \overline{\left(\bar{c}^*_{k-1} \cdot \begin{pmatrix} P_{k-1,1} \\ P_{k-1,2} \end{pmatrix} + \bar{b}^*_k \cdot \begin{pmatrix} P_{k,1} \\ P_{k,2} \end{pmatrix} + \bar{a}^*_k \cdot \begin{pmatrix} P_{k+1,1} \\ P_{k+1,2} \end{pmatrix} \right) \cdot_{L^2} \left(\frac{\partial P_{j,1}}{\partial t}; \frac{\partial P_{j,2}}{\partial t} \right)} =$$

$$= c^*_{k-1} \cdot \overline{\begin{pmatrix} P_{k-1,1} \\ P_{k-1,2} \end{pmatrix} \cdot_{L^2} \left(\frac{\partial P_{j,1}}{\partial t}; \frac{\partial P_{j,2}}{\partial t} \right)} + b^*_k \cdot \overline{\begin{pmatrix} P_{k,1} \\ P_{k,2} \end{pmatrix} \cdot_{L^2} \left(\frac{\partial P_{j,1}}{\partial t}; \frac{\partial P_{j,2}}{\partial t} \right)} +$$

$$+ a^*_k \cdot \overline{\begin{pmatrix} P_{k+1,1} \\ P_{k+1,2} \end{pmatrix} \cdot \left(\frac{\partial P_{j,1}}{\partial t}; \frac{\partial P_{j,2}}{\partial t} \right)} =$$

$$= c^*_{k-1} \cdot I_{k-1,j} + b^*_k \cdot I_{k,j} + a^*_k \cdot I_{k+1,j}.$$

Finally we obtain

$$E_{j,k} = I^*_{k-1,j} \cdot c_{k-1} + I^*_{k,j} \cdot b_k + I^*_{k+1,j} \cdot a_k = \{I^* L\}_{j,k}. \tag{40}$$

In the same manner we obtain the corresponding matrix representation for $\widehat{E}_{j,\alpha;k,\beta}$:

$$\widehat{E}_{j,\alpha;k,\beta} = \int\limits_{\mathbb{T}} \lambda \frac{\partial P_{k,\beta}(\lambda,t)}{\partial t} \overline{P_{j,\alpha}(\lambda,t)} d\rho(\lambda,t) =$$

$$= \int\limits_{\mathbb{T}} \frac{\partial P_{k,\beta}(\lambda,t)}{\partial t} \overline{\frac{1}{\lambda} P_{j,\alpha}(\lambda,t)} d\rho(\lambda,t) = \left(\frac{\partial P_{k,\beta}(\lambda,t)}{\partial t}; \bar{\lambda} P_{j,\alpha}(\lambda,t) \right)_{L_2}.$$

Thus (the same way as in the previous case)

$$\widehat{E}_{j;k} = \overline{\begin{pmatrix} \bar{z}P_{j,1} \\ \bar{z}P_{j,2} \end{pmatrix} \cdot_{L^2} \left(\frac{\partial P_{k,1}}{\partial t}; \frac{\partial P_{k,2}}{\partial t} \right)} =$$

$$= \overline{\left(\bar{a}_{j-1} \cdot \begin{pmatrix} P_{j-1,1} \\ P_{j-1,2} \end{pmatrix} + \bar{b}_j \cdot \begin{pmatrix} P_{j,1} \\ P_{j,2} \end{pmatrix} + \bar{c}_j \cdot \begin{pmatrix} P_{j+1,1} \\ P_{j+1,2} \end{pmatrix} \right) \cdot_{L^2} \left(\frac{\partial P_{k,1}}{\partial t}; \frac{\partial P_{k,2}}{\partial t} \right)} =$$

$$= a_{j-1} \cdot I_{j-1,k} + b_j \cdot I_{j,k} + c_j \cdot I_{j+1,k}.$$

Final result:

$$\widehat{E}_{j;k} = a_{j-1} \cdot I_{j-1,k} + b_j \cdot I_{j,k} + c_j \cdot I_{j+1,k} = \{LI\}_{j,k}. \tag{41}$$

Formulae (40), (41) allow us to write out the following expressions for $E_{j,k}$, $\widehat{E}_{j,k}$:

(a) $E_{j,k} = 0$, $\widehat{E}_{j,k} = a_{j-1} \cdot I_{j-1,k} + b_j \cdot I_{j,k} + c_j \cdot I_{j+1,k}$, $j < k-1$;

(b) $E_{k-1,k} = I^*_{k-1,k-1} \cdot c_{k-1}$, $\widehat{E}_{k-1,k} = a_{k-2} \cdot I_{k-2,k} + b_{k-1} \cdot I_{k-1,k} + c_{k-1} \cdot I_{k,k}$;

(c) $E_{k,k} = I^*_{k-1,k} \cdot c_{k-1} + I^*_{k,k} \cdot b_k$, $\widehat{E}_{k,k} = a_{k-1} \cdot I_{k-1,k} + b_k \cdot I_{k,k}$;

(d) $E_{k+1,k} = I^*_{k-1,k+1} \cdot c_{k-1} + I^*_{k,k+1} \cdot b_k + I^*_{k+1,k+1} \cdot a_k$, $\widehat{E}_{k+1,k} = a_k \cdot I_{k,k}$;

(e) $E_{j,k} = I^*_{k-1,j} \cdot c_{k-1} + I^*_{k,j} \cdot b_k + I^*_{k+1,j} \cdot a_k$, $\widehat{E}_{j,k} = 0$, $j > k+1$.

Consider the element $E_{k+1,k}$ (here calculations are the most complicated; all other coefficients are obtained in the same manner):

$$E_{k+1,k} = I^*_{k-1,k+1} \cdot c_{k-1} + I^*_{k,k+1} \cdot b_k + I^*_{k+1,k+1} \cdot a_k =$$

$$= -\big(\Omega^*_{k-1,k+1} + \widehat{\Omega}_{k-1,k+1} + \Psi^*_{k-1,k+1}\big) \cdot c_{k-1} -$$

$$-\big(\Omega^*_{k,k+1} + \widehat{\Omega}_{k,k+1} + \Psi^*_{k,k+1}\big) \cdot b_k +$$

$$+ \begin{pmatrix} \frac{1}{2}\overline{\Big[-\Omega_{k+1,1;k+1,1} - \widehat{\Omega}^*_{k+1,1;k+1,1} - \Psi_{k+1,1;k+1,1}\Big]} & 0 \\ \overline{\Big[-\Omega_{k+1,1;k+1,2} - \widehat{\Omega}^*_{k+1,1;k+1,2} - \Psi_{k+1,1;k+1,2}\Big]} & \frac{1}{2}\overline{[\Xi_{k+1,2;k+1,2}]} \end{pmatrix} \cdot a_k =$$

$$= (-\{\Omega^* \cdot L\}_{k+1,k} + \Omega^*_{k+1,k+1} \cdot a_k) +$$

$$+(-\{\widehat{\Omega} \cdot L\}_{k+1,k} + \widehat{\Omega}_{k+1,k+1} \cdot a_k) + (-\{\Psi^* \cdot L\}_{k+1,k} + \Psi^*_{k+1,k+1} \cdot a_k) +$$

$$+ \begin{pmatrix} \frac{1}{2}\overline{\Big[-\Omega_{k+1,1;k+1,1} - \widehat{\Omega}^*_{k+1,1;k+1,1} - \Psi_{k+1,1;k+1,1}\Big]} & 0 \\ \overline{\Big[-\Omega_{k+1,1;k+1,2} - \widehat{\Omega}^*_{k+1,1;k+1,2} - \Psi_{k+1,1;k+1,2}\Big]} & \frac{1}{2}\overline{[\Xi_{k+1,2;k+1,2}]} \end{pmatrix} \cdot a_k.$$

Note that

$$\Omega^*_{k+1,k+1} \cdot a_k = \begin{pmatrix} \overline{\Omega_{k+1,1;k+1,1}} & \overline{\Omega_{k+1,2;k+1,1}} \\ \overline{\Omega_{k+1,1;k+1,2}} & \overline{\Omega_{k+1,2;k+1,2}} \end{pmatrix} \begin{pmatrix} a_{k;11} & a_{k;12} \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \overline{\Omega_{k+1,1;k+1,1}} & 0 \\ \overline{\Omega_{k+1,1;k+1,2}} & \overline{\Omega_{k+1,2;k+1,2}} \end{pmatrix} \begin{pmatrix} a_{k;11} & a_{k;12} \\ 0 & 0 \end{pmatrix}.$$

So we obtain the following result (overline can be stripped because $I_{j,\alpha;j,\alpha} \in \mathbb{R}$):

$$E_{k+1,k} = -\{\Omega^* \cdot L + \widehat{\Omega} \cdot L + \Psi^* \cdot L\}_{k+1,k} -$$

$$-\frac{1}{2} \begin{pmatrix} \Big[-\Omega_{k+1,1;k+1,1} - \widehat{\Omega}^*_{k+1,1;k+1,1} - \Psi_{k+1,1;k+1,1}\Big] & 0 \\ 0 & [\Xi_{k+1,2;k+1,2}] \end{pmatrix} \cdot a_k =$$

$$= \{\Xi^* \cdot L\}_{k+1,k} - \frac{1}{2} \begin{pmatrix} \Xi^*_{k+1,1;k+1,1} & 0 \\ 0 & \Xi^*_{k+1,2;k+1,2} \end{pmatrix} \cdot a_k.$$

Compare this result with analogous one-dimensional result contained in [11] (Lemma 3, formula (d)).

Formula (d) for $E_{j,k}$ at $j > k+1$ is being calculated in the same way (its proof is part of the proof for $E_{k+1,k}$). Consider the element $\widehat{E}_{k-1,k}$:

$$\widehat{E}_{k-1,k} = a_{k-2} \cdot I_{k-2,k} + b_{k-1} \cdot I_{k-1,k} + c_{k-1} \cdot I_{k,k} =$$

$$= a_{k-2} \cdot (-\Omega_{k-2,k} - \widehat{\Omega}^*_{k-2,k} - \Psi_{k-2,k}) +$$

$$+ b_{k-1} \cdot (-\Omega_{k-1,k} - \widehat{\Omega}^*_{k-1,k} - \Psi_{k-1,k}) +$$

$$+ c_{k-1} \cdot \begin{pmatrix} \frac{1}{2}\Xi_{k,1;k,1} & \Xi_{k,1;k,2} \\ 0 & \frac{1}{2}\Xi_{k,2;k,2} \end{pmatrix} =$$

$$= (-\{L\Omega\}_{k-1,k} + c_{k-1} \cdot \Omega_{k,k}) + (-\{L\widehat{\Omega}^*\}_{k-1,k} + c_{k-1} \cdot \widehat{\Omega}^*_{k,k}) +$$

$$+ (-\{L\Psi\}_{k-1,k} + c_{k-1} \cdot \Psi_{k,k}) +$$

$$+ c_{k-1} \cdot \begin{pmatrix} \frac{1}{2}\left[-\Omega_{k,1;k,1} - \widehat{\Omega}^*_{k,1;k,1} - \Psi_{k,1;k,1}\right] & \left[-\Omega_{k,1;k,2} - \widehat{\Omega}^*_{k,1;k,2} - \Psi_{k,1;k,2}\right] \\ 0 & \frac{1}{2}\left[-\Omega_{k,2;k,2} - \widehat{\Omega}^*_{k,2;k,2} - \Psi_{k,2;k,2}\right] \end{pmatrix}.$$

Analogously

$$c_{k-1} \cdot \Omega_{k,k} = \begin{pmatrix} 0 & 0 \\ c_{k;21} & c_{k;22} \end{pmatrix} \begin{pmatrix} \Omega_{k,1;k,1} & \Omega_{k,1;k,2} \\ \Omega_{k,2;k,1} & \Omega_{k,2;k,2} \end{pmatrix}.$$

So

$$c_{k-1} \cdot I_{k,k} + c_{k-1} \cdot \Omega_{k,k} + c_{k-1} \cdot \widehat{\Omega}^*_{k,k} + c_{k-1} \cdot \Psi_{k,k} =$$

$$= c_{k-1} \cdot \begin{pmatrix} \frac{1}{2}\left[\Omega_{k,1;k,1} + \widehat{\Omega}^*_{k,1;k,1} + \Psi_{k,1;k,1}\right] & 0 \\ I_{k,2;k,1} & \frac{1}{2}\left[\Omega_{k,2;k,2} + \widehat{\Omega}^*_{k,2;k,2} + \Psi_{k,2;k,2}\right] \end{pmatrix}.$$

From (36) we have $I_{k,2;k,1} = 0$, thus

$$\widehat{E}_{k-1,k} = \{L \cdot \Xi\}_{k-1,k} - \frac{1}{2} c_{k-1} \cdot \begin{pmatrix} \Xi_{k,1;k,1} & 0 \\ 0 & \Xi_{k,2;k,2} \end{pmatrix}.$$

The lemma is proved.

Lemma 5 [analogue of [11], step 1]**.** *The non-trivial entries* a_n, b_n, c_n *of Jacobi matrix* $L(t)$ *satisfy the formulae* (16).

Proof. Using Lemma 2 and Lemma 4 it is easy to verify that

$$\dot{a}_n(t) = \dot{L}_{n+1,n}(t) = \{J_1(t) + \big(\widehat{E}(t) + E(t)\big) + J_3(t)\}_{n+1,n} =$$

$$= \{\Phi + L\Omega + \widehat{\Omega}^* L\}_{n+1;n} + \left[a_n \cdot \begin{pmatrix} \frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2} \\ 0 & \frac{1}{2}\Xi_{n,2;n,2} \end{pmatrix} \right] +$$

$$+\left[\{\Xi^*\cdot L\}_{n+1,n}-\frac{1}{2}\begin{pmatrix}\Xi_{n+1,1;n+1,1} & 0\\ 0 & \Xi_{n+1,2;n+1,2}\end{pmatrix}\cdot a_n\right]+\{L\Psi\}_{n+1;n}=$$

$$=\Phi_{n+1,n}+(a_n\cdot\Omega_{n,n}+b_{n+1}\cdot\Omega_{n+1,n})+$$

$$+\big(\widehat{\Omega}^*_{n-1,n+1}\cdot c_{n-1}+\widehat{\Omega}^*_{n,n+1}\cdot b_n+\widehat{\Omega}^*_{n+1,n+1}\cdot a_n\big)+$$

$$+\left[a_n\cdot\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2}\\ 0 & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\right]+$$

$$+\Big[(\Xi^*_{n-1,n+1}\cdot c_{n-1}+\Xi^*_{n,n+1}\cdot b_n+\Xi^*_{n+1,n+1}\cdot a_n)-$$

$$-\frac{1}{2}\begin{pmatrix}\Xi_{n+1,1;n+1,1} & 0\\ 0 & \Xi_{n+1,2;n+1,2}\end{pmatrix}\cdot a_n\Big]+$$

$$+(a_n\cdot\Psi_{n,n}+b_{n+1}\cdot\Psi_{n+1,n}).$$

Now verify the same for $b_n(t)$:

$$\dot b_n(t)=\dot L_{n,n}(t)=\Big\{J_1(t)+\big(\widehat{E}(t)+E(t)\big)+J_3(t)\Big\}_{n,n}=$$

$$=\{\Phi+L\Omega+\widehat{\Omega}^*L\}_{n,n}+\left[a_{n-1}\cdot\Xi_{n-1,n}+b_n\cdot\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2}\\ 0 & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\right]+$$

$$+\left[\{\Xi^*\}_{n-1,n}\cdot c_{n-1}+\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\cdot b_n\end{pmatrix}\right]+\{L\Psi\}_{n,n}=$$

$$=\{\Phi\}_{n,n}+(a_{n-1}\cdot\Omega_{n-1,n}+b_n\cdot\Omega_{n,n}+c_n\cdot\Omega_{n+1,n})+$$

$$+\big(\widehat{\Omega}^*_{n-1,n}\cdot c_{n-1}+\widehat{\Omega}^*_{n,n}\cdot b_n+\widehat{\Omega}^*_{n+1,n}\cdot a_n\big)+$$

$$+\left[a_{n-1}\cdot\{-\Omega-\widehat{\Omega}^*-\Psi\}_{n-1,n}+b_n\cdot\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2}\\ 0 & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\right]+$$

$$+\left[\{-\Omega^*-\widehat{\Omega}-\Psi^*\}_{n-1,n}\cdot c_{n-1}+\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot b_n\right]+$$

$$+\left\{\frac{1}{2}L\Psi+\frac{1}{2}\Psi L\right\}_{n,n}=$$

$$=\{\Phi\}_{n,n}+(a_{n-1}\cdot\Omega_{n-1,n}+b_n\cdot\Omega_{n,n}+c_n\cdot\Omega_{n+1,n})+$$

$$+\big(\widehat{\Omega}^*_{n-1,n}\cdot c_{n-1}+\widehat{\Omega}^*_{n,n}\cdot b_n+\widehat{\Omega}^*_{n+1,n}\cdot a_n\big)+$$

$$+\left[a_{n-1}\cdot\{-\Omega-\widehat{\Omega}^*-\Psi\}_{n-1,n}+b_n\cdot\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2}\\ 0 & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\right]+$$

$$+\left[\big\{-\Omega^*-\widehat{\Omega}-\Psi^*\big\}_{n-1,n}\cdot c_{n-1}+\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot b_n\right]+$$

$$+\left\{\frac{1}{2}(a_{n-1}\cdot\Psi_{n-1,n}+b_n\cdot\Psi_{n,n}+c_n\cdot\Psi_{n+1,n})+\right.$$

$$\left.+\frac{1}{2}(\Psi_{n,n-1}\cdot c_{n-1}+\Psi_{n,n}\cdot b_n+\Psi_{n,n+1}\cdot a_n)\right\}_{n,n}=$$

$$=\{\Phi\}_{n,n}+(b_n\cdot\Omega_{n,n}+c_n\cdot\Omega_{n+1,n})+\big(\widehat{\Omega}^*_{n-1,n}\cdot c_{n-1}+\widehat{\Omega}^*_{n,n}\cdot b_n+\widehat{\Omega}^*_{n+1,n}\cdot a_n\big)+$$

$$+\left[a_{n-1}\cdot\{-\widehat{\Omega}^*\}_{n-1,n}+b_n\cdot\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & \Xi_{n,1;n,2}\\ 0 & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\right]+$$

$$+\left[\big\{-\Omega^*-\widehat{\Omega}-\Psi^*\big\}_{n-1,n}\cdot c_{n-1}+\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot b_n\right]+$$

$$+\left\{\frac{1}{2}(b_n\cdot\Psi_{n,n}+c_n\cdot\Psi_{n+1,n})+\frac{1}{2}(\Psi_{n,n-1}\cdot c_{n-1}+\Psi_{n,n}\cdot b_n+\Psi_{n,n+1}\cdot a_n)\right\}_{n,n}.$$

The lemma is proved.

Similarly obtain analogous results for c_n:

$$\dot{c}_n(t)=\dot{L}_{n,n+1}(t)=\{J_1(t)+\big(\widehat{E}(t)+E(t)\big)+J_3(t)\}_{n,n+1}=$$

$$=\{\Phi+L\Omega+\widehat{\Omega}^*L\}_{n,n+1}+\left[\{L\cdot\Xi\}_{n,n+1}-\frac{1}{2}c_n\cdot\begin{pmatrix}\Xi_{n+1,1;n+1,1} & 0\\ 0 & \Xi_{n+1,2;n+1,2}\end{pmatrix}\right]+$$

$$+\left[\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot c_n\right]+\{L\Psi\}_{n,n+1}=$$

$$=\big(a_{n-1}\cdot\Omega_{n-1,n+1}+b_n\cdot\Omega_{n,n+1}+c_n\cdot\Omega_{n+1,n+1}\big)+$$

$$+\big(\widehat{\Omega}^*_{n,n}\cdot c_n+\widehat{\Omega}^*_{n+1,n}\cdot b_{n+1}+\widehat{\Omega}^*_{n+2,n}\cdot a_{n+1}\big)+$$

$$+\Bigg[\big(a_{n-1}\cdot\Xi_{n-1,n+1}+b_n\cdot\Xi_{n,n+1}+c_n\cdot\Xi_{n+1,n+1}\big)-$$

$$-\frac{1}{2}c_n\cdot\begin{pmatrix}\Xi_{n+1,1;n+1,1} & 0\\ 0 & \Xi_{n+1,2;n+1,2}\end{pmatrix}\Bigg]+$$

$$+\left[\begin{pmatrix}\frac{1}{2}\Xi_{n,1;n,1} & 0\\ \overline{\Xi_{n,1;n,2}} & \frac{1}{2}\Xi_{n,2;n,2}\end{pmatrix}\cdot c_n\right]+$$

$$+\big(a_{n-1}\cdot\Psi_{n-1,n+1}+b_n\cdot\Psi_{n,n+1}+c_n\cdot\Psi_{n+1,n+1}\big)+\Phi_{n,n+1}.$$

6. Samples. To obtain difference-differential flows like Schur Flow it is sufficient to overwrite the coefficients of multiplication operator L (matrix J) in terms of Verblunsky coefficients. The reader can find particular example in Leonid Golinskii article [15] of how to obtain Schur flow from the appropriate Lax equation.

***Example* 1.** *Let* $\Phi(\lambda,t)\equiv 0$, $\Psi(\lambda,t)=\lambda+\dfrac{1}{\lambda}$. *Then we obtain the case described in* [15] *by L. Golinskii.*

Proof. At $\Phi(\lambda,t)\equiv 0$ we have $\Omega=\Phi(L(t),t)D(t)=\mathbb{O}$, $\widehat{\Omega}=\Phi(L^*(t),t)D(t)=\mathbb{O}$, $\Psi(L(t),t)=L+L^*$, $\Xi=-\Omega-\widehat{\Omega}^*-\Psi=-\Psi=-L-L^*$. Substitute this into (39):

$$\frac{d}{dt}L(t)=\Phi(L(t),t)+\left[L(t),\Omega+I+\frac{1}{2}\Psi\right]=[L,B],$$

where $B=I+\dfrac{1}{2}\Psi=\dfrac{(L+L^*)_- - (L+L^*)_+}{2}$. Compare this with [15] (formulae (1.21), (1.22)). This equation corresponds to Schur flow:

$$\alpha_n'(t)=(1-|\alpha_n|^2)(\alpha_{n+1}(t)-\alpha_{n-1}(t)),\quad t>0. \tag{42}$$

***Example* 2.** *Let* $\Phi(\lambda,t)\equiv 0$, $\Psi(\lambda,t)=\lambda$. *Then we obtain two-dimensional analogue for unitary case of the Toda lattice (that originally was built in one-dimensional case* $\ell_2=\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\ldots$ *for self-adjoint* L*).*

Proof. At $\Phi(\lambda,t)\equiv 0$ we have $\Omega=\Phi(L(t),t)D(t)=\mathbb{O}$, $\widehat{\Omega}=\Phi(L^*(t),t)D(t)=\mathbb{O}$, $\Psi(L(t),t)=L(t)$, $\Xi=-\Omega-\widehat{\Omega}^*-\Psi=-\Psi=-L$. Substitute this into (39):

$$\frac{d}{dt}L(t) = \Phi(L(t),t) + \left[L(t), \Omega + I + \frac{1}{2}\Psi\right] = [L, A],$$

where

$$A = \left(\begin{array}{c|cc|cc} 0 & -\frac{1}{2}c_{0;10} & -\frac{1}{2}c_{0;11} & 0 & 0 \\ \hline \frac{1}{2}a_{0;01} & 0 & -\frac{1}{2}b_{1;01} & 0 & 0 \\ 0 & \frac{1}{2}b_{1;10} & 0 & -\frac{1}{2}c_{1;10} & -\frac{1}{2}c_{1;11} \\ \hline 0 & \frac{1}{2}a_{1;00} & \frac{1}{2}a_{1;01} & 0 & -\frac{1}{2}b_{2;01} \\ 0 & 0 & 0 & \frac{1}{2}b_{2;10} & 0 \end{array}\right).$$

Note that A is not uniquely determined: differential equation doesn't change if we replace A with $A+T$ where T is an arbitrary operator that commutes with L.

Now if we re-write A and L in terms of Verblunsky coefficients

$$L = \mathcal{C}(\{\alpha_n\}) = \left(\begin{array}{c|cc|cc} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_0\rho_1 & & \\ \hline \rho_0 & -\bar{\alpha}_1\alpha_0 & -\alpha_0\rho_1 & 0 & 0 \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_2\rho_3 \\ \hline & \rho_1\rho_2 & -\alpha_1\rho_2 & -\bar{\alpha}_3\alpha_2 & -\alpha_2\rho_3 \\ & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 \end{array}\right) \qquad (43)$$

then we obtain "Toda" flow for unitary case:

$$\alpha_n'(t) = (|\alpha_n|^2 - 1)\alpha_{n-1}. \qquad (44)$$

There are many ways how to prove this. The simplest one is to modify slightly [15] (Theorem 2). Similarly the next example is obtained.

***Example* 3.** *Let* $\Phi(\lambda,t) \equiv 0$, $\Psi(\lambda,t) = \lambda^2$. *Then we obtain the analog for Kac – van Moerbeke lattice.*

Proof. Recall that classical Kac – van Moerbeke lattice for self-adjoint L has the following view:

$$\dot{x}_n(t) = x_n(x_{n+1} - x_{n-1}), \qquad n = 0,1,\ldots, \quad x_{-1} = 0. \qquad (45)$$

In our case Lax equation has the same form as in previous example with $A = I + \frac{1}{2}\Psi$ where $\Psi = L^2$ and $\Xi = -L^2$. In terms of Verblunsky coefficients Kac – van Moerbeke flow is as follows:

$$\alpha_n'(t) = \left(1 - |\alpha_n|^2\right)\left(\alpha_{n+1}\bar{\alpha}_n\alpha_{n-1} - \alpha_{n-2} + |\alpha_{n-1}|^2(\alpha_n + \alpha_{n-2})\right). \qquad (46)$$

The described above theory gives the possibility to build entire families of different flows. And this is the object of further investigations.

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