UDC 517.95

I. Dağadur (Mersin Univ., Turkey)

C_λ-SEMICONSERVATIVE FK-SPACES

C_λ-НАПІВКОНСЕРВАТИВНІ FK-ПРОСТОРИ

We study C_λ-semiconservative FK-spaces for C_λ-methods defined by deleting a set of rows from the Cesáro matrix C_1 and give some characterizations.

Вивчено C_λ-напівконсервативні FK-простори для C_λ-методів, що визначаються видаленням групи рядків із матриці Чезаро C_1, і наведено деякі характеристики.

1. Introduction and notation. The definition of semiconservative FK-space and some properties of this space was given by Snyder and Wilansky in [14]. Ince, in [8], continued to work on Cesáro semiconservative FK-space and to give some characterizations. In Section 2, for an FK-space X, the concepts of C_λ-semiconservative FK-space have been defined. Their relationship to Cesáro semiconservative space and C_λ-semiconservative have also been examined. However, we study the C_λ-semiconservative of the absolute summability domain l_A, and show that if l_A is C_λ-semiconservative, then A cannot be l-replaceable. In Section 3 we study the subspaces $C_\lambda F^+$, $C_\lambda F$, $C_\lambda B$ and $C_\lambda B^+$ of an FK-space X. In Section 4 we solve the problem of characterizing matrices A such that Y_A is C_λ-semiconservative space for given Y.

Let F be an infinite subset of $\mathbb{N}$ and F as the range of a strictly increasing sequence of positive integers, say $F = \{\lambda(n)\}_{n=1}^{\infty}$. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad n = 1, 2, \ldots,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. The basic properties of C_λ-method can be found in [1] and [10].

Let s denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_n(x) = |x_n|$, $n = 1, 2, \ldots,$ and any vector subspace of s is called a sequence space. A sequence space X, with a vector space topology τ, is a K-space provided that the inclusion mapping $i\colon (X, \tau) \to s$, $i(x) = x$ is continuous. If, in addition, τ is complete, metrizable and locally convex then (X, τ) is called an FK-space. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals are continuous. The basic properties of such spaces may be found in [1 – 13, 15].

By c_0, l^∞ we denote the spaces of all number sequences that converge to zero and bounded sequences, respectively. These are FK-spaces under $\|x\| = \sup_n |x_n|$.

As usual, $l_1 = \left\{x \in s\colon \sum_{n=1}^{\infty} |x_n| < \infty\right\}$ is denoted simply by l. $cs = \left\{x \in s\colon \sum_{n=1}^{\infty} x_n \text{ exists}\right\}$, the space of all summable sequences; and bs is as the following:

$$bs = \left\{x \in s\colon \sup_k \left|\sum_{n=1}^{k} x_n\right| < \infty\right\}.$$

The sequence spaces

$$\sigma s(\lambda) = \left\{ x \in s \colon \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j \text{ exists} \right\},$$

$$\sigma b(\lambda) = \left\{ x \in s \colon \sup_n \left| \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

and

$$q(\lambda) := \left\{ x \colon \sum_{j=1}^{\infty} \lambda(j) \left| \triangle^2 x_j \right| < \infty \text{ and } x \in l^\infty \right\}, \qquad q_0(\lambda) := q(\lambda) \cap c_0$$

is FK-space with the norms [2, 3, 5 – 7]

$$\|x\|_{\sigma b(\lambda)} = \sup_n \left| \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j \right|,$$

$$\|x\|_{q(\lambda)} = \sum_{j=1}^{\infty} \lambda(j) \left| \triangle^2 x_j \right| + \sup_n |x_j|,$$

where

$$\triangle x_j = x_j - x_{j+1} \qquad \text{and} \qquad \triangle^2 x_j = \triangle x_j - \triangle x_{j+1}.$$

Throughout the paper e denotes the sequences of ones, $(1, 1, \ldots, 1, \ldots)$; δ^j, $j = 1, 2, \ldots$, the sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with the one in the jth position; ϕ the linear span of the δ^j's. The topological dual of X is denoted by X'. The space X is said to have AD if ϕ is dense in X. A sequence x in a locally convex sequence space X is said the property AK (respectively $\sigma K(\lambda)$) if $x^{(n)} \to x$ $\left(\text{respectively } \dfrac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x^{(k)} \to x \right)$ in X where $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, \ldots) = \sum_{k=1}^{n} x_k \delta^k$. An FK-space X is called Cesáro semiconservative space if $X^f \subset \sigma s$ where $\sigma s := \left\{ x \in s \colon \lim_n \dfrac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \text{ exists} \right\}$ (see [8]). Every AK space is a $\sigma K(\lambda)$. We recall (see [5, 6, 13, 14]) that the f, β, σ, σb, $\sigma(\lambda)$ and $\sigma b(\lambda)$-dual of a subset X of s is defined to be

$$X^f = \left\{ \{ f(\delta^k) \} \colon f \in X' \right\},$$

$$X^\beta = \left\{ x \in s \colon \sum_{k=1}^{\infty} x_k y_k \text{ exists for all } y \in X \right\} =$$

$$= \{ x \in s \colon xy = (x_k y_k) \in cs \text{ for all } y \in X \},$$

$$X^\sigma = \left\{ x \in s \colon \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j y_j \text{ exists for all } y \in X \right\} =$$

$$= \{x \in s \colon xy \in \sigma s \text{ for all } y \in X\},$$

$$X^{\sigma b} = \left\{ x \in s \colon \sup_n \frac{1}{n} \left| \sum_{k=1}^{n} \sum_{j=1}^{k} y_j \right| < \infty \text{ for all } y \in X \right\} =$$

$$= \{x \in s \colon xy \in \sigma b \text{ for all } y \in X\},$$

$$X^{\sigma(\lambda)} = \left\{ x \in s \colon \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j y_j \text{ exists for all } y \in X \right\} =$$

$$= \{x \in s \colon xy \in \sigma s(\lambda) \text{ for all } y \in X\},$$

$$X^{\sigma b(\lambda)} = \left\{ x \in s \colon \sup_n \frac{1}{\lambda(n)} \left| \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} y_j \right| < \infty \text{ for all } y \in X \right\} =$$

$$= \{x \in s \colon xy \in \sigma b(\lambda) \text{ for all } y \in X\},$$

where $xy = (x_n y_n)$. Let E, E_1 be sets of sequences. Then for $k = \beta, \sigma, \sigma b, \sigma(\lambda)$ and $\sigma b(\lambda)$

(a) $E \subset E^{kk}$,

(b) $E^{kkk} = E^k$,

(c) if $E \subset E_1$ then $E_1^k \subset E^k$

holds. Also, if $\phi \subset E \subset E_1$ then $E_1^f \subset E^f$.

We shall be concerned with matrix transformations $y = Ax$, where $x, y \in s$, $A = \{a_{nk}\}_{n,k=1}^{\infty}$ is an infinite matrix with complex coefficients, and

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n = 1, 2, \ldots.$$

The sequence $\{a_{nk}\}_{k=1}^{\infty}$ is called the nth row of A and is denoted by a^n, $n = 1, 2, \ldots$; similarly, the kth column of the matrix A, $\{a_{nk}\}_{n=1}^{\infty}$ is denoted by a^k, $k = 1, 2, \ldots$. For an FK-space Y, we consider the summability domain Y_A defined by

$$Y_A = \{x \in s \colon \ Ax \text{ exists and } Ax \in Y\}.$$

Then Y_A is an FK-space under the seminorms $p_n(x) = |x_n|$, $n = 1, 2, \ldots$;

$$h_n(x) = \sup_m \left| \sum_{k=1}^{m} a_{nk} x_k \right|, \quad n = 1, 2, \ldots, \quad \text{and} \quad (q \circ A)(x) = q(Ax) \quad \text{(see[13])}.$$

2. C_λ-semiconservative FK-spaces. In this section, the concept of C_λ-semiconservative an FK-space X containing ϕ is defined, and several theorems on this subject are given.

Definition 2.1. *An FK-space X is called C_λ-semiconservative space if*

$$X^f \subset \sigma s(\lambda).$$

This means that $\phi \subset X$ and $\left\{ \dfrac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} e^{(k)} \right\}$ is convergent for each $f \in X'$.

For example, c_0 is a C_λ-semiconservative FK-space. Every semiconservative FK-space is a C_λ-semiconservative FK-space. But every C_λ-semiconservative FK-space is not a semiconservative FK-space. An example of FK-space which is C_λ-semiconservative but not semiconservative is given in [8] in case $\lambda(n) = n$.

The theorem below gives us the equivalence of Cesáro semiconservative and C_λ-semiconservative of an FK-space X.

Theorem 2.1. *Let X be an FK-space with $\phi \subset X$ and $X^f \subset bs$. Let $\lambda := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N}$ such that $\limsup_n \dfrac{\lambda(n+1)}{\lambda(n)} = 1$. Then X is C_1-semiconservative if and only if it is C_λ-semiconservative.*

Proof. *Necessity* is trivial.

Sufficiency. Let X be C_λ-semiconservative. Then for each $f \in X'$, we have

$$\lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} f\left(\delta^j\right) \quad \text{exists.}$$

Let $t_k(f) := \sum_{j=1}^{k} f\left(\delta^j\right)$. So, $(t_k(f))$ is C_λ-summable. Since $X^f \subset bs$, for all $f \in X'$, $(t_k(f)) \in$ $\in l^\infty$. Since $\limsup_n \dfrac{\lambda(n+1)}{\lambda(n)} = 1$, by Theorem 2.1 of [10], it is C_1-summable. Therefore, X is a C_1-semiconservative space.

Using the same technique one can get the following theorem.

Theorem 2.2. *Let X be an FK-space with $\phi \subset X$, $X^f \subset bs$ and $\lambda := \{\lambda(n)\}$, $\mu := \{\mu(n)\}$ infinite subsets of $\mathbb{N}$. If $\lim_n \dfrac{\mu(n)}{\lambda(n)} = 1$, then X is C_λ-semiconservative if and only if it is C_μ-semiconservative.*

To see that $\lim_n \dfrac{\mu(n)}{\lambda(n)} = 1$ is not a necessary condition in Theorem 2.2, simply consider the sequences $\lambda(n) = n^2$ and $\mu(n) = n^3$. Then $\lim_n \dfrac{\lambda(n+1)}{\lambda(n)} = \lim_n \dfrac{\mu(n+1)}{\mu(n)} = 1$, and hence, by Theorem 2.1, X is C_λ-semiconservative if and only if it is C_1-semiconservative and X is C_μ-semiconservative if and only if it is C_1-semiconservative. However, $\lim_n \dfrac{\mu(n)}{\lambda(n)} = \dfrac{n^3}{n^2} \neq 1$.

In Theorem 2.1, with $\limsup_n \dfrac{\lambda(n+1)}{\lambda(n)} = 1$ replaced by $\lim_n \dfrac{\lambda(n+1)}{\lambda(n)} = 1$, the following result is easily obtained by Theorem 2.2.

***Corollary* 2.1.** *Let $\lim_n \dfrac{\lambda(n+1)}{\lambda(n)} = 1$. Then X is C_1-semiconservative if and only if it is C_λ-semiconservative.*

The definition of a C_λ-conull FK-space X with $\phi \subset X$, can be given by using C_λ-semiconservativity. A C_λ-semiconservative space X is called C_λ-conull, if

$$f(e) = \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} f\left(\delta^j\right),$$

for all $f \in X'$. A C_λ-semiconservative space need not contain e but C_λ-conull must contain e.

Theorem 2.3. *If X_A is a C_λ-conull FK-space, then it is a C_λ-semiconservative space.*

Proof. Suppose that X_A is C_λ-conull FK-space. Then

$$f(e) = \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} f\left(\delta^j\right),$$

for all $f \in X'_A$. Hence $X_A^f \subset \sigma s(\lambda)$.

We recall that, in [9] it is defined that a matrix A is l-replaceable if there is a matrix $B = (b_{nk})$ with $l_A = l_B$ and $\sum_{n=1}^{\infty} b_{nk} = 1$ for all $k \in \mathbb{N}$.

Theorem 2.4. *If a matrix A is l-replaceable, then l_A is not a C_λ-semiconservative FK-space.*

Proof. If A is l-replaceable, then there is $f \in l'_A$ such that $f\left(\delta^j\right) = 1$ for all $j \in \mathbb{N}$ in [9]. Hence $\lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} f\left(\delta^j\right)$ does not exist since

$$\frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} f\left(\delta^j\right) = \frac{\lambda(n)+1}{2},$$

so l_A is not C_λ-semiconservative space.

Theorem 2.5. (i) *An FK-space that contains a C_λ-semiconservative FK-space must be a C_λ-semiconservative FK-space.*

(ii) *A closed subspace, containing ϕ, of a C_λ-semiconservative FK-space is a C_λ-semiconservative FK-space.*

(iii) *A countable intersection of C_λ-semiconservative FK-spaces is a C_λ-semiconservative FK-spaces.*

The proof is easily obtained from elementary properties of FK-spaces (see [13]).

Theorem 2.6. *Let X be an FK-space containing ϕ. Then*

(i) $X^\beta \subset X^{\sigma(\lambda)} \subset X^{\sigma b(\lambda)} \subset X^f$,

(ii) *if X is a $\sigma K(\lambda)$-space, then $X^f = X^{\sigma(\lambda)}$,*

(iii) *if X is an AD-space, then $X^{\sigma(\lambda)} = X^{\sigma b(\lambda)}$.*

Proof. (ii) Let $v \in X^{\sigma(\lambda)}$ and define $f(x) = \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} v_j x_j$ for $x \in X$. Then $f \in X'$ by the Banach – Steinhaus theorem of [13]. Also

$$f\left(\delta^q\right) = \lim_n \frac{1}{\lambda(n)} (\lambda(n) - (q-1)) v_q = v_q, \quad q < \lambda(n),$$

so $v \in X^f$. Thus $X^{\sigma(\lambda)} \subset X^f$.

Now we show that $X^f \subset X^{\sigma(\lambda)}$. Let $v \in X^f$. Since X is a $\sigma K(\lambda)$-space

$$f(x) = \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j f\left(\delta^j\right) = \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} v_j x_j$$

for $x \in X$, then $v \in X^{\sigma(\lambda)}$. This completes the proof of (ii).

(iii) Let $v \in X^{\sigma b(\lambda)}$ and define $f_n(x) = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} v_j x_j$ for $x \in X$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by [13]. Since

$$\lim_n f_n\left(\delta^q\right) = v_q, \quad q < \lambda(n),$$

then $\phi \subset \{x\colon \lim_n f_n(x) \text{ exists}\}$. Hence $\{x\colon \lim_n f_n(x) \text{ exists}\}$ is closed subspace of X by the Convergence lemma [13]. Since X is an AD space, then $X = \{x\colon \lim_n f_n(x) \text{ exists}\} = \overline{\phi}$ and then $\lim_n f_n(x)$ exists for all $x \in X$. Thus $v \in X^{\sigma(\lambda)}$. The opposite inclusion is trivial.

(i) $\overline{\phi} \subset X$ by hypothesis. Since $\overline{\phi}$ is AD-space, then

$$X^{\sigma b(\lambda)} \subset \left(\overline{\phi}\right)^{\sigma b(\lambda)} = \left(\overline{\phi}\right)^{\sigma(\lambda)} \subset \left(\overline{\phi}\right)^{f} = X^f$$

by (iii) and [13].

Theorem 2.7. *$z^{\sigma(\lambda)}$ is a C_λ-semiconservative space if and only if $z \in \sigma s(\lambda)$.*

Proof. Let $z^{\sigma(\lambda)}$ be a C_λ-semiconservative space. Then $\left(z^{\sigma(\lambda)}\right)^f \subset \sigma s(\lambda)$. Since $z^{\sigma(\lambda)}$ is a $\sigma K(\lambda)$-space by [4], we have $\left(z^{\sigma(\lambda)}\right)^f = \left(z^{\sigma(\lambda)}\right)^{\sigma(\lambda)}$ by Theorem 2.6 (ii). So since

$$\{z\} \in \left(z^{\sigma(\lambda)}\right)^{\sigma(\lambda)} \subset \sigma s(\lambda),$$

we get $z \in \sigma s(\lambda)$.

Now let $z \in \sigma s(\lambda)$. Then $(\sigma s\,(\lambda))^{\sigma(\lambda)} \subset z^{\sigma(\lambda)}$ and hence

$$\left(z^{\sigma(\lambda)}\right)^{\sigma(\lambda)} \subset (\sigma s(\lambda))^{\sigma(\lambda)\sigma(\lambda)} = \sigma s\,(\lambda) \quad \text{in [5]}.$$

Since $z^{\sigma(\lambda)}$ is a $\sigma K\,(\lambda)$-space, then $\left(z^{\sigma(\lambda)}\right)^f = \left(z^{\sigma(\lambda)}\right)^{\sigma(\lambda)} \subset \sigma s(\lambda)$.

It is clear that $\sigma s(\lambda)$ is not *a* C_λ-semiconservative space. Because $\sigma s\,(\lambda) = e^{\sigma(\lambda)}$ and $e \notin \sigma s(\lambda)$.

Now we get following theorem.

Theorem 2.8. *The intersection of all C_λ-semiconservative FK-spaces is q_0.*

Proof. Let the set of all (C_1-semiconservative) C_λ-semiconservative spaces be $(\Gamma(C_1))\ \Gamma(C_\lambda)$. Since every C_1-semiconservative FK-space is C_λ-semiconservative space we get $\Gamma(C_1) \subset \Gamma(C_\lambda)$. Also

$$\cap\{X\colon X \in \Gamma(C_1)\} \subset \cap\{X\colon X \in \Gamma(C_\lambda)\}.$$

On the other hand Theorem 6 of [8] the intersection of all C_1-semiconservative spaces is q_0. Hence $q_0 \subset \cap\{X\colon X \in \Gamma(C_\lambda)\}$. Therefore, by Theorem 5 of [8] we have

$$q_0 \subset \cap\{X\colon X \in \Gamma(C_\lambda)\} \subset \cap\{z^\sigma\colon z \in \sigma s\} = \sigma s^\sigma = q.$$

Also $\cap\{X\colon X \in \Gamma(C_\lambda)\} \subset c_0$, since c_0 is a C_λ-semiconservative space so

$$\cap\{X\colon X \in \Gamma(C_\lambda)\} \subset q \cap c_0 = q_0,$$

where

$$q := \left\{x\colon \sum_{j=1}^{\infty} j\left|\triangle^2 x_j\right| < \infty \ \text{ and } \ x \in l^\infty\right\} \quad \text{and} \quad q_0 = q \cap c_0.$$

Theorem 2.8 is proved.

3. A relationship between the distinguished subsets and C_λ-semiconservative FK-spaces. In this section we shall now study the subspaces $C_\lambda F$, $C_\lambda F^+$, $C_\lambda B$ and $C_\lambda B^+$ of an FK-space X.

Let X be an FK-space with $\phi \subset X$. Then

$$C_\lambda W := C_\lambda W(X) = \left\{ x \in X \colon \ \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x^{(k)} \to x \ \text{(weakly) in } \ X \right\} =$$

$$= \left\{ x \in X \colon \ f(x) = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j f\left(\delta^j\right) \ \text{ for all } \ f \in X' \right\},$$

$$C_\lambda S := C_\lambda S(X) = \left\{ x \in X \colon \ \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x^{(k)} \to x \right\} =$$

$$= \left\{ x \in X \colon x \ \text{ has } \ \sigma K(\lambda) \ \text{ in } \ X \right\},$$

$$C_\lambda F^+ := C_\lambda F^+(X) = \left\{ x \colon \ \lim_n \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} x_j f\left(\delta^j\right) \ \text{ exists for all } \ f \in X' \right\} =$$

$$= \left\{ x \colon \ \{x_n f(\delta^n)\} \in \sigma s(\lambda) \ \text{ for all } \ f \in X' \right\},$$

$$C_\lambda B^+ := C_\lambda B^+(X) = \left\{ x \colon \ \left\{ \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x^{(k)} \right\} \ \text{ is bounded in } \ X \right\} =$$

$$= \left\{ x \colon \ \{x_n f(\delta^n)\} \in \sigma b(\lambda) \ \text{ for all } \ f \in X' \right\}.$$

Also $C_\lambda F = C_\lambda F^+ \cap X$ and $C_\lambda B = C_\lambda B^+ \cap X$.

We note that subspaces $C_\lambda W$ and $C_\lambda S$ are closely related to C_λ-conullity of the FK-space X (see [4]).

The theorems below gives us some characterizations which are analogous to those given in [13] (Chapter 10).

Theorem 3.1. *Let X be an FK-space with $\phi \subset X$, $z \in s$. Then $z \in C_\lambda F^+$ if and only if $z^{-1}X = \{x \colon zx \in X\}$ is a C_λ-semiconservative FK-space, where $zx = \{x_n z_n\}$, in particular $e \in C_\lambda F^+$ if and only if X is C_λ-semiconservative FK-space.*

Proof. Let $f \in \left(z^{-1}X\right)'$. Then $f(x) = \alpha x + g(zx)$, $\alpha \in \phi$, $g \in Y'$, by [13] and

$$f(\delta^n) = \alpha_n + g(z\delta^n) = \alpha_n + g(z_n\delta^n) = \alpha_n + z_n g(\delta^n).$$

Thus, since $\alpha \in \phi \subset \sigma s(\lambda)$ then $\{f(\delta^n)\} \in \sigma s(\lambda)$ if and only if $\{z_n g(\delta^n)\} \in \sigma s(\lambda)$, i.e., $z \in C_\lambda F^+$.

An FK-space is called bounded convex C_λ-semiconservative space if it is a C_λ-semiconservative space and includes $q(\lambda)$.

Theorem 3.2. *Let X be an FK-space with $\phi \subset X$, $z \in s$. Then $z \in C_\lambda F$ if and only if $z^{-1}X$ is bounded convex C_λ-semiconservative FK-space, in particular $e \in C_\lambda F$ if and only if X is bounded convex C_λ-semiconservative FK-space.*

Proof. Let $z \in C_\lambda F$. Then $z \in X$ so $e \in z^{-1}X$ and since $z \in C_\lambda F^+$, $z^{-1}X$ is a C_λ-semiconservative FK-space by Theorem 3.1. Thus $z^{-1}X$ is a bounded convex C_λ-semiconservative FK-space.

Let $z^{-1}X$ be a bounded convex C_λ-semiconservative FK-space. Then $z^{-1}X$ is C_λ-semiconservative FK-space and $e \in z^{-1}X$ so $z \in X$. Thus since $z \in C_\lambda F^+$ by Theorem 3.1 and $z \in X$, then $z \in C_\lambda F$.

Theorem 3.3. *Let X be an FK-space with $\phi \subset X$, $z \in s$. Then $z \in C_\lambda B^+$ if and only if $q_0(\lambda) \subset z^{-1}X$, in particular $e \in C_\lambda B^+$ if and only if $q_0(\lambda) \subset X$.*

Proof. Let $f \in \left(z^{-1}X\right)'$. Then $f(\delta^n) = \alpha_n + z_n g\left(\delta^n\right)$ by [13]. Hence, since $\alpha \in \phi \subset \sigma s(\lambda)$, then $z \in C_\lambda B^+$ if and only if $\{z_n g\left(\delta^n\right)\} \in \sigma b(\lambda)$, i.e., $z \in C_\lambda B^+$.

Theorem 3.4. *Let X be an FK-space with $\phi \subset X$, $z \in s$. Then $z \in C_\lambda B$ if and only if $q(\lambda) \subset z^{-1}X$, in particular $e \in C_\lambda B$ if and only if $q(\lambda) \subset X$.*

Proof. Let $z \in C_\lambda B$. Then $z \in X$ so $e \in z^{-1}X$ and $z \in C_\lambda B^+$. Thus $z^{-1}X \supset q(\lambda)$ by Theorem 3.3.

Let $z^{-1}X \supset q(\lambda)$, then $z^{-1}X \supset q_0(\lambda)$ and $e \in z^{-1}X$. Thus, since $z \in C_\lambda B^+$ by Theorem 3.3 and $z \in X$, then $z \in C_\lambda B$.

4. Matrix domains. In this section we give simple conditions for the subspaces $C_\lambda B$ and $C_\lambda F$ in the FK-space Y_A, which is depend on the choice of the FK-space Y and the matrix A. Also, we solve the problem of characterizing matrices A such that Y_A is C_λ-semiconservative space for given Y.

The theorems below gives us some results which are analogous to those given in [13] (Chapters 9 and 12).

Theorem 4.1. *Let Y be an FK-space and A be a matrix. Then Y_A is a C_λ-semiconservative space if and only if the columns of A are in Y and $\left\{g(a^k)\right\} \in \sigma s(\lambda)$ for each $g \in Y'$, where a^k is the kth column of A, $a_n^k = a_{nk}$.*

Proof. *Necessity.* The columns of A are in$Y$ since $Y_A \supset \phi$ by definition of C_λ -semiconservative space. Given $g \in Y'$, let $f(x) = g(Ax)$ for $x \in Y_A$, so $f \in Y_A'$ by [13] (Theorem 4.4.2). Then $f(\delta^k) = g(a^k)$ and the result follows since $Y_A^f \subset \sigma s(\lambda)$.

Sufficiency. We first note that each row of A belongs to $\sigma s(\lambda)$ since in the hypothesis we may take $g = P_n$, where $P_n(x) = x_n$. This yields

$$\left\{g(a^k)\right\} = \left\{P_n(a_n^k)\right\} = \{a_{nk}\} \in \sigma s(\lambda), \quad k = 1, 2, 3, \ldots .$$

Hence $s_A \supset (\sigma s(\lambda))^\beta$.

Now let $f \in Y_A'$. Then by Theorem 4.4.2 of [13],

$$f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(Ax) \qquad \text{with} \quad g \in Y',$$

$$\alpha \in s_A^{\beta} = \left\{ x\colon \sum_{n=1}^{\infty} x_n y_n \text{ convergent for all } y \in s_A \right\} \subset (\sigma s(\lambda))^{\beta\beta} = \sigma s(\lambda), \quad \text{in [5].}$$

Thus

$$f(\delta^k) = \alpha_k + g(a^k);$$

by the hypotesis and the fact that $\alpha \in \sigma s\,(\lambda)$ we have $\left\{ f(\delta^k) \right\} \in \sigma s\,(\lambda)$. Thus $Y_A^f \subset \sigma s(\lambda)$ and Y_A is C_λ-semiconservative space.

Theorem 4.2. *If Y_A is C_λ-semiconservative space then $A^T \in \left(Y^\beta, \sigma s(\lambda)\right)$, where A^T denotes transpose of matrix A.*

Proof. Since $Y_A \supset q_0$ by Theorem 2.8 then $A \in (q_0,\ Y)$. Hence

$$A^T \in \left(Y^\beta, q_0^f\right) = \left(Y^\beta, \sigma b\right), \quad \text{where} \quad \sigma b = \left\{ x \in w\colon \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

by [13] (Theorem 8.3.8]. Let $z \in Y^\beta$ and define $g \in Y'$ by $g(y) = zy$ using the Banach–Steinhaus theorem [13], where $zy = \sum_{k=1}^{\infty} z_k y_k$. Let $f(x) = g(Ax)$ so that $f \in Y_A'$ by [13] (Theorem 4.4.2). Hence $\left\{ f(\delta^k) \right\} \in \sigma s(\lambda)$. But

$$f(\delta^k) = \sum_{n=1}^{\infty} z_n a_{nk} = \left(A^T z\right)_k$$

so $\left(A^T z\right) \in \sigma s(\lambda)$.

Theorem 4.3. *Let Y be an FK-space with AK. Then Y_A is C_λ-semiconservative space if and only if the columns of A belong to Y and $A^T \in \left(Y^\beta, \sigma s(\lambda)\right)$.*

Proof. *Necessity* is trivial by Theorem 4.2.

Sufficiency. Let $g \in Y',\ z_n = g\,(\delta^n)$. Then $z \in Y^f = Y^\beta$ by [13] (Theorem 7.2.7), so $\left(A^T z\right) \in \sigma s(\lambda)$. But

$$\left(A^T z\right)_k = \sum_{n=1}^{\infty} z_n a_{nk} = g\left(\sum_{n=1}^{\infty} a_{nk} \delta^n \right) = g(a^k)$$

since Y has AK. Hence we get$g(a^k) \in \sigma s(\lambda)$. Then Y_A is C_λ-semiconservative space by Theorem 4.1.

Definition 4.1. *A matrix A is called C_λ-semiconservative if c_A is C_λ-semiconservative space.*

This definition is given because summability theory deals with spaces of the form c_A and with FK-spaces whose properties generalize those of such spaces. It would be nice if we can extend theorems about conservative spaces to C_λ-semiconservative spaces.

Theorem 4.4. *A is C_λ-semiconservative if and only if*

(i) *a has convergent columns, i.e., $c_A \supset \phi$,*

(ii) *$a \in \sigma s(\lambda)$, where $a = \{a_k\}$, $a_k = \lim_n a_{nk}$,*

(iii) $A^T \in (l, \sigma s(\lambda))$.

Proof. *Necessity*. (i) is by Definition 4.1; to prove (ii) apply Theorem 4.1 with $g := \lim$; (iii) is by Theorem 4.2.

Sufficiency. Let $g \in c'$. Then $g(y) = \chi \lim y + \sum_{n=1}^{\infty} t_n y_n$, $t \in l$ by [13]. If we take $y = Ax$; $x = \delta^k$ in here we obtained $g\left(a^k\right) = \chi \lim a_{nk} + (tA)_k$, where $(tA)_k = \sum_{n=1}^{\infty} t_n a_{nk}$. Since $g\left(a^k\right) \in \sigma s(\lambda)$ from (ii) and (iii) then by Theorem 4.1. the result is obtained.

Theorem 4.5. *The following are equialent for an FK-space X.*

(i) *If $A \in (X,\ X)$ then X_A is C_λ-semiconservative space.*

(ii) *X is C_λ-semiconservative space.*

Proof. (i) implies (ii): Take $A = I$.

(ii) implies (i): If $A \in (X,\ X)$ then $X \subset X_A$, hence X_A is C_λ-semiconservative space by Theorem 2.5.

Theorem 4.6. *Let $z \in s$, Y be an FK-space, and A be a matrix such that $\phi \subset Y_A$ i.e., the columns of A belong to Y. Then the following propositions are equivalent in Y_A:*

(i) $z \in C_\lambda B^+$,

(ii) $\left\{\dfrac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)}\right\}$ *is bounded in Y,*

(iii) $Y_{Az} \supset q_0(\lambda)$ *where the matrix Az is $(a_{nk} z_k)$,*

(iv) $\left\{z_k g(a^k)\right\} \in \sigma b(\lambda)$ *for each $g \in Y'$, where a^k is kth column of A.*

Proof. (i) $\Leftrightarrow$ (iii): $z \in C_\lambda B^+$ if and only if $z^{-1} Y_A \supset q_0(\lambda)$, where

$$z^{-1} Y_A = \{x \colon zx \in Y_A\}, \qquad zx = \{x_n z_n\} \Leftrightarrow Y_{Az} \supset q_0(\lambda)$$

by $z^{-1} Y_A = Y_{Az}$ and Theorem 3.3.

(iii) $\Leftrightarrow$ (iv): Since $q_0(\lambda)$ is AD space and by hypothesis then

$$Y_{Az}^f \subset (q_0(\lambda))^f$$

by [13] (Theorem 8.6.1). Hence $f(\delta^k) = \alpha_k + g(a_n^k z_k)$ for each $f \in Y'_{Az}$ with $\alpha \in s_{Az}^\beta$, $g \in Y'$ by [13] (Theorem 4.4.2). Since

$$\alpha \in s_{Az}^\beta \subset Y_{Az}^\beta \subset \sigma b\,(\lambda)$$

then $\left\{f(\delta^k)\right\} \in \sigma b(\lambda) \Leftrightarrow \left\{z_k g(a^k)\right\} \in \sigma b\,(\lambda)$ for each $g \in Y'$.

(ii) $\Leftrightarrow$ (iv): (iv) is true if and only if

$$\left\{ g\left(\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)}\right)\right\}$$

is bounded for each $g \in Y'$ by [13] (Theorem 8.0.2), where

$$g\left(\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)}\right) = g\left(\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} \sum_{k=1}^{p} a_{nk} z_k\right) = \frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} \sum_{k=1}^{p} z_k g(a_n^k).$$

Theorem 4.7. *Assume that $z \in s$, $(Y, q\ )$ is an FK-space, and A is a matrix such that $\phi \subset Y_A$ i.e., the columns of A belong to Y. Then the following propositions are equivalent in Y_A :*

(i) $z \in C_\lambda F^+$,

(ii) $\left\{ \frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \right\}$ *is weakly Cauchy in* Y, *i.e.,* $\left\{ g\left(\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \right) \right\}$ *is convergent for each* $g \in Y'$,

(iii) Y_{Az} *is* C_λ*-semiconservative space,*

(iv) $\left\{ z_k g(a^k) \right\} \in \sigma s(\lambda)$ *for each* $g \in Y'$.

Proof. (i) $\Leftrightarrow$ (ii): $z \in C_\lambda F^+ \Leftrightarrow z^{-1} Y_A$ is C_λ-semiconservative space if and only if Y_{Az} is C_λ-semiconservative space by Theorem 3.1.

(iii) $\Leftrightarrow$ (ii): Since the k th column of Az is $z_k a^k$ and by Theorem 4.1, this equivalent is trivial.

(iii) $\Leftrightarrow$ (iv): By Theorem 4.1, since the k th column of Az is $z_k a^k$.

Theorem 4.8. *Let* Y *be an* FK*-space such that weakly convergent sequences are convergent in the* FK*-topology, let* A *be a row finite matrix with* $\phi \subset Y_A$. *Then* $C_\lambda S = C_\lambda W = C_\lambda F = C_\lambda F^+$ *in* Y_A.

Proof. If $z \in C_\lambda F^+$, $\left\{ \frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \right\}$ is weakly Cauchy in Y by Theorem 4.7, hence Cauchy [13] (Theorem 12.0.2), hence convergent say $\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \to y$. However $\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \to$ $\to z$ in s_A since this is a $\sigma K(\lambda)$ space because of s_A is an AK space [13]. Thus $\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \to Az$ in s. But $\frac{1}{\lambda(r)} \sum_{p=1}^{\lambda(r)} Az^{(p)} \to y$ in s since Y is an FK-space hence $y = Az$ so $z \in C_\lambda S$ by [4].

1. *Armitage D. H., Maddox I. J.* A new type of Cesáro mean // Analysis. – 1989. – **9**. – P. 195 – 204.
2. *Bosanquet L. S.* Note on convergence and summability factors // J. London Math. Soc. – 1945. – **20**. – P. 39 – 48.
3. *Buntinas M.* Convergent and bounded Cesáro sections in FK-spaces // Math. Z. – 1971. – **121**. – S. 191 – 200.
4. *Dağadur İ.* C_λ-conull FK-spaces // Demonstr. math. – 2002. – **35**. – P. 835 – 848.
5. *Goes G., Goes S.* Sequences of bounded variation and sequences of Fourier coefficients, I // Math. Z. – 1970. – **118**. – S. 93 – 102.
6. *Goes G.* Sequences of bounded variation and sequences of Fourier coefficients, II // J. Math. Anal. and Appl. – 1972. – **39**. – P. 477 – 494.
7. *Goes G.* Summan Von FK-Räumen Funktionale Abstchnıttskkonvergenz und Umkehrsätz // Tohoku Math. J. – 1974. – **26**. – P. 487 – 504.
8. *İnce H. G.* Cesáro semiconservative FK-spaces // Math. Communs. – 2009. – **14**. – P. 157 – 165.
9. *Macphail M. S., Orhan C.* Analysis. – 1989. – **9**. – P. 317 – 322.
10. *Osikiewicz J. A.* Equivalance results for Cesáro submethods // Analysis. – 2000. – **20**. – P. 35 – 43.
11. *Wilansky A.* Modern methods in topological vector spaces. – New York: McGraw Hill, 1978.
12. *Wilansky A.* Functional analysis. – Blaisdell Press, 1964.
13. *Wilansky A.* Summability through functional analysis. – North Holland, 1984.
14. *Snyder A. K., Wilansky A.* Inclusion theorems and semiconservative FK-space // Rocky Mountain J. Math. – 1972. – **2**. – P. 595 – 603.
15. *Zeller K.* Theorie der Limitierungsverfahren. – Berlin etc., 1958.

Received 06.05.10,
after revision — 11.01.12