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IMPULSIVE DIFFERENTIAL INCLUSIONS INVOLVING EVOLUTION OPERATORS IN SEPARABLE BANACH SPACES

ІМПУЛЬСНІ ДИФЕРЕНЦІАЛЬНІ ВКЛЮЧЕННЯ, ЩО МІСТЯТЬ ОПЕРАТОРИ В СЕПАРАБЕЛЬНИХ БАНАХОВИХ ПРОСТОРАХ

We present some results on the existence of mild solutions and study the topological structure of the sets of solutions for the following first-order impulsive semilinear differential inclusions with initial and boundary conditions:

$$\begin{aligned} & y'(t) - A(t)y(t) \in F(t, y(t)), \quad \text{for a.e.} \quad t \in J\backslash\{t_1, \ldots, t_m, \ldots\}, \\ & y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \qquad k = 1, \ldots, \\ & y(0) = a \end{aligned}$$

and

$$\begin{aligned} & y'(t) - A(t)y(t) \in F(t, y(t)), \quad \text{for a.e.} \quad t \in J\backslash\{t_1, \ldots, t_m, \ldots\}, \\ & y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \qquad k = 1, \ldots, \\ & Ly = a, \end{aligned}$$

where $J = \mathbb{R}_+$, $0 = t_0 < t_1 < \ldots < t_m < \ldots; (m \in \mathbb{N})$, $\lim_{k\to\infty} t_k = \infty$, $A(t)$ is the infinitesimal generator of a family of evolution operator $U(t, s)$ on a separable Banach space E, and F is a set-valued mapping. The functions I_k characterize the jump of solutions at the impulse points t_k, $k = 1, \ldots$. The mapping $L\colon PC_b \to E$ is a bounded linear operator. We also investigate the compactness of the set of solutions, some regularity properties of the operator solutions, and the absolute retractness.

Наведено деякі результати про існування м'яких розв'язків та вивчено топологічну будову множин розв'язків для наступних імпульсних напівлінійних диференціальних включень першого порядку з початковими та граничними умовами:

$$\begin{aligned} & y'(t) - A(t)y(t) \in F(t, y(t)) \quad \text{для майже кожного} \quad t \in J\backslash\{t_1, \ldots, t_m, \ldots\}, \\ & y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \qquad k = 1, \ldots, \\ & y(0) = a \end{aligned}$$

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де $J = \mathbb{R}_+$, $0 = t_0 < t_1 < \ldots < t_m < \ldots; (m \in \mathbb{N})$, $\lim_{k\to\infty} t_k = \infty$, $A(t)$ — інфінітезимальний генератор сім'ї операторів еволюції $U(t, s)$ на сепарабельному банаховому просторі E та F — багатозначне відображення. Функції I_k характеризують стрибки розв'язків в точках імпульсної дії t_k, $k = 1, \ldots$. Відображення $L\colon PC_b \to E$ є обмеженим лінійним оператором. Також досліджено компактність множини розв'язків, деякі властивості регулярності операторних розв'язків та абсолютну ретрактність.

1. Introduction. Differential equations with impulses were considered for the first time by Milman and Myshkis [47] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [34]. Many phenomena and evolution processes in the field of physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. (See for instance [2, 42, 43] and the references therein.) These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, chemical technology, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Various mathematical results (existence, asymptotic behavior, ...) have been obtained so far (see [4, 10, 12, 44, 51, 54, 55] and the references therein).

Given a real separable Banach space E with norm $\|\cdot\|$, consider the following problem:

$$\begin{aligned} & y'(t) - A(t)y(t) \in F(t, y(t)), \quad \text{for a.e.} \quad t \in J\backslash\{t_1, \ldots, t_m, \ldots\}, \\ & \Delta y_{t=t_k} = I_k(y(t_k^-)), \qquad\qquad k = 1, \ldots, \\ & y(0) = a \in E, \end{aligned} \tag{1}$$

where $J = \mathbb{R}_+$ $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} \ldots$ $(m \in \mathbb{N})$, $\lim_{k\to\infty} t_k = \infty$. $F\colon J \times E \to \mathcal{P}(E)$ is a multivalued map, $A(t)$ is the infinitesimal generator of a family of evolution $\{U(t, s)\}$. We always assume that the operator $A(t)$ is closed and densely defined in its domain $D(A(t))$, which is independent of t, $I_k \in C(E, E)$, $k = 1, \ldots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of $y(t)$ at $t = t_k$, respectively.

Later, we study the following impulsive boundary-value problems:

$$\begin{aligned} & y'(t) - A(t)y(t) \in F(t, y(t)), \quad \text{for a.e. } t \in J\backslash\{t_1, \ldots, t_m, \ldots\}, \\ & y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \qquad\qquad k = 1, \ldots, \\ & Ly = a, \end{aligned} \tag{2}$$

where $L\colon PC_b \to E$ is a bounded linear operator.

Many properties of solutions for differential equations and inclusions, such as stability or oscillation, require global properties of solutions. This is the main motivation to look for sufficient conditions that ensure global existence of solutions for impulsive differential equations and inclusions. In this direction, some questions have been discussed by Baghli and Benchohra [7 – 9], Graef and Ouahab [26, 28], Guo [30, 31], Guo and Liu [32], Henderson and Ouahab [35 – 37], Marino et al. [46], Ouahab [49, 50], Stamov and Stamova [56], Weng [60], and Yan [61, 62].

In case the space E is finite-dimensional and J is compact interval, some existence results of solutions for problems (1) and (2) in the particular case $Ay = \lambda y$, $\lambda \in \mathbb{R}$, have been obtained in [11, 13, 27]. Very recently some existence results and solutions sets on unbounded interval was studied by Djebali et al. [1, 21, 22]. For infinite dimensional space and A is infinitesimal generator of a C_0-semigroup and $Ly = a - y(0) + y(b)$. Problems of existence and solutions sets of above problems on bounded interval was solved by Djebali et al. [19, 20, 23].

The goal in this work is to complement and extend some recent results to the case of infinite-dimensional spaces; moreover the right-hand side nonlinearity may be either convex or nonconvex. Our approach here is based on a nonlinear alternative for compact u.s.c. maps. Then, we present some existence results and investigate the compactness of solution set, some regularity of operator solutions and absolute retract (in short AR) of solution is also proved.

2. Preliminaries. In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $(E, \|\cdot\|)$ be a separable Banach space, $J = [0, \infty)$ an interval in $\mathbb{R}$ and $C_b(J, E)$ the Banach space of all continuous and bounded functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| \colon t \in J\}.$$

$B(E)$ refers to the Banach space of linear bounded operators from E into E with the norm

$$\|N\|_{B(E)} = \sup\{\|N(y)\| \colon \|y\| = 1\}.$$

A function $y\colon J \to E$ is called measurable provided for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J \colon y(t) \in U\}$ is Lebesgue measurable. A measurable function $y\colon J \to E$ is Bochner integrable if $\|y\|$ is Lebesgue integrable. For properties of the Bochner integral, see, e.g., Yosida [58]. In what follows, $L^1(J, E)$ denotes the Banach space of functions $y\colon J \longrightarrow E$, which are Bochner integrable with norm

$$\|y\|_1 = \int_0^\infty \|y(t)\| dt.$$

Denote by $\mathcal{P}(E) = \{Y \subset E \colon Y \neq \varnothing\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ closed}\}$, $\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ bounded}\}$, $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ convex}\}$, $\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ compact}\}$.

2.1. Multivalued analysis. Let (X, d) and (Y, ρ) be two metric spaces and $G\colon X \to \mathcal{P}_{cl}(Y)$ be a multivalued map. A single-valued map $g : X \to Y$ is said to be a selection of G and we write $g \subset G$ whenever $g(x) \in G(x)$ for every $x \in X$. G is called *upper semicontinuous* (*u.s.c. for short*) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $G^{-1}(V) = \{x \in X,\ G(x) \cap V \neq \varnothing\}$ is closed for any closed set V in Y. Equivalently, G is *u.s.c.* if the set $G^{+1}(V) = \{x \in X,\ G(x) \subset V\}$ is open for any open set V in Y.

G is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $G(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that $G(A) = \bigcup\{G(x),\ x \in A\} \subset K$. G is compact if $G(X)$ is relatively compact. It is called locally compact if, for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $G(U)$ is relatively compact. G is quasicompact if, for each subset $A \subset X$, compact, $G(A)$ is relatively compact.

The following two results are easily deduced from the limit properties.

Lemma 2.1 (see, e.g., [6], Theorem 1.4.13)**.** *If $G\colon X \longrightarrow \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x \to x_0} G(x) = G(x_0).$$

Lemma 2.2 (see, e.g., [6], Lemma 1.1.9)**.** *Let* $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ *be a sequence of subsets where* K *is compact in the separable Banach space* X*. Then*

$$\overline{\mathrm{co}}\,(\limsup_{n\to\infty} K_n) = \bigcap_{N>0} \overline{\mathrm{co}}\,(\bigcup_{n\ge N} K_n),$$

where $\overline{\mathrm{co}}\,C$ *refers to the closure of the convex hull of* C*.*

Definition 2.1. *A multivalued map* $F\colon [0,\infty) \to \mathcal{P}_{cl}(Y)$ *is said measurable provided for every open* $U \subset Y$*, the set* $F^{+1}(U)$ *is Lebesgue measurable.*

We have the following lemma.

Lemma 2.3 [16, 25]**.** *The mapping* F *is measurable if and only if for each* $x \in Y$*, the function* $\zeta\colon J \to [0,+\infty)$ *defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x-y\|\colon y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski – Ryll – Nardzewski selection theorem.

Lemma 2.4 ([25], Theorem 19.7)**.** *Let* Y *be a separable metric space and* $F\colon J \to \mathcal{P}(Y)$ *a measurable multivalued map with nonempty closed values. Then* F *has a measurable selection.*

Lemma 2.5 [20, 63]**.** *Let* $F\colon [0,b] \to \mathcal{P}_{cp}(Y)$ *be a measurable multivalued map and* $u\colon [0,b] \to \to E$ *a measurable function. Then there exists a measurable selection* f *of* F *such that for a.e.* $t \in [0,b]$,

$$|u(t) - f(t)| \le d(u(t), F(t)).$$

We denote the graph of G to be the set $\mathcal{G}r(G) = \{(x,y) \in X \times Y,\ y \in G(x)\}$.

Definition 2.2. *G is closed if* $\mathcal{G}r(G)$ *is a closed subset of* $X \times Y$*, i.e., for every sequences* $(x_n)_{n\in\mathbb{N}} \subset X$ *and* $(y_n)_{n\in\mathbb{N}} \subset Y$*, if* $x_n \to x_*,\ y_n \to y_*$ *as* $n \to \infty$ *with* $y_n \in F(x_n)$*, then* $y_* \in G(x_*)$*.*

We recall the following two results; the first one is classical.

Lemma 2.6 ([18], Proposition 1.2)**.** *If* $G\colon X \to \mathcal{P}_{cl}(Y)$ *is u.s.c., then* $\mathcal{G}r(G)$ *is a closed subset of* $X \times Y$*. Conversely, if* G *is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.7 [20, 40]**.** *If* $G\colon X \to \mathcal{P}_{cp}(Y)$ *is quasicompact and has a closed graph, then* G *is u.s.c.*

Given a separable Banach space $(E, \|\cdot\|)$, for a multivalued map $F\colon J \times E \to \mathcal{P}(E)$, denote

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{\|v\|\colon\ v \in F(t,x)\}.$$

Definition 2.3. *A multivalued map* F *is called a Carathéodory function if*

(a) *the function* $t \mapsto F(t,x)$ *is measurable for each* $x \in E$;

(b) *for a.e.* $t \in J$*, the map* $x \mapsto F(t,x)$ *is upper semicontinuous. Furthermore,* F *is* L^1- *Carathéodory if it is locally integrably bounded, i.e., for each positive* r*, there exists* $h_r \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} \le h_r(t), \ \text{for a.e. } t \in J \text{ and all } \|x\| \le r.$$

For each $x \in C(J,E)$, the set

$$S_{F,x} = \left\{ f \in L^1(J,E) \colon f(t) \in F(t,x(t)) \text{ for a.e. } t \in J \right\}$$

is known as the set of selection functions.

***Remark* 2.1.** (a) For each $x \in C(J,E)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t,x(t))$ is convex for a.e. $t \in J$.

(b) From [38] (see also [45] when E is finite-dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \mapsto \inf\{\|v\| \colon v \in F(t,x(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t,x(t))\|_{\mathcal{P}}$ belongs to $L^1(J)$; this particularly holds true when F is $L^1 -$ Carathéodory. For the sake of completeness, we refer also to Theorem 1.3.5 in [40] which states that $S_{F,x}$ contains a measurable selection whenever x is measurable and F is a Carathéodory function.

Consider the Hausdorff pseudometric $H_d \colon \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ defined by

$$H_d(A,B) = \max\left\{ \sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a \in A} d(a,b)$, $d(a,B) = \inf_{b \in B} d(a,b)$. $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space and $(\mathcal{P}_{b,cl}(E), H_d)$ is a metric space space (see [41]). Also, notice that if $x_0 \in E$, then

$$d(x_0, A) = \inf_{x \in A} d(x_0, x) \quad \text{while} \quad H_d(\{x_0\}, A) = \sup_{x \in A} d(x_0, x).$$

Definition 2.4. *A multivalued operator $N \colon E \to \mathcal{P}_{cl}(E)$ is called*

(a) *γ-Lipschitz if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \textit{for each} \quad x,\ y \in E,$$

(b) *a contraction if it is γ-Lipschitz with $\gamma < 1$.*

Notice that if N is $\gamma-$Lipschitz, then

$$\forall\, x, y \in E \colon H_d(F(x), F(y)) \le \gamma d(x,y).$$

For further readings and details on multivalued analysis, we refer to the books by Andres and Górniewicz [3], Aubin and Cellina [5], Aubin and Frankowska [6], Deimling [18], Górniewicz [25], Hu and Papageorgiou [38, 39], Kamenskii et al. [40], and Tolstonogov [57].

3. Evolution family.

Definition 3.1. *A family of operators $\{U(t,s)\}_{t \ge s} \subset B(E)$, with $t, s \in \mathbb{R}$ or $t, s \in \mathbb{R}_+$, is called an evolution family if satisfying the conditions*:

(i) $U(t,s) = U(t,s) \circ U(s,\tau)$, *for $t \ge s \ge \tau$,*

(ii) $U(t,t) = I$; *here I denotes the identity operator in E,*

(iii) *for each $x \in E$, the function $(t,s) \to U(t,s)x$ is continuous for $t \ge s$.*

In what follows, for the family $\{A(t), t \in J\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions:

(i) the domain $D(A(t))$ is independent of t and is dense in E,

(ii) for $t \ge 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $\operatorname{Re} \lambda \le 0$, and there is a constant M independent λ and t such that

$$\|R(\lambda, A(t))\|_{B(E)} \leq M(1+|\lambda|)^{-1}, \quad \text{for} \quad \operatorname{Re}\lambda \leq 0,$$

(iii) there exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(s))A^{-1}(\theta)\|_{B(E)} \leq L|t-\theta|^{\alpha}, \quad \text{for} \quad t, s, \theta \in J,$$

(iv) the mapping $(s, b] \ni t \to U(t, s) \in B(E)$ is continuous with respect to the uniform operator topology of $B(E)$. Moreover, this continuity is uniform with respect to s lying in sets bounded away from t, i.e., as long as $t - s \geq \beta$ for any fixed $\beta > 0$.

Definition 3.2. *The solution operator $U(t, s)$ is called exponentially bounded if there are constants $L(U) > 0$ and $\omega \geq 0$ such that*

$$\|U(t,s)\|_{B(E)} \leq L(U)e^{-\omega(t-s)}, \quad t, s \geq 0.$$

More details on evolution families can be fond in (see Engel and Nagel [24]) and Pazy [52].

4. Existence results. Consider the Banach space $PC_b = \{y \in PC(J, E)\colon y \text{ is bounded}\}$, where

$$PC(J,E) = \{y\colon J \to E,\ y_k \in C((t_k, t_{k+1}], E),\ k = 0, \ldots,$$

$$y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k) = y(t_k^-) \text{ for } k = 1, \ldots\}$$

and $y_k := y_{|(t_k, t_{k+1}]}$. Endowed with the norm

$$\|y\|_b = \sup\{\|y(t)\|\colon t \in J\},$$

PC_b is a Banach space. Next we define what we mean by a solution to problem (1).

Definition 4.1. *A function $y \in PC$ is said to be a mild solution of problem* (1) *if there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J, and*

$$y(t) = U(t,0)a + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,s)I_k(y(t_k^-)), \quad \textit{for a.e.} \quad t \in J.$$

In what follows assume that the evolution family is exponentially bounded.

5. Existence and compactness of solutions set. In this section, we present a global existence result and prove the compactness of solution set for the problem (1) by using a nonlinear alternative for multivalued maps combined with a compactness argument. The nonlinearity is u.s.c. with respect to the second variable and satisfies a Nagumo growth condition in all this part assume that E is a reflexive Banach space. Hereafter we assume that the Carathéodory multivalued map $F\colon J \times E \to \to \mathcal{P}(E)$ has compact and convex values.

Theorem 5.1. *The impulsive functions $I_k \in C(E, E)$ satisfy*

$(\mathcal{H}_1)$ *there exist $c_k, d_k > 0$ such that*

$$\|I_k(x)\| \leq c_k\|x\| + d_k, \quad \textit{for every} \quad x \in E,\ k = 1, 2, \ldots,$$

with

$$L(U)\sum_{k=1}^{\infty} c_k < 1 \quad \textit{and} \quad \sum_{k=1}^{\infty} d_k < \infty;$$

($\mathcal{H}_2$) *there exist a continuous nondecreasing function* $\psi\colon [0,\infty) \longrightarrow (0,\infty)$ *and* $p \in L^1(J,\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|), \quad \textit{for a.e.} \quad t \in J \quad \textit{and each} \quad x \in \mathbb{R}^n,$$

with

$$\int\limits_0^\infty m(s)ds < \int\limits_c^\infty \frac{du}{\psi(u)},$$

where

$$m(s) = \frac{L(U)p(s)}{1 - L(U)\sum_{k=1}^\infty c_k} \quad \textit{and} \quad c = \frac{L(U)\left(\|a\| + \sum_{k=1}^\infty d_k\right)}{1 - L(U)\sum_{k=1}^\infty c_k};$$

($\mathcal{H}_3$) *for every* $t - s > 0$, $U(t,s)$ *is compact;*

($\mathcal{H}_4$) *for every* $M > 0$ *and* $\varepsilon > 0$ *there exist an* $L^1(J,E)$*–function* $b = b(M,\varepsilon)$ *such that*

$$F(t,y) - b \subset \varepsilon B(0,M), \textit{ for a.a. } t \geq T \textit{ and all } y \in B(0,M),$$

where

$$B(0,M) = \{y \in E\colon \|y\| \leq M\}.$$

Then problem (1) *has at least one solution. Moreover, the solution set* $S(a)$ *is compact and the multivalued map* $S\colon a \to S(a)$ *is u.s.c.*

First, recall the well-known nonlinear alternative of Leray – Schauder for multivalued maps (see, e.g., [29, 25]).

Lemma 5.1. *Let* X *be a Banach space with* $C \subset X$ *a convex. Assume* U *is a relatively open subset of* C *with* $0 \in U$ *and* $G\colon \overline{U} \to \mathcal{P}_{cp,c}(X)$ *be an upper semicontinuous and compact map. Then either,*

(a) *there is a point* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda G(u)$, *or*

(b) G *has a fixed point in* $\overline{U}$.

Definition 5.1. *Let* E *be a Banach space. A sequence* $(v_n)_{n\in\mathbb{N}} \subset L^1([a,b],E)$ *is said to be semicompact if*

(a) *it is integrably bounded, i.e., there exists* $q \in L^1([a,b],\mathbb{R}^+)$ *such that*

$$\|v_n(t)\| \leq q(t), \quad \textit{for a.e.} \quad t \in [a,b] \quad \textit{and every} \quad n \in \mathbb{N},$$

(b) *the image sequence* $(v_n(t))_{n\in\mathbb{N}}$ *is relatively compact in* E *for a.e.* $t \in [a,b]$.

The following important result follows from Dunford – Pettis theorem (see [40], Proposition 4.2.1).

Lemma 5.2. *Every semicompact sequence* $L^1([a,b],E)$ *is weakly compact in* $L^1([a,b],E)$.

When the nonlinearity takes convex values, Mazur's lemma, may be useful:

Lemma 5.3 [58]. *Let* E *be a normed space and* $(x_k)_{k\in\mathbb{N}} \subset E$ *a sequence weakly converging to a limit* $x \in E$. *Then there exists a sequence of convex combinations* $y_m = \sum_{k=1}^m \alpha_{mk}x_k$ *with* $\alpha_{mk} > 0$ *for* $k = 1,2,\ldots,m$ *and* $\sum_{k=1}^m \alpha_{mk} = 1$ *which converges strongly to* x.

The following compactness criterion on unbounded domains is a simple extension of a compactness criterion in $C_b(J,E)$ (see [17, p. 62; 53]).

Lemma 5.4. *A set $\mathcal{M} \subset PC_b$ is relatively compact if it satisfies the following conditions*:

(a) *$\mathcal{M}$ is uniformly bounded in $PC_b(J,E)$*,

(b) *the functions belonging to $\mathcal{M}$ are almost equicontinuous on J, i.e., equicontinuous on every compact interval of J*,

(c) *the functions from $\mathcal{M}$ are equiconvergent, that is, given $\varepsilon > 0$, there exist $T(\varepsilon) > 0$ and $\delta(\epsilon) > 0$ such that if $x, y \in \mathcal{M}$ with $\|x(T) - y(T)\| \leq \delta(\epsilon)$, then $|x(t) - y(t)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in \mathcal{M}$*,

(d) *for every $t \in J$, the set $\{x(t) \colon x \in \mathcal{M}\}$ is relatively compact.*

Proof [Proof of Theorem 5.1].

***Step* 1.** Existence of solutions. Consider the operator $N \colon PC_b \to \mathcal{P}(PC_b)$ defined for $y \in PC_b$ by

$$N(y) = \left\{ h \in PC_b \colon\ h(t) = \begin{cases} U(t,0)a + \int\limits_0^t U(t,s)v(s)ds + \\ + \sum\limits_{0<t_k<t} U(t,t_k)I_k(y(t_k)), \ \text{for a.e. } t \in J, \end{cases} \right\} \tag{3}$$

where $v \in S_{F,y} = \{v \in L^1(J,E) \colon v(t) \in F(t, y(t)),\ \text{a.e. } t \in J\}$. Note that, from [59], Theorem 5.10 or [45], the set $S_{F,y}$ is nonempty if and only if the mapping $t \mapsto \inf\{\|v\| \colon v \in F(t, y(t))\}$ belongs to $L^1(J)$. It is further bounded if and only if the mapping $t \mapsto \|F(t, y(t))\|_{\mathcal{P}}$ belongs to $L^1(J)$; this particularly holds true when F satisfies $(\mathcal{H}_2)$. Moreover, fixed points of the operator N are mild solutions of problem (1). We shall show that N satisfies the assumptions of Lemma 5.1. Finally notice that since $S_{F,y}$ is convex (because F has convex values), then N takes convex values.

***Claim* 1.** $N(PC_b) \subset PC_b$. Indeed, if $y \in PC_b$ and $h \in N(y)$ then there exists $v \in S_{F,y}$ such that

$$h(t) = U(t,0)a + \int\limits_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)), \quad \text{for a.e.} \quad t \in J. \tag{4}$$

Since $v \in L^1(J)$, we have

$$\|h(t)\| \leq L(U)\|a\| + \int\limits_0^t \|F(s, y(s))\|_{\mathcal{P}} ds + L(U) \sum_{0<t_k<t} \|I_k(y(t_k))\| \leq$$

$$\leq L(U)\|a\| + \int\limits_0^t p(s)\psi(\|y(s)\|)ds + L(U) \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k).$$

Hence

$$\|h\|_{PC_b} \leq L(U)\|a\| + L(U)\psi(\|y\|_{PC_b}) \int\limits_0^\infty p(s)ds + L(U)\sum_{k=1}^\infty c_k\|y\|_{PC_b} + \sum_{k=1}^\infty d_k.$$

This shows that N sends bounded sets into bounded sets in PC_b.

***Claim* 2.** N sends bounded sets in PC_b into almost equicontinuous sets of PC_b.

Let $r > 0$, $B_r := \{y \in PC_b \colon \|y\|_\infty \leq r\}$ be a bounded set in PC_b, $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and $y \in B_r$. For each $h \in N(y)$, we have

$$\|h(\tau_2) - h(\tau_1)\| \leq \|U(\tau_1, 0) - U(\tau_2, 0)\|_{B(E)}\|a\| +$$

$$+L(U)\int\limits_{\tau_1}^{\tau_2} \|v(s)\|ds \int\limits_0^{\tau_1} \|U(\tau_1, s) - U(\tau_2, s)\|_{B(E)}\|v(s)\|ds +$$

$$+L(U)\sum_{\tau_1 < t_k < \tau_2} \|I_k(y(t_k))\| \leq$$

$$\leq L(U)\psi(r)\int\limits_{\tau_1}^{\tau_2} p(s)ds + L(U)\sum_{\tau_1 < t_k < \tau_2} (c_k r + d_k).$$

Since $\sum_{k=1}^\infty c_k < \infty$, $\sum_{k=1}^\infty d_k < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, the right-hand term tends to zero as $|\tau_1 - \tau_2| \to 0$, proving equicontinuity for the case where $t \neq t_i^-$, $i = 1, \ldots$. To prove equicontinuity at $t = t_i$ for some $i \in \mathbb{N}^*$, fix $\varepsilon_0 > 0$ such that $\{t_j \colon j \neq i\} \cap [t_i - \varepsilon_0, t_i + \varepsilon_0] = \varnothing$. Then for each $0 < \varepsilon < \varepsilon_0$, we have the estimates

$$\|h(t_i) - h(t_i - \varepsilon)\| \leq \|U(t_i - \epsilon, 0) - U(t_i, 0)\|_{B(E)}\|a\| + \int\limits_{t_i - \varepsilon}^{t_i} \|v(s)\|ds \leq$$

$$\leq \|U(t_i - \epsilon, 0) - U(t_i, 0)\|_{B(E)}\|a\| + \psi(r)\int\limits_{t_i - \varepsilon}^{t_i} p(s)ds.$$

Since $p \in L^1(J, \mathbb{R}^+)$, the right-hand term tends to 0 as $\varepsilon \to 0$. The equicontinuity at t_i^+, $i = 1, \ldots,$ is proved in the same way. Now we to show that N maps B_q into a precompact set in E. Let $0 < t \leq b$ and let $0 < \varepsilon < t$. For $y \in B_q$, define

$$h_\varepsilon(t) = U(t - \varepsilon, 0)a + U(\varepsilon, 0)\int\limits_0^{t-\varepsilon} U(t - \varepsilon, 0)f(s)ds +$$

$$+U(\varepsilon, 0)\sum_{0 < t_k < t - \varepsilon} U(t - \varepsilon, t_k)I_k(y(t_k)).$$

Then

$$\|h(t)-h_\varepsilon(t)\| \leq \int\limits_{t-\varepsilon}^{t} \|U(t,s)\|_{B(E)} p(s)ds + \sum_{t-\varepsilon<t_k<t} \|U(t,t_k)\|_{B(E)}(c_k r + d_k),$$

which tends to 0 as $\varepsilon \to 0$. Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t)\colon h \in N(y)\}$. This set is then precompact in E.

***Claim* 3.** We now show the stability of the set $N(\overline{B}(0,r))$, i.e., we show that for every $\varepsilon > 0$, there exist $T(\varepsilon), \delta(\epsilon) > 0$ such that if $h \in N(x)$ and $h_* \in N(y)$ with $x, y \in B(0,r)$ and $\|h(T(\varepsilon)) - h_*(T(\varepsilon))\| \leq \delta(\epsilon)$, then $\|h(t) - h_*(t)\| \leq T(\varepsilon)$ for every $t \geq T$ and each $h \in N(\overline{B}(0,r))$. Fix $\varepsilon > 0$ and choose $T_1 = T_1(\varepsilon)$ such that

$$F(t,y) - b(t) \subset \frac{\varepsilon\alpha}{8L(U)} B(0,1), \quad \text{for a.a.} \quad t \geq T_1 \quad \text{and all} \quad y \in B(0,1).$$

Since $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$ there exist k_0 such that

$$L(U) \sum_{k=k_0}^{\infty} (c_k r + d_k) \leq \frac{\varepsilon}{8}.$$

From $(\mathcal{H}_2)$ we have

$$F(t,y) \subset p(t)\psi(r)B(0,1), \quad \text{for a.a.} \quad t \in [0,T_2], \quad T_2 = \max(T_1, k_0).$$

We choose $T = T(\varepsilon) \geq T_2$ so large that for $t \geq T$ and $s \leq T_2$ we have

$$\|U(t,s)\|_{B(E)} \leq L(U)e^{-\alpha(T-T_2)} < \frac{\epsilon}{8(\|p\|_{L^1}+1)(\psi(r)+1)\left(r\sum_{k=1}^{\infty} c_k + \sum_{k=1}^{\infty} d_k + 1\right)}.$$

Hence

$$\left\|\sum_{0<t_k<t} U(t,t_k)[I_k(x(t_k)) - I_k(y(t_k))]\right\| \leq 2L(U) \sum_{0<t_k<k_0} e^{-\alpha(T-t_k)}(c_k r + d_k) +$$

$$+ 2L(U) \sum_{k=k_0}^{\infty} (c_k r + d_k).$$

Then

$$\left\|\sum_{0<t_k<t} U(t,t_k)[I_k(x(t_k)) - I_k(y(t_k))]\right\| \leq \frac{\varepsilon}{2}. \tag{5}$$

Letting $h \in N(x)$ and $h_* \in N(y)$ for some $x, y \in \overline{B}(0,r)$, there exist $v_1 \in S_{F,x}$ and $v_2 \in S_{F,y}$ such that

$$h(t) = U(t,0)a + \int\limits_0^t U(t,s)v_1(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k))$$

and

$$h_*(t) = U(t,0)a + \int_0^t U(t,s)v_2(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)).$$

Consequently, for $t \geq T$,

$$\left\| \int_0^t U(t,s)[v_1(s) - v_2(s)]ds \right\| \leq \left\| \int_{[0,t]\cap\{s\leq T_2\}} U(t,s)[v_1(s) - v_2(s)]ds \right\| +$$

$$+ \left\| \int_{[0,t]\cap\{s>T_2\}} U(t,s)[v_1(s) - v_2(s)]ds \right\| \leq$$

$$\leq \left\| \int_{[0,t]\cap\{s\leq T_2\}} U(t,s)[v_1(s) - b(s)]ds \right\| +$$

$$+ \left\| \int_{[0,t]\cap\{s\leq T_2\}} U(t,s)[v_2(s) - b(s)]ds \right\| +$$

$$+ \left\| \int_{[0,t]\cap\{s>T_2\}} U(t,s)[v_1(s) - v_2(s)]ds \right\| \leq \frac{\varepsilon}{4L(U)} \int_0^t L(U)e^{-\alpha(t-s)}ds +$$

$$+ \int_{[0,t]\cap\{s>T_2\}} \|U(t,s)\|_{B(E)}\|v_1(s)\|ds + \int_{[0,t]\cap\{s>T_2\}} \|U(t,s)\|_{B(E)}\|v_2(s)\|ds \leq$$

$$\leq \frac{\varepsilon}{4L(U)} \int_0^t L(U)e^{-\alpha(t-s)}ds + \frac{\varepsilon\|p\|_{L^1}\psi(r)}{4(\|p\|_{L^1}+1)(\psi(r)+1)\left(r\sum_{k=1}^{\infty} c_k + \sum_{k=1}^{\infty} d_k + 1\right)}.$$

Hence

$$\left\| \int_0^t U(t,s)[v_1(s) - v_2(s)]ds \right\| \leq \frac{\varepsilon}{2}. \tag{6}$$

From (5) and (6) we conclude that

$$\|h(t) - h_*(t)\| \leq \varepsilon, \quad \text{for each} \quad t \geq T(\epsilon).$$

Then $N(B(0,r))$ is equiconvergent. With Lemma 5.4 and Claims $1-3$, we conclude that N is completely continuous.

***Claim* 4.** N is u.s.c.

To this end, we show that N has a closed graph. Let $h_n \in N(y_n)$ such that $h_n \longrightarrow h$ and $y_n \longrightarrow y$, as $n \to +\infty$. Then there exists $M > 0$ such that $\|y_n\| \leq M$. We shall prove that $h \in N(y)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that for each $t \in J$

$$h_n(t) = U(t,0)a + \int_0^t U(t,s)v_n(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k)).$$

$(\mathcal{H}_2)$ implies that $v_n(t) \in p(t)\psi(M)B(0,1)$. Then $(v_n)_{n\in\mathbb{N}}$ is integrably bounded in $L^1(J,E)$ by using that E is reflexive space we conclude that $\{v_n\}$ is semicompact. By Lemma 5.2, there exists a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,E)$. Moreover, the mapping $\Gamma\colon L^1(J,E) \to PC_b(J,E)$ defined by

$$\Gamma(g)(t) = \int_0^t U(t,s)g(s)ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [14, 15]. Therefore for a.e. $t \in J$, $y_n(t)$ converge to $y(t)$ and by continuity of I_k, we get

$$h(t) = U(t,0)a + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)).$$

It remains to prove that $v \in F(t,y(t))$, for a.e. $t \in J$. Lemma 5.3 yields the existence of $\alpha_i^n \geq \geq 0$, $i = 1,\ldots,k(n)$, such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\cdot) = = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\cdot)$ converges strongly to v in L^1. Using Lemma 2.2, we obtain that

$$v(t) \in \bigcap_{n\geq 1} \overline{\{g_k(t)\colon k \geq n\}}, \quad \text{for a.e.} \quad t \in J \subset$$

$$\subset \bigcap_{n\geq 1} \overline{\text{co}}\{v_k(t),\ k \geq n\} \subset$$

$$\subset \bigcap_{n\geq 1} \overline{\text{co}}\{\bigcup_{k\geq n} F(t,y_k(t))\} = \overline{\text{co}}(\limsup_{k\to\infty} F(t,y_k(t))). \tag{7}$$

However, the fact that the multivalued $x \to F(.,x)$ is u.s.c. and has compact values together with Lemma 2.1 imply that

$$\limsup_{n\to\infty} F(t,y_n(t)) = F(t,y(t)), \quad \text{for a.e.} \quad t \in J.$$

This with (7) yield that $v(t) \in \overline{\text{co}}\, F(t,y(t))$. Finally $F(.,.)$ has closed, convex values, hence $v(t) \in \in F(t,y(t))$, for a.e. $t \in J$. Thus $h \in N(y)$, proving that N has a closed graph. Finally, with Lemma 2.7 and the compactness of N, we conclude that N is u.s.c.

***Claim* 5.** A priori bounds on solutions.

Let $y \in PC_b$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda U(t,0)a + \lambda \int_0^t U(t,s)v(s)ds + \lambda \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \quad \text{for a.e.} \quad t \in J.$$

Arguing as in Claim 1, we get

$$\|y(t)\| \leq L(U)\|a\| + L(U) \int_0^t p(s)\psi(\|y(s)\|)ds + L(U) \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k), \quad \text{for a.e.} \quad t \in J.$$

Letting $\alpha(t) = \sup\{\|y(s)\| \colon s \in [0,t]\}$ and using the increasing character of ψ, we get

$$\alpha(t) \leq L(U)\|a\| + L(U) \int_0^t p(s)\psi(\alpha(s))ds + L(U) \sum_{0<t_k<t} (c_k\alpha(t) + d_k).$$

Hence

$$\alpha(t) \leq \frac{L(U)}{1 - L(U)\sum_{k=1}^{\infty} c_k} \left(\|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^{\infty} d_k \right).$$

Denoting by $\beta(t)$ the right-hand side of the above inequality, we have

$$\|y(t)\| \leq \alpha(t) \leq \beta(t), \quad t \in J,$$

as well as

$$\beta(0) = \frac{L(U)\left(\|a\| + \sum_{k=1}^{\infty} d_k\right)}{1 - L(U)\sum_{k=1}^{\infty} c_k},$$

and

$$\beta'(t) = \frac{L(U)p(t)\psi(\alpha(t))}{1 - L(U)\sum_{k=1}^{\infty} c_k} \leq \frac{L(U)p(t)\psi(\beta(t))}{1 - L(U)\sum_{k=1}^{\infty} c_k}.$$

From $(\mathcal{H}_1)$, this implies that for $t \in J$

$$\Gamma(z(t)) = \int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq \frac{L(U)}{1 - L(U)\sum_{k=1}^{\infty} c_k} \int_0^{\infty} p(s)ds < \int_{\beta(0)}^{\infty} \frac{ds}{\psi(s)} = \Gamma(+\infty).$$

Thus

$$\beta(t) \leq \Gamma^{-1}\left(\frac{L(U)\|p\|_{L^1}}{1 - L(U)\sum_{k=1}^{\infty} c_k} \right), \quad \text{for every} \quad t \in J,$$

where $\Gamma(z) = \displaystyle\int_{\beta(0)}^{z} \frac{du}{\psi(u)}$. As a consequence

$$\|y\|_{PC_b} \leq \Gamma^{-1}\left(\frac{L(U)\|p\|_{L^1}}{1 - L(U)\sum_{k=1}^{\infty} c_k}\right) := \widetilde{M}.$$

Finally, let

$$U := \{y \in PC_b \colon \|y\|_{PC_b} < \widetilde{M} + 1\}$$

and consider the operator $N \colon \overline{U} \to \mathcal{P}_{cv,cp}(PC_b)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the multivalued version of the nonlinear alternative of Leray – Schauder (Lemma 5.1), N has a fixed point y in U which is a solution of problem (1).

***Step* 2.** Compactness of the solution set.

For each $a \in E$, let

$$S(a) = \{y \in PC_b \colon y \text{ is a solution of problem } (1)\}.$$

From Step 1, there exists $\widetilde{M}$ such that for every $y \in S(a)$, $\|y\|_{PC_b} \leq \widetilde{M}$. Since N is completely continuous, $N(S(a))$ is relatively compact in PC_b. Let $y \in S(a)$; then $y \in N(y)$ hence $S(a) \subset \overline{N(S(a))}$. It remains to prove that $S(a)$ is a closed subset in PC_b. Let $\{y_n \colon n \in \mathbb{N}\} \subset S(a)$ be such that $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists v_n such that $v_n(t) \in \in F(t, y_n(t))$, a.e. $t \in J$ and

$$y_n(t) = U(t,0)a + \int_0^t U(t,s)v_n(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k)). \tag{8}$$

Arguing as in Claim 4, we can prove that there exists v such that $v(t) \in F(t, y(t))$ and

$$y(t) = U(t,0)a + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)), \quad \text{for a.e.} \quad t \in J.$$

Therefore $y \in S(a)$ which yields that $S(a)$ is closed, hence compact subset in PC_b. Finally, we prove that $S(.)$ is u.s.c. by proving that the graph of $S(.)$

$$\Gamma_S := \{(a,y) \colon \ y \in S(a)\}$$

is closed. Let $(a_n, y_n) \in \Gamma_S$ be such that $(a_n, y_n) \to (a, y)$ as $n \to \infty$. Since $y_n \in S(a_n)$, there exists $v_n \in L^1(J,E)$ such that

$$y_n(t) = U(t,0)a_n + \int_0^t U(t,s)v_n(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k)), \quad t \in J.$$

Arguing as in Claim 4, we can prove that there exists $v \in S_{F,y}$ such that

$$y(t) = U(t,0)a + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)), \quad \text{for a.e.} \quad t \in J.$$

Thus, $y \in S(a)$. Now, we show that $S(.)$ maps bounded sets into relatively compact sets of PC_b. Let B be a compact set in E and let $\{y_n\} \subset S(B)$. Then there exists $\{a_n\} \subset B$ such that

$$y_n(t) = U(t,0)a_n + \int_0^t U(t,s)v_n(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y_n(t_k)), \quad t \in J,$$

where $v_n \in S_{F,y_n},\ n \in \mathbb{N}$. Since $\{a_n\} \subset B$ is a bounded sequence and B is compact set, there exists a subsequence of $\{a_n\}$ converging to a. As in Claims 2, 3, we can show that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous on every compact of J and equiconvergent at ∞. As a consequence of Lemma 5.4, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. By a similar argument of Claim 4, we can prove that

$$y(t) = U(t,0)a + \int_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k)), \quad \text{for a.e.} \quad t \in J,$$

where $v \in S_{F,y}$. Thus, $y \in \overline{S(B)}$. This implies that $S(.)$ is u.s.c., ending the proof of Theorem 5.1.

***Remark* 5.1.** $(\mathcal{R}_1)$ We can replace the compactness of $U(.,.)$ by $F(J \times E)$ is compact.

$(\mathcal{R}_2)$ We can replace the reflexivity of E by assuming that there exist $P_M : J \to \mathcal{P}_{wkc}(E)$ such that

$$F(t,x) \subset P_M(t), \quad \text{for a.a.} \quad t \in J.$$

6. Existence result in Fréchet space. In this section, we prove existence of solutions under Hausdorff–Lipschitz conditions. Let E be a Fréchet space with the topology generated by a family of seminorms $\|\cdot\|_n$. First, we start with the following definition.

Definition 6.1. *A multivalued map $F : E \to E$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$, there exists $k_n \in (0,1)$ such that*

(a) *$H_{d_n}(F(x), F(y)) \le k_n|x-y|_n$ for all $x, y \in E$, where H_d is the Hausdorff distance,*

(b) *for every $x \in E$ and every $\varepsilon > 0$, there exists $y \in F(x)$ such that*

$$\|x-y\|_n \le d_n(x, F(x)) + \varepsilon, \quad \textit{for every} \quad n \in \mathbb{N}.$$

A subset $A \subset E$ is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|x|_n \le M_n$, for every $x \in A$. Our main tool will be the following nonlinear alternative of Frigon for multivalued contractions [33]:

Lemma 6.1. *Let E be a Fréchet space, $U \subset E$ an open neighborhood of the origin, and let $N : \overline{U} \to E$ be a bounded admissible multivalued contraction. Then either one of the following statements holds*:

(a) *N has a fixed point,*

(b) *there exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

We now present our second existence result for problem (1).

Theorem 6.1. *Let $(E, \|\cdot\|)$ be a Banach space. Suppose the multivalued map $F : J \times E \to$ $\to \mathcal{P}_{cp}(E)$ is such that $t \to F(t,.)$ is a measurable and*

$(\mathcal{HF}_3)$ *for each $k = 1, 2, \ldots$, there exist $l_k \in L^1(J_k, \mathbb{R}^+)$ such that*

$$H_d(F(t,x),F(t,y)) \leq l_k(t)\|x-y\|, \quad \textit{for} \quad x,y \in E \quad \textit{and a.e.} \quad t \in J_k$$

and

$$F(t,0) \subset l_k(t)\overline{B}(0,1), \quad \textit{for a.e.} \quad t \in J_k,$$

$(\mathcal{HF}_4)$ $\sum_{k=1}^{\infty} \|I_k(0)\| < \infty$ *and there exist constants* $c_k \geq 0$ *such that* $L(U)\sum_{k=1}^{\infty} c_k < 1$ *and*

$$\|I_k(x) - I_k(y)\| \leq c_k\|x-y\|, \quad \textit{for each} \quad x,y \in E.$$

Then problem (1) *has at least one mild solution.*

Here and hereafter $J_k = [0, t_k]$.

***Remark* 6.1.** (a) Note that $(\mathcal{HF}_4)$ implies $(\mathcal{H}_1)$ with $d_k = \|I_k(0)\|$.

(b) $(\mathcal{HF}_3)$ implies that the nonlinearity F has at most linear growth

$$\|F(t,x)\|_{\mathcal{P}} \leq l_k(t)(1+\|x\|), \quad l_k \in L^1(J_k, \mathbb{R}^+), \quad \text{for a.e.} \quad t \in J_k, \quad x \in E$$

and thus $(\mathcal{H}_2)$ is satisfied. However, F is not Carathéodory and may take nonconvex values.

Proof. We begin by defining a family of seminorms on PC, thus rendering PC a Fréchet space. Let τ be a sufficiently large real parameter, say

$$\frac{1}{\tau} + L(U)\sum_{k=1}^{\infty} c_k < 1.$$

For each $n \in \mathbb{N}$, define in PC the seminorms

$$\|y\|_n = \sup\{e^{-\tau L_n(t)}\|y(t)\| \colon 0 \leq t \leq t_n\},$$

where

$$L_n(t) = \int\limits_0^t l_n(s)ds.$$

Thus $PC = \bigcap_{n\geq 1} PC_n$, where $PC_n = \{y \colon J_n \to E$ such that y is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist and $y(t_k^-) = y(t_k)$, $k = 1, 2, \ldots, n-1\}$. Then PC is a Fréchet space with the family of seminorms $\{\|\cdot\|_n\}$. In order to transform problem (1) into a fixed point problem, we define the operator $N \colon PC \to \mathcal{P}(PC)$ by

$$N(y) = \left\{ h \in PC \colon h(t) = U(t,0)a + \int\limits_0^t U(t,s)v(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \ \text{a.e. } t \in J \right\},$$

where $v \in S_{F,y} = \{v \in L^1_{\text{loc}}(J,E) \colon v(t) \in F(t,y(t)), \ t \in J\}$. Clearly, the fixed points of the operator N are solutions of problem (1). We use the Frigon nonlinear alternative to prove that N has a fixed point.

***Step* 1.** A priori estimates.

Given $t \in J_n$, let $y \in \lambda N(y)$ for some $\lambda \in (0,1]$. Then there exists $v \in S_{F,y}$ such that

$$\|y(t)\| \leq L(U)\|a\| + L(U)\int\limits_0^t \|v(s)\|ds + L(U)\sum_{0<t_k<t}\|I_k(y(t_k))\| \leq$$

$$\leq L(U)\|a\| + L(U)\int\limits_0^t l_n(s)(1+\|y(s)\|)ds +$$

$$+L(U)\sum_{k=1}^n c_k\|y(t_k^-)\| + L(U)\sum_{k=1}^n \|I_k(0)\|.$$

Consider the function μ defined on J_n by

$$\mu(t) = \sup\{\|y(s)\|\colon 0 \leq s \leq t\}.$$

By the previous inequality, we have for $t \in J_n$

$$\mu(t) \leq \frac{L(U)}{1 - L(U)\sum_{k=1}^n c_k}\left(\|a\| + \sum_{k=1}^n \|I_k(0)\| + \int\limits_0^t l_n(s)(1+\mu(s))ds\right).$$

Let us take the right-hand side of the above inequality as $\beta(t)$. Then we have

$$\beta(0) = \frac{L(U)(\|a\| + \sum_{k=1}^\infty \|I_k(0)\|)}{1 - L(U)\sum_{k=1}^n c_k} = c,$$

$$\mu(t) \leq \beta(t), \quad t \in J_n,$$

and

$$\beta'(t) = \frac{L(U)l_n(t)(1+\mu(t))}{1 - \sum_{k=1}^\infty c_k} \leq \frac{L(U)l_n(t)(1+\beta(t))}{1 - L(U)\sum_{k=1}^\infty c_k}, \quad t \in J_n.$$

Integrating we get

$$\int\limits_{\beta(0)}^{\beta(t)} \frac{ds}{1+s} \leq \frac{L(U)}{1 - L(U)\sum_{k=1}^\infty c_k}\int\limits_0^{t_n} l_n(s)ds =: M_n.$$

Hence $\beta(t) \leq K_n := (1+\beta(0))e^{M_n}$ and as a consequence

$$\|y(t)\| \leq \mu(t) \leq \beta(t) \leq K_n, \quad t \in J_n.$$

Therefore

$$\|y\|_n \leq K_n \quad \forall\, n \in \mathbb{N}^*.$$

Let

$$U = \{y \in PC\colon \|y\|_n < K_n + 1, \ \text{for all } n \in \mathbb{N}\}.$$

Clearly that U is open set in PC.

***Step* 2.** We show $N\colon \overline{U} \to \mathcal{P}_{cl}(PC)$ is an admissible multivalued contraction, where $U \subset PC$ is some open subset. Firstly we prove that there exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_n, \quad \text{for each} \quad y, \overline{y} \in PC_n.$$

Let $y, \overline{y} \in PC_n$ and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h(t) = U(t,0)a + \int_0^t U(t,s)v(s)\,ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \quad \text{for a.e.} \quad t \in J_n.$$

$(\mathcal{HF}_3)$ implies that

$$H_d(F(t,y(t)), F(t,\overline{y}(t)) \leq l_n(t)\|y(t) - \overline{y}(t)\|, \quad \text{for a.e.} \quad t \in J_n.$$

Hence, there is some $w \in F(t, \overline{y}(t)$ such that

$$\|v(t) - w\| \leq l_n(t)\|y(t) - \overline{y}(t)\|, \quad t \in J_n.$$

Consider the multivalued map $U_n\colon J_n \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$U_n(t) = \{w \in F(t, y(t))\colon \|v(t) - w\| \leq l_n(t)\|y(t) - \overline{y}(t)\|, \;\; \text{for a.e.} \;\; t \in J_n\}.$$

Then $U_n(t)$ is a nonempty set and Theorem III.4.1 in [16] tells us that U_n is measurable. Moreover, the multivalued intersection operator $V_n(.) = U_n(.) \cap F(., \overline{y}(.))$ is measurable. Therefore, by Lemma 2.5, there exists a function $t \mapsto \overline{v}_n(t)$, which is a measurable selection for V_n, that is $\overline{v}_n(t) \in$ $\in F(t, \overline{y}(t))$ and

$$\|v(t) - \overline{v}_n(t)\| \leq l_n(t)\|y(t) - \overline{y}(t)\|, \quad \text{for a.e.} \quad t \in J_n.$$

Define $\overline{h}$ by

$$\overline{h}(t) = U(t,0)a + \int_0^t U(t,s)\overline{v}_n(s)\,ds + \sum_{0<t_k<t} U(t,t_k)I_k(\overline{y}(t_k^-)).$$

Then we have, for $t \in J_n$,

$$\|h(t) - \overline{h}(t)\| \leq L(U)\int_0^t \|v(s) - \overline{v}_n(s)\|\,ds + L(U)\sum_{0<t_k<t} \|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))\| \leq$$

$$\leq L(U)\int_0^t l_n(s)\|y(s) - \overline{y}(s)\|ds + L(U)\sum_{0<t_k<t} c_k\|y(t_k) - \overline{y}(t_k)\| \leq$$

$$\leq \int_0^t L(U)l_n(s)e^{\tau L_n(s)}e^{-\tau L_n(s)}\|y(s) - \overline{y}(s)\|ds +$$

$$+ \sum_{0<t_k<t} c_k e^{\tau L_n(t)}e^{-\tau L_n(t)}\|y(t_k) - \overline{y}(t_k)\| \leq$$

$$\leq \int_0^t l_n(s)e^{\tau L_n(s)}ds\|y-\overline{y}\|_n + L(U)\sum_{0<t_k<t} c_k e^{\tau L(t)}\|y-\overline{y}\|_n \leq$$

$$\leq \int_0^t \frac{1}{\tau}(e^{\tau L_n(s)})' ds\|y-\overline{y}\|_n + \sum_{k=1}^n c_k e^{\tau L_n(t)}\|y-\overline{y}\|_n \leq$$

$$\leq e^{\tau L_n(t)}\left(\frac{1}{\tau}+\sum_{k=1}^n c_k\right)\|y-\overline{y}\|_n.$$

Thus

$$e^{-\tau L_n(t)}\|h(t)-\overline{h}(t)\| \leq \left(\frac{1}{\tau}+L(U)\sum_{k=1}^n c_k\right)\|y-\overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at the estimate

$$H_{d_n}(N(y),N(\overline{y})) \leq \left(\frac{1}{\tau}+L(U)\sum_{k=1}^n c_k\right)\|y-\overline{y}\|_n.$$

In addition, since F is compact valued, we can prove that N has compact values too. Let $x\in\overline{U}$ and $\varepsilon>0$. If $x\notin N(x)$, then $d_n(x,N(x))\neq 0$. Since $N(x)$ is compact, there exists $y\in N(x)$ such that $d_n(x,N(x))=\|x-y\|_n$ and we have

$$\|x-y\|_n \leq d_n(x,N(x))+\varepsilon.$$

If $x\in N(x)$, then we may take $y=x$. Therefore N is an admissible operator contraction.

Clearly, U is a open subset of PC and there is no $y\in\partial U$ such that $y\in\lambda N(y)$ and $\lambda\in(0,1)$. By Lemma 6.1 and Steps 1, 2, N has at least one fixed point y solution to problem (1).

In this part we prove that if we work in Banach space and F is globally Lipschitz, the solution set of the problem (2) is an absolute retract.

Definition 6.2. *A space X is called an absolute retract* (*in short $X\in AR$*) *provided that for every space Y, every closed subset $B\subseteq Y$ and any continuous map $f\colon B\to X$, there exists a continuous extension $\widetilde{f}\colon Y\to X$ of f over Y, i.e., $\widetilde{f}(x)=f(x)$ for every $x\in B$. In other words, for every space Y and for any embedding $f\colon X\longrightarrow Y$, the set $f(X)$ is a retract of Y.*

Proposition 6.1 [48]**.** *Let C be a closed, convex subset of a Banach space E and let $N\colon C\to$ $\to\mathcal{P}_{cp,cv}(C)$ be a contraction multivalued map. Then* Fix (N) *is a nonempty, compact AR-space.*

Our contribution is the following theorem.

Theorem 6.2. *Let $F\colon J\times E\to\mathcal{P}_{cp,cv}(E)$ be multivalued. Assume the following conditions*:

$(\overline{\mathcal{A}}_1)$ *the function $F\colon J\times E\to\mathcal{P}_{cp}(E)$ satisfies:*

for fixed y, the multifunction $t\mapsto F(t,y)$ is measurable,

$(\overline{\mathcal{A}}_2)$ *there exists $p\in L^1(\mathbb{R}_+,(0,\infty))$ such that*

$$H_d(F(t,z_1),F(t,z_2)) \leq p(t)\|z_1-z_2\|, \quad \textit{for all} \quad z_1,z_2\in E$$

and

$$F(t,0) \subset p(t)B(0,1), \quad \textit{for a.e.} \quad t \in J,$$

are satisfied.

Then the solution set $S(a) \in AR$.

Proof. It is clear that all solutions of problem (1) are fixed points of the multivalued operator N defined in Theorem 5.1. Using the fact that $F(.,.)$ has convex and compact values and by $(\mathcal{HF}_4)$,$(\overline{\mathcal{A}}_1)$ $(\overline{\mathcal{A}}_2)$, for every $y \in \Omega$ we have $N(y) \in \mathcal{P}_{cv,cp}(\Omega)$. By some Bielecki-type norm on Ω we can prove that N is a contraction. Hence, from Proposition 6.1, the solution set $S(a) = \operatorname{Fix}(N)$ is a nonempty, compact AR-space.

7. Boundary-value problem on unbounded interval. The following conditions will be needed in the sequel

$(\mathcal{C}_1)$ The operator L_*, defined by

$$L_* x = L(U(.,0)x), \quad x \in E,$$

has bounded inverse $L_*^{-1} \colon E \to E$.

Lemma 7.1. *Let $f \colon J \to E$ be a continuous function. If $y \in PC$ is a solution of the problem*

$$\begin{aligned} & y'(t) - A(t)y(t) = f(t), \qquad \textit{for a.e.} \quad t \in J, \\ & y(t_k^-) - y(t_k) = I_k(y(t_k)), \qquad t \neq t_k, \quad k = 1, \ldots, \\ & Ly = a, \end{aligned} \tag{9}$$

then it is given by

$$y(t) = U(t,0)L_*^{-1}\left(a - \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)) - \int_0^t U(t,s)f(s)ds\right) +$$

$$+ \int_0^t U(t,s)f(s))ds + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \quad \textit{for} \quad t \in J. \tag{10}$$

Proof. Let y be a solution of problem (9), then

$$y(t) = U(t,0)y(0) + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k) + \int_0^t U(t,s)f(s)ds.$$

Since $L_* y = L(U(.,0)y)$ and $Ly = a$ thus

$$L(U(t,0)y(0) + \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k) + \int_0^t U(t,s)f(s)ds) = a.$$

Hence

$$L_*\left(y(0)+\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k)+\int\limits_0^t U(t,s)f(s)ds\right)=a.$$

This implies that

$$y(0)=L_*^{-1}\left(a-\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k)-\int\limits_0^t U(t,s)f(s)ds\right).$$

Hence we obtain

$$y(t)=U(t,0)L_*^{-1}\left(a-\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k))-\int\limits_0^t U(t,s)f(s)ds\right)+$$

$$+\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k))+\int\limits_0^t U(t,s)f(s)ds, \quad \text{for} \quad t\in J,$$

proving the lemma. This lemma leads us to the definition of a mild solution.

Definition 7.1. *A function $y\in PC_b$ is said to be a mild solution of problem* (2) *if there exists $f\in L^1(J,E)$ such that $f(t)\in F(t,y(t))$ a.e. on J, and*

$$y(t)=U(t,0)L_*^{-1}\left(a-\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k))-\int\limits_0^t U(t,s)f(s)ds\right)+$$

$$+\int\limits_0^t U(t,s)f(s))ds+\sum_{0<t_k<t}U(t,t_k)I_k(y(t_k^-)), \quad \textit{for} \quad t\in J.$$

Consider the following assumptions:

$(\mathcal{B}_1)$ Let E be reflexive Banach space and $F\colon J\times E\longrightarrow\mathcal{P}_{cp,cv}(E)$ is an integrably bounded multivalued map, i.e., there exists $p\in L^1(J,E)$ such that

$$\|F(t,x)\|_{\mathcal{P}}\le p(t), \quad \text{for every} \quad x\in E \quad \text{and a.e.} \quad t\in J.$$

$(\mathcal{B}_2)$ There exist constants $a_k,b_k>0$ and $\alpha\in[0,1)$ such that

$$\|I_k(x)\|\le a_k\|x\|^\alpha+b_k \quad \text{for every} \quad x\in E, \quad k=1,\ldots,$$

with

$$\sum_{k=1}^\infty a_k<\infty, \quad \sum_{k=1}^\infty b_k<\infty.$$

One can relax assumption $(\mathcal{B}_1)$ with the following sublinear growth condition:

$(\mathcal{B}_1')$ there exist $p,q\in L^1(J,\mathbb{R}_+)$ and $\beta\in[0,1-\alpha)$ such that

$$\|F(t,x)\|_{\mathcal{P}}\le q(t)+p(t)\|x\|^\beta, \quad \text{for every} \quad x\in E \quad \text{and a.e.} \quad t\in J.$$

Theorem 7.1. *Assume $F\colon J \times E \to \mathcal{P}_{cp,cv}(E)$ is a Carathéodory map satisfying $(\mathcal{H}_3), (\mathcal{H}_4)$, $(\mathcal{C}_1)$ and $(\mathcal{B}_1)$ – $(\mathcal{B}_3)$. Then problem* (2) *has at least one solution. If further E is a reflexive space, then the solution set is compact in PC.*

Proof. It is clear that all solutions of problem (2) are fixed points of the multivalued operator $N_1\colon PC_b \to \mathcal{P}(PC_b)$ defined by

$$N_1(y) := \left\{ h \in PC_b \colon h(t) = \begin{cases} U(t,0)L_*^{-1}\left(a - \sum\limits_{0<t_k<t} U(t,t_k)I(y(t_k^-)) - \int\limits_0^t U(t,s)f(s)ds\right) + \\ + \int\limits_0^t U(t,s)f(s))ds + \sum\limits_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \ \text{for } t \in J \end{cases} \right\},$$

where $f \in S_{F,y}$. Notice that the set $S_{F,y}$ is nonempty (see Remark 2.1 (b)). Since, for each $y \in PC_b$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and therefore N_1 has convex values. As in Theorem 5.1 we can prove that the operator N_1 is completely continuous and u.s.c. Now we prove only the priori bounded of solution for the problem (2). Let $y \in PC_b$ such that $y \in N_1(y)$. Then there exists $f \in S_{F,y}$ such that

$$y(t) = \lambda U(t,0)L_*^{-1}\left(a - \sum_{0<t_k<t}^{\infty} U(t,t_k)I(y(t_k)) - \int\limits_0^t U(t,s)f(s)ds\right) +$$

$$+\lambda \int\limits_0^t U(t,s)f(s))ds + \lambda \sum_{0<t_k<t} U(t,t_k)I_k(y(t_k^-)), \quad \text{for } t \in J, \ \lambda \in (0,1).$$

Then

$$\|y(t)\| \le L^2(U)\|L_*^{-1}\|_{B(E)}\left(\sum_{k=1}^{\infty}(a_k\|y(t_k)\|^{\alpha} + b_k) + \int\limits_0^b \|f(s)\|ds\right) +$$

$$+L(U)\int\limits_0^t \|f(s)\|ds + L(U)\sum_{k=1}^{\infty}(a_k\|y(t_k^-)\|^{\alpha} + b_k),$$

and so

$$\|y\|_{PC} \le L(U)\|L_*^{-1}\|_{B(E)}\left(\sum_{k=1}^{\infty}(a_k\|y\|_{PC}^{\alpha} + b_k) + \|p\|_{L^1}\right) +$$

$$+L(U)\|p\|_{L^1} + L(U)\sum_{k=1}^{m}(a_k\|y\|_{PC}^{\alpha} + b_k).$$

If $\|y\|_{PC} > 1$, then since $0 \le \alpha < 1$, we have

$$\|y\|_{PC}^{1-\alpha} \le L(U)\|L_*^{-1}\|_{B(E)}\left(\sum_{k=1}^{\infty}(a_k + b_k) + \|p\|_{L^1}\right) +$$

$$+L(U)\|p\|_{L^1} + L(U\sum_{k=1}^{\infty}(a_k + b_k).$$

Hence

$$\|y\|_{PC} \le \left(L^2(U)\|L_*^{-1}\|_{B(E)} \left(\sum_{k=1}^{m}(a_k + b_k) + \|p\|_{L^1} \right) + \right.$$

$$\left. + L(U)\|p\|_{L^1} + L(U)\sum_{k=1}^{\infty}(a_k + b_k) \right)^{\frac{1}{1-\alpha}} := \overline{M}.$$

Therefore

$$\|y\|_{PC} \le \max(1, \overline{M}) := \widetilde{M}.$$

Let

$$U := \{y \in PC_b \colon \|y\|_{PC} < \widetilde{M} + 1\},$$

and consider the operator $N\colon \overline{U} \to \mathcal{P}_{cv,cp}(PC_b)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N_1(y)$ for some $\lambda \in (0, 1)$. As a consequence of the Leray – Schauder nonlinear alternative (Lemma 5.1), we deduce that N has a fixed point y in U which is a solution of problem (2). Also by the sam method used in Theorem 5.1 we can prove that the solution set of the problem (2) is compact. In this part we present existence result of problem (2) with right-hand side not necessarily convex.

Theorem 7.2. *Assume that the conditions* $(\mathcal{HF}_3)$, $(\mathcal{HF}_4)$ *are satisfied. If*

$$L(U)(\|L_*^{-1}\|_{B(E)}L(U) + 1)\sum_{k=1}^{\infty} c_k < 1,$$

then the problem (2) *has at least one solution.*

Proof. By the same method as used in Theorem 6.1 we can prove that N_1 has at least one fixed point which is solution of the problem (2) in PC_b.

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