UDC 517.9

Z. Kamont (Techn. Univ. Gdańsk, Poland)

INFINITE SYSTEMS OF HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

НЕСКІНЧЕННІ СИСТЕМИ ГІПЕРБОЛІЧНИХ ФУНКЦІОНАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We consider initial problems for infinite systems of first order partial functional differential equations. The unknown function is the functional argument in equations and the partial derivations appear in a classical sense. A theorem on the existence of a solution and continuous dependence upon initial data is proved. The Cauchy problem is transformed into a system of functional integral equations. The existence of a solution of this system is proved by using integral inequalities and the iterative method. Infinite differential systems with deviated argument and differential integral systems can be derived from a general model by specializing given operators.

Розглядаються початкові задачі для нескінченних систем функціональних диференціальних рівнянь першого порядку з частинними похідними. Доведено теорему про існування розв'язку та неперервну залежність від початкових даних. Задачу Коші трансформовано у систему функціональних інтегральних рівнянь. Існування розв'язку цієї системи доведено за допомогою інтегральних нерівностей та ітераційного методу. Нескінченні диференціальні системи з аргументом, що відхиляється, та диференціальні інтегральні системи можна отримати із загальної моделі шляхом спеціалізації заданих операторів.

1. Introduction. For any metric spaces U and V we denote by $C(U, V)$ the class of all continuous functions from U to V. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $E_0 = [-r_0, 0] \times R^n$, $E = [0, a] \times R^n$ and $B = [-r_0, 0] \times [-r, r] \subset R^{1+n}$ where $r_0 \in R_+$, $R_+ = [0, +\infty)$, $a > 0$ and $r = (r_1, \ldots, r_n) \in R^n_+$. Given a function $v : E_0 \cup E \to R$ and a point $(t, x) \in E$. We define a function $v_{(t,x)} : B \to R$ by $v_{(t,x)}(\tau, y) = v(t + \tau, x + y)$, $(\tau, y) \in B$. The function $v_{(t,x)}$ is the restriction of v to the set $[t - r_0, t] \times [x - r, x + r]$ and this restriction is shifted to the set B. Let Q be an arbitrary set of indices and

$$X = \left\{p = \{p_k\}_{k \in Q} : p_k \in R \text{ for } k \in Q \text{ and } |p| = \sup\{|p_k| : k \in Q\} < +\infty\right\}.$$

We use the same symbol $|\cdot|$ to denote the absolute value of a real number and the norm in the Banach space X. For $x = (x_1, \ldots, x_n) \in R^n$ we put $\|x\| = |x_1| + \ldots + |x_n|$. Let $\zeta = \{\zeta_{[k]}\}_{k \in Q}$ where $\zeta_{[k]} = (\zeta_{1,k}, \ldots, \zeta_{n,k}) \in R^n$ for $k \in Q$. For simplicity of notation we write

$$\|\zeta\| = \sup\left\{\|\zeta_{[k]}\| : k \in Q\right\}.$$

For a function $z = \{z_k\}_{k \in Q}$, $z_k : E_0 \cup E \to R$, and for a point $(t, x) \in E$ we denote $z_{(t,x)} = \{(z_k)_{(t,x)}\}_{k \in Q}$.

We will use the symbol $|\cdot|_0$ to denote the supremum norm in the space $C(B, R)$. Let $C^0(B, X)$ be the space of continuous functions $w : B \to X$, $w = \{w_k\}_{k \in Q}$, with the finite supremum norm

$$\|w\|_0 = \sup\left\{|w_k|_0 : k \in Q\right\}.$$

Let $\psi = (\psi_0, \psi') = (\psi_0, \psi_1, \ldots, \psi_n)$ be a given function where $\psi_0 : [0, a] \to [0, a]$, $\psi' : E \to R^n$. Write $\psi(t, x) = (\psi_0(t), \psi'(t, x))$. Set $\Omega = E \times C^0(B, X) \times R^n$ and suppose that the functions

$$f = \{f_k\}_{k\in Q}, \quad f_k : \Omega \to R, \quad \text{and} \quad \varphi = \{\varphi_k\}_{k\in Q}, \quad \varphi_k : E_0 \to R,$$

are given. We consider the system of functional differential equations

$$\partial_t z_k(t,x) = f_k(t, x\, z_{\psi(t,x)}, \partial_x z_k(t,x)), \quad k \in Q, \tag{1}$$

where $\partial_x z_k = (\partial_{x_1} z_k, \ldots, \partial_{x_n} z_k)$, with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0. \tag{2}$$

Note that $z_{(t,x)} = \left\{(z_k)_{(t,x)}\right\}_{k\in Q}$ is the restriction of z to the set $[t-r_0, t] \times [x-r, x+r]$ and this restriction is shifted to the set B. Then

$$z_{\psi(t,x)} = \left\{(z_k)_{(\psi_0(t), \psi'(t,x))}\right\}_{k\in Q}$$

denotes the restriction of z to the set $[\psi_0(t)-r_0, \psi(t)] \times [\psi'(t,x)-r, \psi'(t,x)+r] \subset R^{1+n}$ with a domain shifted to B.

Now we give examples of systems (1). Given the functions

$$F = \{F_k\}_{k\in Q}, \quad F_k : E \times X \times R^n \to R, \quad \varphi = \{\varphi_k\}_{k\in Q}, \quad \varphi_k : E_0 \to R, \tag{3}$$

and

$$\alpha = \{\alpha_k\}_{k\in Q}, \quad \alpha_k : [0,a] \to R,$$

$$\beta = \{\beta_{[k]}\}_{k\in Q}, \quad \beta_{[k]} = (\beta_{1,k}, \ldots, \beta_{n,k}) : E \to R^n.$$

Suppose that $\psi_0(t) = t$ for $t \in [0,a]$, $\psi'(t,x) = x$ for $(t,x) \in E$ and

$$f_k(t,x,w,q) = F_k(t, x, w(\alpha_k(t)-t, \beta_{[k]}(t,x)-x), q) \quad \text{on} \quad \Omega \tag{4}$$

where $k \in Q$. Then (1) is equivalent to the infinite system of differential equations with a deviated argument

$$\partial_t z_k(t,x) = F_k(t, x, z(\alpha_k(t), \beta_{[k]}(t,x)), \partial_x z_k(t,x)), \quad k \in Q. \tag{5}$$

Now we consider differential integral systems. Suppose that the function $\psi = (\psi_0, \psi')$ satisfies the conditions: $\psi_0 \in C([0,a], R)$ and $0 \le \psi_0(t) \le t$ for $t \in [0, a]$ and $\psi' = (\psi_1, \ldots, \psi_n) \in C(E, R^n)$. Write

$$D[t,x] = \left\{(\tau, y) \in R^{n+1} : \psi_0(t) - r_0 \le \tau \le \psi_0(t),\ \psi'(t,x) - r \le y \le \psi'(t,x) + r\right\},$$

where $(t,x) \in E$. Let F and φ be given by (3) and

$$f_k(t,x,w,q) = F_k\left(t, x, \int_B w(\tau,y)\,d\tau\,dy,\ q\right), \quad k \in Q, \tag{6}$$

where

$$\int_B w(\tau,y)\,d\tau\,dy = \left\{\int_B w_k(\tau,y)\,d\tau\,dy\right\}_{k\in Q}.$$

It is easily seen that

$$f_k(t, x, z_{\psi(t,x)}, q) = F_k\left(t, x, \int_{D[t,x]} z(\tau,y)\,d\tau\,dy,\ q\right), \quad k \in Q,$$

where

$$\int_{D[t,x]} z(\tau, y)\,d\tau\,dy = \left\{ \int_{D[t,x]} z_k(\tau, y)\,d\tau\,dy \right\}_{k\in Q}.$$

Then (1) is equivalent to the infinite system of differential integral equations

$$\partial_t z(t,x) = F_k\left(t, x, \int_{D[t,x]} z(\tau, y)\,d\tau\,dy,\ \partial_x z_k(t,x) \right), \quad k \in Q. \tag{7}$$

Existence results for systems (5) and (7) with initial condition (2) are given in Section 6.

We will consider weak solutions of problem (1), (2). A function $v = \{v_k\}_{k\in Q}$, v_k: $[-b_0, c]\times R^n \to R$, where $0 < c \le a$, is a solution of the above problem provides

(i) $v_{(t,x)} \in C^0(B, X)$ for $(t,x)\in[0,c]\times R^n$ and the derivatives $\partial_x v = \{\partial_x v_k\}_{k\in Q}$ exist on $[0,c]\times R^n$;

(ii) for each $k\in Q$ and $x\in R^n$ the function $v_k(\cdot, x): [0,c]\to R$ is absolutely continuous on $[0,c]$;

(iii) for every $k\in Q$ and $x\in R^n$ equation (1) is satisfied for almost all $t\in[0,c]$ and condition (2) holds.

In the paper we prove that under suitable assumptions on f, ψ and φ there exists a weak solution of (1), (2) which is local with respect to t.

In this time numerous papers were published concerning various problems for first order partial functional differential equations. The following questions were considered: functional differential inequalities generated by initial or mixed problems and their applications, existence theory of classical or weak solutions of equations of finite systems with initial or initial boundary conditions, approximate solutions of functional differential problems. It is not our aim to show a full review of papers concerning the above problems. We consider the questions of the existence of solutions only.

Classical solutions of initial problems have been considered in [1 – 3]. Existence results presented in these papers are based on a method of successive approximations which was introduced by T. Ważewski for systems without functional dependence [4]. Nonlinear equations with first order partial derivatives have the following property: any classical solution exists locally with respect to t. This leads in a natural way to weak or generalized solutions. Weak solutions of differential integral equations have been studied in [5]. Existence results to quasilinear functional differential hyperbolic systems in the second canonical form and for Carathéodory solutions can be found in [6]. The method of bicharacteristics and functional integral inequalities are used. Existence results given in [7] are also based on the theory of bicharacteristics. Solutions are local with respect to the first variable. Nonlinear equations with a functional dependence and Carathéodory solutions of initial problems have been considered in [8, 9]. The case when the unknown function depends on two variables has been investigated. The solutions are global with respect to both variables. A difference method has been used for obtaining the existence results. The general case is not solved.

Nonlinear equations and solutions in the Cinquini – Cibrario sense have been considered in [10, 11]. Existence results are based on a method of linearization which was introduced and widely studied in nonfunctional setting by S. Cinquini and M. Cibrario [12, 13]. Existence results based on iterative methods for parabolic functional differential equations can be found in [14 – 16].

For further bibliography concerning existence results for functional differential equations or finite systems see [11, 17].

Infinite systems of first order partial differential functional equations were first trated in [18, 19]. The paper [19] deals with an infinite system of weakly coupled differential equations. This means that every equation contains the vector of unknown functions and the derivatives of only one function. For initial problems the following questions have been discussed: error bound for approximate solutions, uniqueness of the solutions and its continuous dependence on the right-hand sides of the system and on the initial functions. The results are obtained under the assumptions that given functions satisfy the Lipschitz condition in a suitable function space. The existence result for the Cauchy problem related to an infinite system of first order functional differential equations has been proved in [18]. The iterative method is used an classical solutions have been considered. Cinquini – Cibrario solutions have been obtained in [20] for a class of infinite systems of functional differential equations with initial conditions. The method of bicharacteristics is used.

Existence results for infinite systems of parabolic functional differential equations can be found in [21].

The aim of the paper is to prove a theorem on the local existence of the weak solution and continuous dependence upon initial function for initial problem (1), (2). We use the method of bicharacteristics. The Cauchy problem is transformed into an infinite system of functional integral equations. We prove the existence of the solution of this system by using the iterative method and simple results on integral inequalities. As a consequence of the main theorem we obtain existence theorems for infinite systems with a retarded argument and for differential integral systems.

The paper is a continuation of [1] and [10] and it gives a generalization of results of papers [1, 3, 18, 20].

2. Function spaces. Let $L([0,a],R_+)$ be the class of all functions $\gamma\colon [0,a]\to R_+$ which are integrable on $[0,a]$. We will denote by $C^1(B,X)$ the set of all functions $w: B\to X$, $w=\{w_k\}_{k\in Q}$, of the variables $(\tau,y)=(\tau,y_1,\ldots,y_n)$ such that $w\in C^0(B,X)$, the derivatives

$$\partial_y w = \{\partial_y w_k\}_{k\in Q}, \qquad \partial_y w_k = (\partial_{y_1} w_k,\ldots,\partial_{y_n} w_k),$$

exist, $\partial_y w_k\in C(B,R^n)$ for $k\in Q$ and

$$\|w\|_1 = \|w\|_0 + \sup\left\{\left\|\partial_y w(\tau,y)\right\|: (\tau,y)\in B\right\} < +\infty.$$

Let $C^{1.L}(B,X)$ be the class of all functions $w\in C^1(B,X)$ such that

$$\|w\|_L = \sup\left\{\frac{\left\|\partial_y w(\tau,y)-\partial_y w(\tau,\bar y)\right\|}{\|y-\bar y\|}: (\tau,y),(\tau,\bar y)\in B,\ y\neq\bar y\right\} < +\infty.$$

We define $\|w\|_{1,L}=\|w\|_1+\left\|\partial_y w\right\|_L$ where $w\in C^{1.L}(B,X)$.

Given $s=(s_0,s_1,s_2)\in R_+^3$, we denote by $C^{1.L}[E_0,s]$ the set of all functions $\varphi\in C(E_0,X)$, $\varphi=\{\varphi_k\}_{k\in Q}$, such that

(i) the partial derivatives $\partial_x\varphi(t,x)=\{\partial_x\varphi_k(t,x)\}_{k\in Q}$ exist and $\partial_x\varphi_k\in C(E_0,R^n)$ for $k\in Q$;

(ii) for (t,x), $(t,\bar x)\in E_0$ we have

$$|\varphi(t,x)| \le s_0, \qquad \|\partial_x\varphi(t,x)\| \le s_1,$$

$$\|\partial_x\varphi(t,x)-\partial_x\varphi(t,\bar x)\| \le s_2\|x-\bar x\|.$$

Let $\varphi\in C^{1.L}[E_0,s]$ be given and let $0<c\le a$, $d=(d_0,d_1,d_2)$, $\lambda=(\lambda_0,\lambda_1)$ where $d_i\ge s_i$ for $i=0,1,2$, and $\lambda_i\in L([0,a],R_+)$ for $i=0,1$. We denote by $C^{1.L}_{\varphi.c}[d,\lambda]$ the set of all functions $z:[-r_0,c]\times R^n\to X$, $z=\{z_k\}_{k\in Q}$, such that $z(t,x)=\varphi(t,x)$ on E_0, $z\in C([-r_0,c]\times R^n,X)$ and

(i) there exists $\partial_x z(t,x) = \{\partial_x z_k(t,x)\}_{k\in Q}$ on $[0,c]\times R^n$;

(ii) $|z(t,x)| \le d_0$ and $\|\partial_x z(t,x)\| \le d_1$ on $[-r_0,c]\times R^n$;

(iii) for $t,\ \bar t \in [0,c]$, $x,\ \bar x \in R^n$ we have

$$|z(t,x) - z(\bar t,x)| \le \left|\int_t^{\bar t} \lambda_0(\tau)d\tau\right|,$$

$$\|\partial_x z(t,x) - \partial_x z(\bar t,\bar x)\| \le \left|\int_t^{\bar t} \lambda_1(\tau)d\tau\right| + d_2\|x-\bar x\|.$$

Let $\varphi \in C^{1,L}[E_0,s]$ be given and let $p=(p_0,p_1)\in R_+^2$ and $\mu \in L([0,c],R_+)$. We denote by $C^{0,L}_{\partial\varphi,c}[p,\mu]$ the class of all functions

$$u = \{u_{[k]}\}_{k\in Q}, \quad u_{[k]} = (u_{1,k},\ldots,u_{n,k}) : [-r_0,c]\times R^n \to R^n,$$

such that $u_{[k]}(t,x) = \partial_x\varphi_k(t,x)$ on E_0 for $k\in Q$ and

(i) $\|u(t,x)\| \le p_0$ on $[0,c]\times R^n$;

(ii) for (t,x), $(\bar t,\bar x)\in[0,c]\times R^n$ we have

$$\|u(t,x) - u(\bar t,\bar x)\| \le \left|\int_t^{\bar t} \mu(\tau)d\tau\right| + p_1\|x-\bar x\|.$$

Write $E_t = [-r_0,c]\times R^n$ where $0\le t\le a$. For a function $z\in C(E_t,X)$, $z=\{z_k\}_{k\in Q}$, we put

$$\|z\|_{0,t} = \sup\{|z(\tau,y)| : (\tau,y)\in E_t\}.$$

For a function $u=\{u_{[k]}\}_{k\in Q}$, $u_{[k]}\in C(E_t,R^n)$, and for $0\le t\le c$ we put

$$\|u\|_t = \sup\{\|u(\tau,y)\| : (\tau,y)\in E_t\}.$$

Suppose that $z\in C(E_t,X)$ and the derivatives $\partial_x z = \{\partial_x z_k\}_{k\in Q}$ exist and $\partial_x z_k \in C(E_t,R^n)$. Then we put

$$\|z\|_{1,t} = \|z\|_{0,t} + \|\partial_x z\|_t$$

where $0\le t\le a$. For $t=0$ we will write $\|z\|_0$ instead of $\|z\|_{0,0}$.

We will prove that under suitable assumptions of f, ψ, and φ and for sufficiently small c, $0<c\le a$, there exists a solution $\tilde z$ of problem (1), (2) such that $\tilde z \in C^{1,L}_{\partial\varphi,c}[d,\lambda]$ and $\partial_x\tilde z \in C^{0,L}_{\partial\varphi,c}[p,\mu]$.

3. Bicharacteristics of nonlinear systems. Write $\Omega_I = E\times C^1(B,X)\times R^n$ and $\Omega_{I,L} = E\times C^{1,L}(B,X)\times R^n$. Denote by Ξ the set of all functions $\alpha : [0,a]\times R_+ \to R_+$ such that $\alpha(\cdot,t)\in L([0,a],R_+)$ for $t\in R_+$ and the function $\alpha(t,\cdot)\in C(R_+,R_+)$ is nondecreasing on R_+ for almost all $t\in[0,a]$. We will need the following assumptions.

Assumption $H[\partial_q f]$. Suppose that

1) the derivatives

$$\partial_q f = \{\partial_q f_k\}_{k\in Q}, \quad \partial_q f_k = (\partial_{q_1} f_k,\ldots,\partial_{q_n} f_k),$$

exist on Ω_I and $\partial_q f_k(\cdot, x, w, q) \in L([0, a], R^n)$ for $k \in Q$, $(x, w, q) \in R^n \times C^1(B, X) \times R^n$;

2) there exist functions α, $\gamma \in \Xi$ such that

$$\|\partial_q f(t, x, w, q)\| \leq \alpha(t, \|w\|_1) \quad \text{on} \quad \Omega_I$$

and

$$\|\partial_q f(t, \bar{x}, w+h, \bar{q}) - \partial_q f(t, x, w, q)\| \leq \gamma(t, \|w\|_{1,L})[\|x - \bar{x}\| + \|h\|_1 + \|q - \bar{q}\|]$$

where $(t, x, w, q) \in \Omega_{I,L}$, $\bar{x}, \bar{q} \in R^n$, and $h \in C^1(B, X)$.

Assumption $H[\psi]$. Suppose that the function $\psi = (\psi_0, \psi')$ satisfies the conditions:

1) $\psi_0 \in C([0, a], R)$ and $0 \leq \psi_0(t) \leq t$ for $t \in [0, a]$ and $\psi' \in C(E, R^n)$;

2) the derivatives

$$\partial_x \psi' = \left[\partial_{x_j} \psi_i\right]_{i,j=1,\ldots,n}$$

exist on E;

3) there are $a_0, a_1 \in R_+$ and $\kappa \in L([0, a], R_+)$ such that

$$|\psi_0(t) - \psi_0(\bar{t})| \leq \left|\int_t^{\bar{t}} \kappa(\tau) d\tau\right| \quad \text{for} \quad t, \bar{t} \in [0, a],$$

and

$$\|\partial_x \psi'(t, x)\| \leq a_0, \quad \|\partial_x \psi'(t, x) - \partial_x \psi'(\bar{t}, \bar{x})\| \leq \left|\int_t^{\bar{t}} \kappa(\tau) d\tau\right| + a_1 \|x - \bar{x}\|$$

on E.

Suppose that

$$\varphi \in C^{1.L}[E_0, s], \quad z \in C^{1.L}_{\varphi,c}[d, \lambda] \quad \text{and} \quad u \in C^{0.L}_{\partial\varphi,c}[p, \mu], \quad u = \{u_{1,k}, \ldots, u_{n,k}\}_{k \in Q}.$$

Let $k \in Q$ be fixed and consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f_k(\tau, \eta(\tau), z_{\psi(\tau,\eta(\tau))}, u_{[k]}(\tau, \eta(\tau))), \quad \eta(t) = x, \tag{8}$$

and denote by $g_k[z, u_{[k]}](\cdot, t, x)$ its solution. The function $g_k[z, u_{[k]}]$ is the k-th bicharacteristic of system (1) corresponding to $(z, u_{[k]})$. The set of all bicharacteristics of (1) corresponding to $(z, u) \in C^{1.L}_{\varphi,c}[d, \lambda] \times C^{0.L}_{\partial\varphi,c}[p, \mu]$ we denote by $g[z, u] = \{g_k[z, u_{[k]}]\}_{k \in Q}$.

Lemma 1. *Suppose that Assumptions* $H[\partial_q f]$ *and* $H[\psi]$ *are satisfied and let*

$$\varphi, \bar{\varphi} \in C^{1.L}[E_0, s], \quad z \in C^{1.L}_{\varphi,c}[d, \lambda], \quad \bar{z} \in C^{1.L}_{\bar{\varphi},c}[d, \lambda],$$

$$u \in C^{0.L}_{\partial\varphi,c}[p, \mu], \quad \bar{u} \in C^{0.L}_{\partial\bar{\varphi},c}[p, \mu],$$

where $0 < c \leq a$, *be given. Then the solutions* $g[z, u](\cdot, t, x)$ *and* $g[\bar{z}, \bar{u}](\cdot, t, x)$ *exist on* $[0, c]$, *they are unique and we have the estimates*

$$\|g[z, u](\tau, t, x) - g[z, u](\tau, \bar{t}, \bar{x})\| \leq \left[\int_t^{\bar{t}} \alpha(\xi, \bar{d}) d\xi + \|x - \bar{x}\|\right] \exp\left[\bar{d}\left|\int_t^{\tau} \gamma(\xi, |d|) d\xi\right|\right] \tag{9}$$

where $|d| = d_0 + d_1 + d_2$, $\tilde{d} = d_0 + d_1$, $\bar{d} = 1 + (d_1 + d_2)a_0 + p_1$ *and*

$$\|g[z,u](\tau,t,x) - g[\bar{z},\bar{u}](\tau,t,x)\| \le$$

$$\le \left|\int_t^\tau \gamma(\xi,|d|)\left[\|z-\bar{z}\|_{1,\xi} + \|u-\bar{u}\|_\xi\right]d\xi\right| \exp\left[\bar{d}\left|\int_t^\tau \gamma(\xi,|d|)d\xi\right|\right] \qquad (10)$$

where (τ,t,x), $(\tau,\bar{t},\bar{x}) \in [0,c]\times[0,c]\times R^n$.

Proof. Our proof starts with the observation that

$$\left\|z_{(\tau,y)} - z_{(\tau,\bar{y})}\right\|_1 \le (d_1+d_2)\|y-\bar{y}\|, \qquad (11)$$

$$\left\|z_{(\tau,y)}\right\|_1 \le \tilde{d}, \quad \left\|z_{(\tau,y)}\right\|_{1,L} \le |d|, \qquad (12)$$

and

$$\left\|z_{(\tau,y)} - \bar{z}_{(\tau,\bar{y})}\right\|_1 \le (d_1+d_2)\|y-\bar{y}\| + \|z-\bar{z}\|_{1,\tau}, \qquad (13)$$

$$\|u(\tau,y) - \bar{u}(\tau,\bar{y})\| \le p_1\|y-\bar{y}\| + \|u-\bar{u}\|_\tau, \qquad (14)$$

where (τ,y), $(\tau,\bar{y}) \in [0,c]\times R^n$. The existence and uniqueness of the solution of Cauchy problem (8) follows from classical theorems. On this purpose note that the right-hand side od the differential system satisfies the Carathéodory assumptions and the following Lipschitz condition holds on $[0,c]\times R^n$:

$$\left\|\partial_q f_k(\tau,y,z_{\psi(\tau,y)},u(\tau,y)) - \partial_q f_k(\tau,\bar{y},z_{\psi(\tau,\bar{y})},u(\tau,\bar{y}))\right\| \le \bar{d}\gamma(\tau,|d|)\|y-\bar{y}\|.$$

We conclude from Assumptions $H[\partial_q f]$, $H[\psi]$ and (8), (11) – (14) that the integral inequality

$$\|g[z,u](\tau,t,x) - g[z,u](\tau,\bar{t},\bar{x})\| \le \left|\int_t^{\bar{t}} \alpha(\xi,\tilde{d})d\xi\right| + \|x-\bar{x}\| +$$

$$+ \bar{d}\left|\int_t^\tau \gamma(\xi,|d|)\|g[z,u](\xi,t,x) - g[z,u](\xi,\bar{t},\bar{x})\|d\xi\right|$$

is satisfied. By virtue of Gronwall inequality estimate (9) follows. According to Assumptions $H[\partial_q f]$, $H[\psi]$ and (8), (11) – (14), we have that

$$\|g[z,u](\tau,t,x) - g[\bar{z},\bar{u}](\tau,t,x)\| \le \left|\int_t^\tau \gamma(\xi,|d|)\left[\|z-\bar{z}\|_{1,\xi} + \|u-\bar{u}\|_\xi\right]d\xi\right| +$$

$$+ \left|\bar{d}\int_t^\tau \gamma(\xi,|d|)\|g[z,u](\xi,t,x) - g[\bar{z},\bar{u}](\xi,t,x)\|d\xi\right|.$$

From Gronwall inequality we deduce (10) and Lemma 1 is proved.

4. Integral functional equations. We denote by $CL(B,X)$ the set of all linear and continuous functions defined on $C^0(B,X)$ and taking values in X. We endow $CL(B,X)$ by the usual norm $\|\cdot\|_*$. We formulate further assumptions on f.

Assumption $H[f,\partial_x f,\partial_w f]$. Suppose that

1) for each $k \in Q$ and $(x,w,q) \in R^n \times C^0(B,X) \times R^n$ the function $f_k(\cdot,x,w,q) : [0,a] \to R$ is measurable and there is $\tilde{\gamma} \in \Xi$ such that $\|f(t,x,w,q)\| \le \tilde{\gamma}(t,\|w\|_0)$ on Ω;

2) there exist on Ω_I the partial derivatives

$$\partial_x f = \{\partial_x f_k\}_{k\in Q}, \quad \partial_x f_k = (\partial_{x_1} f_k, \ldots, \partial_{x_n} f_k),$$

and the Fréchet derivatives $\partial_w f = \{\partial_w f_k\}_{k\in Q}$;

3) for $(t, x, w, q) \in \Omega_I$ we have the estimates

$$\|\partial_x f(t,x,w,q)\| \le \alpha(t, \|w\|_1), \quad \|\partial_x f(t,x,w,q)\|_* \le \alpha(t, \|w\|_1),$$

4) for $(t, x, w, q) \in \Omega_{I,L}$ and for $\bar{x}$, $\bar{q} \in R^n$, $h \in \Omega_I$ we have the estimates

$$\|\partial_x f(t,\bar{x}, w+h, \bar{q}) - \partial_x f(t,x,w,q)\| \le \gamma(t, \|w\|_{1,L})[\|x-\bar{x}\| + \|h\|_1 + \|q-\bar{q}\|]$$

and

$$\|\partial_w f(t,\bar{x}, w+h, \bar{q}) - \partial_w f(t,x,w,q)\|_* \le \gamma(t, \|w\|_{1,L})[\|x-\bar{x}\| + \|h\|_1 + \|q-\bar{q}\|].$$

***Remark* 1.** For simplicity of notations we have assumed the same estimation for the derivatives $\partial_x f$, $\partial_w f$, $\partial_q f$. We have assumed also the Lipschitz condition for these derivatives with the same coefficient.

Now we formulate an infinite system of integral functional equations wich are generated by (1), (2). Given $\varphi \in C^{1,L}[E_0, s]$, $z \in C^{1,L}_{\varphi,c}[d,\lambda]$, $u \in C^{0,L}_{\partial\varphi,c}[p,\mu]$ where $0 < c \le a$ and $u = \{u_{1,k}, \ldots, u_{n,k}\}_{k\in Q}$. Let $1 \le i \le n$ be fixed. Set $u_{i,Q} = \{u_{i,k}\}_{k\in Q}$. For simplicity of notation we write also

$$\delta_k(\tau,t,x) = (\tau, g_k[z, u_{[k]}](\tau,t,x))$$

and

$$P_k(\tau,t,x) = (\delta_k(\tau,t,x), z_{\psi(\delta_k(\tau,t,x))}, u_{[k]}(\delta_k(\tau,t,x))).$$

Let us denote by F_k and $G_{i,k}$, $1 \le i \le n$, $k \in Q$, the operators given by

$$F_k[z,u](t,x) = \varphi_k(\delta_k(0,t,x)) +$$
$$+ \int_0^t \left[f_k(P_k(\tau,t,x)) - \sum_{l=1}^n \partial_{q_l} f_k(P_k(\tau,t,x)) u_{l,k}(\delta_k(\tau,t,x)) \right] d\tau,$$

and

$$G_{i,k}[z,u](t,x) = \partial_{x_i} \varphi_k(\delta_k(0,t,x)) +$$
$$+ \int_0^t \{\partial_{x_i} f_k(P_k(\tau,t,x)) + \partial_w f_k(P_k(\tau,t,x)) w_{i,k}[u]\} d\tau,$$

where

$$w_{i,k}[u] = \sum_{j=1}^n \partial_{x_i} \psi_j(\delta_k(\tau,t,x)) (u_{j,Q})_{\psi(\delta_k(\tau,t,x))}$$

and $k \in Q$. Moreover we put

$$G_{[k]}[z,u](t,x) = (G_{1,k}[z,u](t,x), \ldots, G_{n,k}[z,u](t,x)), \quad k \in Q,$$

and

$$F[z,u](t,x) = \{F_k[z,u](t,x)\}_{k\in Q}, \quad G[z,u](t,x) = \{G_{[k]}[z,u](t,x)\}_{k\in Q}.$$

We will consider the following system of integral functional equations

$$z = F[z,u], \quad u = G[z,u] \tag{15}$$

with the initial conditions

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0, \qquad u(t,x) = \partial_x\varphi(t,x) \quad \text{on} \quad E_0. \tag{16}$$

***Remark* 2.** Integral functional system (15) is obtained in the following way. We introduce first an additional unknown function $u = \partial_x z$, $u = \{u_{1,k},\dots,u_{n,k}\}_{k\in Q}$ in system (1). Then we consider the following linearization of a such obtained system with respect to u:

$$\partial_t z(t,x) = f_k(U_k) + \sum_{j=1}^{n}\partial_{q_j} f_k(U_k)[\partial_{x_j} z_k(t,x) - u_{j,k}(t,x)], \qquad k \in Q, \tag{17}$$

where $U_k = (t, x, z_{\psi(t,x)}, u_{[k]}(t,x))$. By virtue of (1) we get the differential system for the unknown function u:

$$\partial_t u_{i,k}(t,x) = \partial_{x_i} f_k(U_k) + \partial_w f_k(U_k)\sum_{j=1}^{n}\partial_{x_i}\psi_j(t,x)\tilde{w}_j[z] +$$

$$+ \sum_{j=1}^{n}\partial_{q_j} f_k(U_k)\partial_{x_j} u_{j,k}(t,x), \qquad i = 1,\dots,n, \quad k \in Q, \tag{18}$$

where

$$\tilde{w}_j[z] = \left\{(\partial_{x_j} z_k)_{\psi(t,x)}\right\}_{k\in Q}.$$

Finally we put $\partial_x z = u$ in (18).

Note that the systems (17), (18) have the following property: the differential equations of bicharacteristics for (17) and (18) are the same and they have the form (8). If we consider systems (17), (18) along the bicharacteristics $g_k[z, u_{[k]}](\cdot, t, x)$ then we obtain

$$\frac{d}{d\tau} z_k(\delta_k(\tau,t,x)) = f_k(P_k(\tau,t,x)) - \sum_{i=1}^{n}\partial_{q_i} f_k(P_k(\tau,t,x))u_{i,k}(\delta_k(\tau,t,x)), \quad k \in Q,$$

and

$$\frac{d}{d\tau} u_{i,k}(\delta_k(\tau,t,x)) = \partial_{x_i} f_k(P_k(\tau,t,x)) + \partial_w f_k(P_k(\tau,t,x))w_{i,k}[u],$$

$$i = 1,\dots,n, \quad k \in Q.$$

By integrating the above relations on $[0,t]$ with respect to τ we get system (15).

The proof of the existence of the solution of problem (15), (16) is based on the following method of successive approximations. Suppose that $\varphi \in C^{1,L}[E_0, s]$ and that Assumptions $H[\partial_q f]$, $H[\psi]$, $H[f, \partial_x f, \partial_q f]$ are satisfied. We define the sequence

$$\{z^{(m)}, u^{(m)}\}, \qquad z^{(m)} = \{z^{(m)}\}_{k\in Q}, \qquad u^{(m)} = \left\{u^{(m)}_{1,k}, u^{(m)}_{2,k},\dots,u^{(m)}_{n,k}\right\}_{k\in Q},$$

in the following way. We put first

$$z^{(0)}(t,x) = \varphi(t,x) \quad \text{on} \quad E_0, \qquad z^{(0)}(t,x) = \varphi(0,x) \quad \text{on} \quad [0,c]\times R^n, \tag{19}$$

$$u^{(0)}(t,x) = \partial_x\varphi(t,x) \quad \text{on} \quad E_0, \qquad u^{(0)}(t,x) = \partial_x\varphi(0,x) \quad \text{on} \quad [0,c]\times R^n. \tag{20}$$

Then $z^{(0)} \in C^{1,L}_{\varphi,c}[d,\lambda]$ and $u^{(0)} \in C^{0,L}_{\partial\varphi,c}[p,\mu]$. If $z^{(m)} \in C^{1,L}_{\varphi,c}[d,\lambda]$ and $u^{(m)} \in$ $\in C^{0,L}_{\partial\varphi,c}[p,\mu]$ are known functions then $u^{(m+1)}$ is the solution of the problem

$$u = G^{(m)}[u], \quad u(t,x) = \partial_x \varphi(t,x) \quad \text{on} \quad E_0, \tag{21}$$

where

$$G^{(m)} = \left\{G^{(m)}_{[k]}\right\}_{k \in Q}, \quad G^{(m)}_{[k]} = \left(G^{(m)}_{1,k}, \ldots, G^{(m)}_{n,k}\right),$$

$$G^{(m)}_{i,k}[u](t,x) = \partial_{x_i} \varphi_k(\delta^{(m)}(0,t,x)) +$$

$$+ \int_0^t \left[\partial_{x_i} f_k(P^{(m)}_k(\tau,t,x)) + \partial_w f_k(P^{(m)}_k(\tau,t,x)) \overline{w}_{i,k}[u^{(m)}]\right] d\tau \tag{22}$$

and

$$\overline{w}_{i,k}[u^{(m)}] = \sum_{j=1}^n \partial_{x_i} \psi_j(\delta^{(m)}_k(\tau,t,x)) \left(u^{(m)}_{j,Q}\right)_{\psi(\delta^{(m)}_k(\tau,t,x))},$$

$$\delta^{(m)}_k(\tau,t,x) = \left(\tau, g_k[z^{(m)}, u_{[k]}](\tau,t,x)\right),$$

$$P^{(m)}_k(\tau,t,x) = \left(\delta^{(m)}_k(\tau,t,x), z^{(m)}_{\psi(\delta^{(m)}_k(\tau,t,x))}, u_{[k]}(\delta^{(m)}_k(\tau,t,x))\right).$$

The function $z^{(m+1)}$ is defined by

$$z^{(m+1)}(t,x) = \varphi(t,x) \quad \text{on} \quad E_0,$$

$$z^{(m+1)}(t,x) = F[z^{(m)}, u^{(m+1)}](t,x) \quad \text{on} \quad [0,c] \times R^n. \tag{23}$$

We wish to emphasize that the main difficulty in carrying out of this construction is the problem of the existence of the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$.

Remark 3. Note that problem $u = G[z^{(m)}, u]$, $u = \partial_x \varphi$ on E_0 and (21) are not identical. System (21) is obtained in the following way. Suppose that $z^{(m)} \in C^{1,L}_{\varphi,c}[d,\lambda]$ and $u^{(m)} \in C^{0,L}_{\partial\varphi,c}[p,\mu]$ are known functions, we consider system (18) with $z^{(m)}$ instead of z, i. e.

$$\partial_t u_{i,k}(t,x) = \partial_{x_i} f_k(U^{(m)}_k) + \partial_w f_k(U^{(m)}_k) \sum_{j=1}^n \partial_{x_i} \psi_j(t,x) \tilde{w}_j[z^{(m)}] +$$

$$+ \sum_{j=1}^n \partial_{q_j} f_k(U^{(m)}_k) \partial_{x_j} u_{i,k}(t,x), \quad i = 1, \ldots, n, \quad k \in Q,$$

where $U^{(m)}_k = (t, x, z^{(m)}_{\psi(t,x)}, u_{[k]}(t,x))$. If we assume that $\partial_x z^{(m)} = u^{(m)}$ (see Lemma 2), then by integrating the above system along bicharacteristics $g_k[z^{(m)}, u_{[k]}](\cdot,t,x)$ we obtain (21).

5. Existence of solutions of initial problems. We first prove that the sequence $\{z^{(m)}, u^{(m)}\}$ exists and that $\partial_x z^{(m)}(t,x) = u^{(m)}(t,x)$ on $[0,c] \times R^n$ provide $c \in (0, a]$ is sufficiently small.

Lemma 2. *Suppose that* $\varphi \in C^{1,L}[E_0, s]$ *and that Assumptions* $H[\partial_q f]$, $H[\psi]$ *and* $H[f, \partial_x f, \partial_w f]$ *are satisfied.*

Then there are $c \in (0, a]$ *and* $d \in R^3_+$, $p \in R^2_+$, μ, λ_0, $\lambda_1 \in L([0,c], R_+)$ *such that for any* $m \geq 0$ *we have*

(I_m) $z^{(m)}$ *and* $u^{(m)}$ *are defined on* $[-r_0, c] \times R^n$ and $z^{(m)} \in C^{1,L}_{\varphi,c}[d,\lambda]$, $u^{(m)} \in C^{0,L}_{\partial\varphi,c}[p,\mu]$;

(II_m) $\partial_x z^{(m)}(t,x) = u^{(m)}(t,x)$ on $[0,c]\times R^n$.

Proof. Let $d=(d_0,d_1,d_2)\in R_+^3$ and $p=(p_0,p_1)\in R_+^2$ be such that

$$d_i > s_i \quad \text{for} \quad i=0,1,2, \quad \text{and} \quad p_0 = d_1, \quad p_1 = d_2. \tag{24}$$

We define the functions Γ, $\tilde{\Gamma}:[0,a]\to R_+$ by

$$\Gamma(t) = \exp\left[\bar{d}\int_0^t \gamma(\xi,|d|)d\xi\right]\Bigg\{s_2+\bar{d}(1+a_0p_0)\int_0^t\gamma(\xi,|d|)d\xi + $$

$$+ (p_0a_1+p_1a_0)\int_0^t\alpha(\xi,\tilde{d})d\xi\Bigg\},$$

$$\tilde{\Gamma}(t) = \exp\left[\bar{d}\int_0^t\gamma(\xi,|d|)d\xi\right]\left\{s_1+(\bar{d}+p_1)\int_0^t\alpha(\xi,\tilde{d})d\xi+\bar{d}p_0\int_0^t\gamma(\xi,|d|)d\xi\right\},$$

and

$$\tilde{\gamma}(t) = \gamma(\xi,|d|) + \alpha(\xi,\tilde{d}), \qquad \bar{q} = \Gamma(a)+1+a_0+p_0.$$

Suppose that the constant $c\in(0,a]$ is small enough to satisfy

$$a_0\int_0^c\alpha(\xi,\tilde{d})d\xi < 1, \quad \Gamma(c)\le p_1,$$

$$\left(1-a_0\int_0^c\alpha(\xi,\tilde{d})d\xi\right)^{-1}\left[s_1+\int_0^c\alpha(\xi,\tilde{d})d\xi\right]\le p_0, \tag{25}$$

and

$$s_0+\int_0^c\left[\tilde{\gamma}(\xi,d_0)+p_0\alpha(\xi,\tilde{d})\right]d\xi \le d_0. \tag{26}$$

Write

$$\lambda_0(t) = \tilde{\gamma}(t,d_0)+p_0\alpha(t,\tilde{d})+\tilde{\Gamma}(c)\,\alpha(t,\tilde{d}) \tag{27}$$

and

$$\lambda_1(t) = \mu(t) = [\Gamma(c)+1+a_0p_0]\alpha(t,\tilde{d}). \tag{28}$$

We will prove (I_m) and (II_m) by induction. It follows from (19), (20) that conditions (I_0) and (II_0) are satisfied. Supposed now that conditions (I_m) and (II_m) hold for a given $m\ge 0$ we will prove that there exists the solution $u^{(m+1)}\in C^{0,L}_{\partial\varphi,c}[p,\mu]$ of (21) and that $z^{(m+1)}$ given by (21) is an element of the space $C^{1,L}_{\varphi,c}[d,\lambda]$.

We claim that

$$G^{(m)}:\ C^{0,L}_{\partial\varphi,c}[p,\mu]\to C^{0,L}_{\partial\varphi,c}[p,\mu]. \tag{29}$$

Indeed, it follows that for $u\in C^{0,L}_{\partial\varphi,c}[p,\mu]$ we have

$$\left\|G^{(m)}[u](t,x)\right\| \le s_1+(1+a_0p_0)\int_0^t\alpha(\xi,\tilde{d})d\xi$$

and

$$\left\| G^{(m)}[u](t,x) - G^{(m)}[u](\bar{t},\bar{x}) \right\| \le [\Gamma(c) + 1 + a_0 p_0] \left| \int_t^{\bar{t}} \alpha(\xi, \bar{d}) d\xi \right| + \Gamma(c) \| x - \bar{x} \|$$

on $[0, c] \times R^n$. The above inequalities and (25), (28) imply (29).

It follows that for $u, \bar{u} \in C^{0,L}_{\partial\varphi,c}[p, \mu]$ we have

$$\left\| G^{(m)}[u] - G^{(m)}[\bar{u}] \right\|_t \le \bar{q} \int_0^t \bar{\gamma}(\tau) \| u - \bar{u} \|_\tau d\tau, \quad t \in [0, c]. \tag{30}$$

For $u, \bar{u} \in C^{0,L}_{\partial\varphi,c}[p, \mu]$ we put

$$[|u - \bar{u}|] = \sup \left\{ \| u - \bar{u} \|_t \exp\left[-2\bar{q} \int_0^t \bar{\gamma}(\tau) d\tau \right] : \ t \in [0, c] \right\}.$$

It follows from (30) that

$$\left\| G^{(m)}[u] - G^{(m)}[\bar{u}] \right\|_t \le \bar{q} [|u - \bar{u}|] \int_0^t \bar{\gamma}(\tau) \exp\left[2\bar{q} \int_0^\tau \bar{\gamma}(\xi) d\xi \right] d\tau \le$$

$$\le \frac{1}{2} [|u - \bar{u}|] \exp\left[2\bar{q} \int_0^t \bar{\gamma}(\tau) d\tau \right], \quad t \in [0, c],$$

and consequently

$$\left[\left| G^{(m)}[u] - G^{(m)}[\bar{u}] \right| \right] \le \frac{1}{2} [|u - \bar{u}|].$$

Then the Banach fixed point theorem shows that there exists exactly one $u^{(m+1)} \in C^{0,L}_{\partial\varphi,c}[p, \mu]$ satisfying (21). Now we prove that the function $z^{(m+1)}$ given by (23) satisfies (II_{m+1}). For each $k \in Q$ consider the function

$$\Delta_k(t, x, \bar{x}) = z_k^{(m+1)}(t, \bar{x}) - z_k^{(m+1)}(t, x) - \sum_{j=1}^n u_{j,k}^{(m+1)}(t, x)(\bar{x}_j - x_j), \tag{31}$$

where $t \in [0, c]$, $x, \bar{x} \in R^n$. It follows from (21), (23) that

$$\Delta_k(t, x, \bar{x}) = F_k[z^{(m)}, u^{(m+1)}](t, \bar{x}) -$$

$$- F_k[z^{(m)}, u^{(m+1)}](t, x) - \sum_{j=1}^n (\bar{x}_j - x_j) G_{j,k}^{(m)}[u^{(m+1)}](t, x).$$

We conclude from Lemma 1 and (22), (23) that for each $k \in Q$ there exists $\tilde{C}_k \in R_+$ such that

$$|\Delta_k(t, x, \bar{x})| \le \tilde{C}_k \| x - \bar{x} \|^2 \quad \text{for} \quad t \in [0, c], \quad x, \bar{x} \in R^n.$$

See [20] and [11] (Chapter 4) for more details concerning the method of the proof. The above estimates and (31) imply (II_{m+1}).

Now we prove that $z^{(m+1)} \in C^{1,L}_{\varphi,c}[s, \lambda]$. Of course $z^{(m+1)}$ is continuous on $[-r_0, c] \times R^n$ and $z^{(m+1)}(t, x) = \varphi(t, x)$ on E_0. Moreover, it follows from (24), (25) and (II_{m+1}) that

$$\left\| \partial_x z^{(m+1)}(t, x) \right\| \le d_1$$

and

$$\left\|\partial_x z^{(m+1)}(t,x)-\partial_x z^{(m+1)}(\bar{t},\bar{x})\right\| \le \left|\int_t^{\bar{t}} \lambda_1(\xi)d\xi\right| + d_2\|x-\bar{x}\|$$

on $[0, c]\times R^n$. Our assumptions and (26), (27) imply also the following inequalities

$$\left\|z^{(m+1)}(t,x)\right\| \le d_0, \qquad \left\|z^{(m+1)}(t,x)-z^{(m+1)}(\bar{t},x)\right\| \le \left|\int_t^{\bar{t}} \lambda_0(\xi)d\xi\right|$$

on $[0, c]\times R^n$. This completes the proof of Lemma 2.

Now we prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are convergent if the constant $c \in (0, a]$ is sufficiently small.

Lemma 3. *Suppose that*

1) $\varphi \in C^{1,L}[E_0, s]$ *and Assumptions* $H[\partial_q f]$, $H[\psi]$, $H[f, \partial_x f, \partial_w f]$ *are satisfied*;

2) *conditions* (24) – (28) *hold.*

Then there is $c \in (0, a]$ *such that the sequences* $\{z^{(m)}\}$ *and* $\{u^{(m)}\}$ *are uniformly convergent on* $[0, c]\times R^n$.

Proof. Write $\tilde{c} = 1 + a_0 p_0 + \Gamma(c)$ and

$$G(t) = [1 + a_0 p_0 + \Gamma(c)]\gamma(t, |d|) + a_0\alpha(t, \tilde{d}).$$

From (21) we conclude that

$$\left\|u^{(m+1)} - u^{(m)}\right\|_t \le \tilde{c}\int_0^t \gamma(\xi, |d|)\left\|z^{(m)} - z^{(m-1)}\right\|_{1,\xi} d\xi +$$

$$+ \int_0^t G(\xi)\left\|u^{(m+1)} - u^{(m)}\right\|_\xi d\xi, \quad t \in [0, c],$$

and consequently

$$\left\|u^{(m+1)} - u^{(m)}\right\|_t \le \tilde{c}\exp\left[\int_0^t G(\xi)d\xi\right]\int_0^t \gamma(\xi, |d|)\left\|z^{(m)} - z^{(m-1)}\right\|_{1,\xi} d\xi, \tag{32}$$

where $t \in [0, c]$. Let

$$\tilde{G}(t) = \exp\left[\bar{d}\int_0^t \gamma(\xi, |d|)d\xi\right]\left\{s_1 + \bar{d}(1+p_0)\int_0^t \gamma(\xi, |d|)d\xi + p_1\int_0^t \alpha(\xi, \bar{p})d\xi\right\} + 1 + p_0$$

and

$$C^* = \tilde{G}(a) + \tilde{c}\exp\left[\int_0^a G(\xi)d\xi\right]\int_0^a \left[\tilde{G}(a)\gamma(\xi, |d|) + \alpha(\xi, \tilde{d})\right]d\xi,$$

$$\bar{c} = 2\max\left\{C^*, \ \tilde{c}\exp\left[\int_0^c G(\xi)d\xi\right]\right\}.$$

An easy computation shows that

$$\left\|z^{(m+1)} - z^{(m)}\right\|_{0,t} \le \tilde{G}(a)\int_0^t \gamma(\xi, |d|)\left\|z^{(m)} - z^{(m-1)}\right\|_{1,\xi} d\xi +$$

$$+ \int_0^t \left[\tilde{G}(a)\gamma(\xi,|d|) + \alpha(\xi,\tilde{d})\right] \left\| u^{(m+1)} - u^{(m)} \right\|_\xi d\xi \le$$

$$\le \frac{\bar{c}}{2}\int_0^t \gamma(\xi,|d|)\left[\left\| z^{(m)} - z^{(m-1)} \right\|_{0,\xi} + \left\| u^{(m)} - u^{(m-1)} \right\|_\xi\right] d\xi, \quad t \in [0, c]. \tag{33}$$

Estimates (32) and (33) imply

$$\left\| u^{(m+1)} - u^{(m)} \right\|_t + \left\| z^{(m+1)} - z^{(m)} \right\|_{0,t} \le$$

$$\le \bar{c}\int_0^t \gamma(\xi,|d|)\left[\left\| u^{(m)} - u^{(m-1)} \right\|_\xi + \left\| z^{(m)} - z^{(m-1)} \right\|_{0,\xi}\right] d\xi. \tag{34}$$

From Assumptions $H[\partial_q f]$ and $H[f, \partial_x f, \partial_w f]$ it follows that there is $\overline{C} \in R_+$ such that

$$\left\| u^{(1)} - u^{(0)} \right\|_t + \left\| z^{(1)} - z^{(0)} \right\|_{0,t} \le \overline{C} + \int_0^t \left[\tilde{\gamma}(\xi, d_0) + (1 + p_0 + a_0 p_0)\alpha(\xi, \tilde{d})\right] d\xi,$$

where $t \in [0, c]$. Suppose that $c \in (0, a]$ is such a small constant that

$$\bar{c}\int_0^c \gamma(\xi,|d|)d\xi = \delta < 1. \tag{35}$$

We conclude from (34) that

$$\left\| u^{(m+1)} - u^{(m)} \right\|_t + \left\| z^{(m+1)} - z^{(m)} \right\|_{0,t} \le \delta\left[\left\| u^{(m)} - u^{(m-1)} \right\|_t + \left\| z^{(m)} - z^{(m-1)} \right\|_{0,t}\right]$$

where $t \in [0, c]$. This gives the uniform convergence of the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ on $[0, c] \times R^n$, which proves Lemma 3.

We are able now to state the main result on the existence of weak solutions.

Theorem 1. *Suppose that*

1) $\varphi \in C^{1,L}[E_0, s]$ *and Assumptions* $H[\partial_q f]$, $H[\psi]$, $H[f, \partial_x f, \partial_w f]$ *are satisfied;*

2) *conditions* (24) – (28) *and* (35) *hold.*

Then there exists a solution $v = \{v_k\}_{k \in Q}$, $v_k : [-r_0, c] \times R^n \to R$, *of problem* (1), (2). *Moreover* $v \in C^{1,L}_{\varphi,c}[d, \lambda]$ *and* $\partial_x v \in C^{0,L}_{\partial\varphi,c}[p, \mu]$.

If $\tilde{\varphi} \in C^{1,L}[E_0, s]$ *and* $\tilde{v} \in C^{1,L}_{\tilde{\varphi},c}[d, \lambda]$ *is a solution of system* (1) *with the initial condition*

$$z(t, x) = \tilde{\varphi}(t, x) \quad on \quad E_0 \tag{36}$$

then there exists $\tilde{\kappa} \in C([0, c], R_+)$ *such that*

$$\| v - \tilde{v} \|_{0,t} \le \tilde{\kappa}(t) \| \varphi - \tilde{\varphi} \|_{1,0}, \quad t \in [0, c]. \tag{37}$$

Proof. It follows from Lemma 3 that two functions $v \in C^{1,L}_{\varphi,c}[d, \lambda]$ and $u \in C^{0,L}_{\partial\varphi,c}[p, \mu]$ exist such that $\{z^{(m)}\}$ converges to v and $\{u^{(m)}\}$ converges to u uniformly on $[0, c] \times R^n$ and $\partial_x v = u$. Write

$$\rho_k(\tau, t, x) = (\tau, g_k[v, \partial_x v_k](\tau, t, x)),$$

$$\theta_k(\tau, t, x) = (\rho_k(\tau, t, x), v_{\psi(\rho_k(\tau, t, x))}, \partial_x v_k(\rho_k(\tau, t, x))).$$

Thus we get that

$$\tilde{z}_k(t,x) = \varphi_k(\rho_k(0,t,x)) +$$
$$+ \int_0^t \Big[f_k(\theta_k(\tau,t,x)) - \sum_{i=1}^n \partial_{q_i} f_k(\theta_k(\tau,t,x)) \partial_{x_i} v_k(\theta_k(\tau,t,x)) \Big] d\tau, \tag{38}$$

where $(t,x) \in [0,c] \times R^n$, $k \in Q$, and $g[v,\partial_x v](\cdot,t,x) = \{g_k[v,\partial_x v_k](\cdot,t,x)\}_{k\in Q}$ satisfies the integral system

$$g_k[v,\partial_x v_k](\tau,t,x) = x + \int_\tau^t \partial_q f_k(\theta_k(\tau,t,x)) d\tau, \quad k \in Q. \tag{39}$$

Now we prove that v is a weak solutions of (1), (2). Let $k \in Q$ be fixed. For a given $x \in R^n$ let us put $y = g[v,\partial_x v_k](0,t,x)$. It follows from Lemma 1 that

$$g_k[v,\partial_x v_k](\tau,t,x) = g_k[v,\partial_x v_k](\tau,0,y) \quad \text{for} \quad \tau,\ t \in [0,c].$$

Write

$$\tilde{\rho}_k(t,y) = (t, g_k[v,\partial_x v_k](\tau,0,y))$$

and

$$\tilde{\theta}_k(t,y) = (\tilde{\rho}_k(t,y), v_{\psi(\tilde{\rho}_k(t,y))}, \partial_x v_k(\tilde{\rho}_k(t,y))).$$

Then relations (38) are equivalent to

$$v_k(\tilde{\rho}_k(t,y)) = \varphi(0,y) + \int_0^t \Big[f_k(\tilde{\theta}(\tau,y)) - \sum_{l=1}^n \partial_{q_l} f_k(\tilde{\theta}(\tau,y)) \partial_{x_l} \tilde{u}_k(\tilde{\rho}_k(t,y)) \Big] d\tau, \tag{40}$$

where $(t,y) \in [0,c] \times R^n$, $k \in Q$. The relations

$$y = g_k[v,\partial_x v_k](0,t,x) \quad \text{and} \quad x = g_k[v,\partial_x v_k](t,0,y)$$

are equivalent for $x,\ y \in R^n$. By differentiating (40) with respect to t and by putting again $x = g_k[v,\partial_x v_k](t,0,y)$ we obtain that v_k satisfies equation in (1) for almost all $t \in [0,c]$ with fixed $x \in R^n$. Since v satisfies (16) then initial condition (2) is satisfied.

Now we prove (37). An easy computation shows that

$$\|\partial_x v - \partial_x \tilde{v}\|_t \le \int_0^t [G(\xi) + \tilde{c}\gamma(\xi,|d|)] \|\partial_x v - \partial_x \tilde{v}\|_\xi d\xi +$$
$$+ \tilde{c}\int_0^t \gamma(\xi,|d|) \|v - \tilde{v}\|_{0,\xi} d\xi + \|\partial_x \varphi - \partial_x \tilde{\varphi}\|_0 \tag{41}$$

and

$$\|v - \tilde{v}\|_{0,t} \le \|\varphi - \tilde{\varphi}\|_0 + \int_0^t G^*(\xi) \Big[\|v - \tilde{v}\|_{1,\xi} + \|\partial_x v - \partial_x \tilde{v}\|_\xi \Big] d\xi +$$
$$+ \int_0^t \alpha(\xi,\tilde{d}) \|\partial_x v - \partial_x \tilde{v}\|_\xi d\xi, \tag{42}$$

where

$$G^* = (\tilde{\Gamma}(c) + p_0)\gamma(t,|d|) + \alpha(t,\tilde{d}).$$

According to (41) and Gronwall inequality we have

$$\|\partial_x v - \partial_x \bar{v}\|_t \le \left[\bar{c}\int_0^t \gamma(\xi, |d|)\|v - \bar{v}\|_{0,\xi}\, d\xi + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_0\right] +$$

$$+ \exp\left[\int_0^t (G(\xi) + \bar{c}\gamma(\xi, |d|))\, d\xi\right]. \tag{43}$$

Write

$$\bar{b} = \int_0^c \left(2G^*(\xi) + \alpha(\xi, \bar{d})\right) d\xi \exp\left[\int_0^c (G(\xi) + \bar{c}\gamma(\xi, |d|))\, d\xi\right]$$

and $A = \max\{1, \bar{b}\}$. It follows from (42) and (43) that the integral inequality

$$\|v - \bar{v}\|_{0,t} \le A\|\varphi - \bar{\varphi}\|_{1,0} + \int_0^t \left[G^*(\xi) + \bar{b}\bar{c}\gamma(\xi, |d|)\right]\|v - \bar{v}\|_{0,\xi}\, d\xi$$

is satisfied for $t \in [0, c]$. Then we get (37) with

$$\tilde{\kappa}(t) = A \exp\int_0^t \left[G^*(\xi) + \bar{b}\bar{c}\gamma(\xi, |d|)\right] d\xi, \quad t \in [0, c].$$

This completes the proof of Theorem 1.

Remark 4. Suppose that all the assumptions of Theorem 1 are satisfied. If v and $\bar{v}$ are the solutions of (1), (2) and (1), (36) respectively then we can estimate the norm $\|v - \bar{v}\|_{1,t}$, $0 \le t \le c$, in the following way: there is $\bar{\kappa} \in C([0, c], R_+)$ such that

$$\|v - \bar{v}\|_{1,t} \le \overline{\kappa}(t)\|\varphi - \bar{\varphi}\|_{1,0}, \quad t \in [0, c]. \tag{44}$$

Indeed, it follows from (41), (42) that there is $\overline{\mu} \in L([0, c], R_+)$ such that

$$\|v - \bar{v}\|_{1,t} \le \|\varphi - \bar{\varphi}\|_{1,0} + \int_0^t \overline{\mu}(\xi)\|v - \bar{v}\|_{1,\xi}\, d\xi,$$

where $t \in [0, c]$. Then we get (44) for

$$\bar{\kappa}(t) = \exp\left[\int_0^t \overline{\mu}(\xi)\, d\xi\right].$$

Remark 5. If we assume in Theorem 1 that f is continuous with respect to t, then we get classical solutions of Cauchy problem (1), (2).

6. Applications of the main theorem. Now we formulate the existence result for problem (5), (2).

Assumption $H[F]$. Suppose that the function

$$F = \{F_k\}_{k \in Q}, \qquad F_k : E \times X \times R^n \to R,$$

of the variables (t, x, p, q) satisfies the conditions:

1) for each $k \in Q$ and $(x, p, q) \in R^n \times X \times R^n$ the function $F_k(\cdot, x, w, q)$ is measurable and there is $\tilde{\gamma} \in L([0, a], R_+)$ such that $\|F(t, x, w, q)\| \le \tilde{\gamma}(t)$ on $E \times X \times \times R^n$;

2) the derivatives $\partial_x F = \{\partial_x F_k\}_{k \in Q}$, $\partial_q F = \{\partial_q F_k\}_{k \in Q}$ and the Frechét derivatives $\partial_p F = \{\partial_p F_k\}_{k \in Q}$ exist on $E \times X \times R^n$ and there is $\bar{\alpha} \in L([0, a], R_+)$ such that

$$\|\partial_x F(t,x,p,q)\| \le \tilde{\alpha}(t), \quad \left\|\partial_p F(t,x,p,q)\right\| \le \tilde{\alpha}(t),$$
$$\left\|\partial_q F(t,x,p,q)\right\| \le \tilde{\alpha}(t)$$

on $E \times X \times R^n$;

3) there is $\tilde{\beta} \in L([0,a],R_+)$ such that the terms

$$\|\partial_x F(t,x,p,q) - \partial_x F(t,\bar{x},\bar{p},\bar{q})\|, \quad \left\|\partial_p F(t,x,p,q) - \partial_x F(t,\bar{x},\bar{p},\bar{q})\right\|,$$
$$\left\|\partial_q F(t,x,p,q) - \partial_q F(t,\bar{x},\bar{p},\bar{q})\right\|$$

are bounded from above by

$$\tilde{\beta}(t)\,[\|x-\bar{x}\| + |p-\bar{p}| + \|q-\bar{q}\|].$$

Assumption $H[\alpha,\beta]$. Suppose that the functions

$\alpha = \{\alpha_k\}_{k\in Q}$, $\alpha_k : [0,a] \to R$, $\beta = \{\beta_{[k]}\}_{k\in Q}$, $\beta_{[k]} = (\beta_{1,k},\ldots,\beta_{n,k}) : E \to R^n$, satisfy the conditions:

1) $\alpha_k \in C([0,a],R)$, $\beta_{[k]} \in C(E,R^n)$ and $-r_0 \le \alpha_k(t) - t \le 0$ for $t \in [0,a]$, $-r \le \beta_{[k]}(t,x) - x \le r$ on E where $k \in Q$;

2) the derivatives $\partial_x \beta_{[k]} = [\partial_{x_i} \beta_{j,k}]_{i,j=1,\ldots,n}$ exist on E and

$$\left\|\partial_x \beta_{[k]}(t,x)\right\| \le L_0, \quad \left\|\partial_x \beta_{[k]}(t,x) - \partial_x \beta_{[k]}(t,\bar{x})\right\| \le L_1 \|x-\bar{x}\|,$$

where $k \in Q$ and $L_0,\ L_1 \in R_+$.

Theorem 2. *Suppose that Assumptions* $H[F]$, $H[\alpha,\beta]$ *are satisfied and* $\varphi \in C^{1,L}[E_0,s]$.

Then there exist $(d_0,d_1,d_2) \in R_+^3$, $c \in (0,a]$, $\lambda_0, \lambda_1 \in L([0,c],R_+)$ *and a function* $v \in C^{1,L}_{\varphi,c}[d,\lambda]$ *such that* v *is a weak solution of* (5), (2).

Moreover, if $\tilde{\varphi} \in C^{1,L}[E_0,s]$ *and* $\tilde{v} \in C^{1,L}_{\tilde{\varphi},c}[d,\lambda]$ *is a solution of* (5), (36) *then there is* $\tilde{\kappa} \in C([0,c],R_+)$ *such that estimate* (37) *holds.*

Proof. We apply Theorem 1 to the function f given by (4). Suppose that $(t,x,w,q) \in \Omega_I$. Then we have

$$\partial_{x_i} f_k(t,x,w,q) = \partial_{x_i} F_k(t,\, x,\, w(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x),\, q) +$$
$$+\ \partial_p F_k(t,\, x,\, w(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x),\, q) \times$$
$$\times \left[\sum_{j=1}^{n} \partial_{y_j} w(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x)\,(\partial_{x_i}\beta_{j,k}(t,x) - \delta_{ij})\right], \quad i = 1,\ldots,n,$$

where δ_{ij} is the Kronecker symbol and

$$\partial_w f_k(t,x,w,q)h =$$
$$= \partial_p F_k(t,\, x,\, w(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x),\, q)\; h(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x),$$
$$\partial_q f_k(t,x,w,q) = \partial_q F_k(t,\, x,\, w(\alpha_k(t)-t,\, \beta_{[k]}(t,x)-x),\, q)$$

where $k \in Q$. An easy computation shows that

$$\|\partial_x f(t,x,p,q)\| \le \tilde{\alpha}(t)\left[1 + (1+L_0)\|w\|_1\right],$$
$$\|\partial_w f(t,x,p,q)\|_* \le \tilde{\alpha}(t), \quad \left\|\partial_q f(t,x,p,q)\right\| \le \tilde{\alpha}(t)$$

on Ω_I. Moreover the following Lipschitz condition is satisfied

$$\|\partial_x f(t,\bar{x},w+h,\bar{q})-\partial_x f(t,x,w,q)\| \le \gamma^*(t,\|w\|_{1,L})\left[\|x-\bar{x}\|+\|h\|_1+\|q-\bar{q}\|\right]$$

where $(t,x,w,q)\in\Omega_{I,L}$, $\bar{x},\bar{q}\in R^n$, $h\in C^1(B,X)$, and

$$\gamma^*(t,s) = \tilde{\beta}(t)\left[1+(1+L_0)s\right]^2 + \tilde{\alpha}(t)\left[(1+L_0)^2+L_1\right]s.$$

It is easy to see that the derivatives $\partial_q f$ and $\partial_w f$ satisfy the Lipschitz condition established in Assumption $H[\partial_q f]$ and $H[f,\partial_x f,\partial_w f]$. Hence the assertion follows as an immediate consequence of Theorem 1.

Now we formulate the existence result for problem (7), (2).

Theorem 3. *Suppose that Assumptions* $H[\psi]$ *and* $H[F]$ *are satisfied and* $\varphi\in C^{1,L}[E_0,s]$.

Then there exist

$$(d_0,d_1,d_2)\in R_+^3,\quad c\in(0,a],\quad \lambda_0,\lambda_1\in L([0,c],R_+)$$

and a function $v\in C^{1,L}_{\varphi,c}[d,\lambda]$ *such that* v *is a weak solution of* (7), (2).

Moreover, if $\bar{\varphi}\in C^{1,L}[E_0,s]$ *and* $\bar{v}\in C^{1,L}_{\bar{\varphi},c}[d,\lambda]$ *is a solution of* (7), (36) *then there is* $\bar{\kappa}\in C([0,c],R_+)$ *such that estimate* (37) *holds.*

Proof. It is easy to check that the function f given by (6) satisfies Assumptions $H[\partial_q f]$ and $H[f,\partial_x f,\partial_w f]$. Hence the assertion follows from Theorem 1.

***Remark* 6.** Note that the results of the papers [18, 20] are not applicable to system (7) in the case when Q is the set of natural numbers.

Let F and φ be given by (3) and

$$\psi_0:[0,a]\to[0,a],\qquad \psi':E\to R^n,\qquad \psi=(\psi_0,\psi'),$$

and

$$f_k(t,x,w,q) = F_k(t,x\,w(0,0),q),\qquad k\in Q.$$

Then system (1) is equivalent to the infinite system of differential equations with a deviated argument

$$\partial_t z_k(t,x) = F_k(t,x,z(\psi(t,x)),\partial_x z_k(t,x)),\qquad k\in Q. \tag{45}$$

Now we formulate the existence result for problem (45), (2) with $r_0=0$.

Theorem 4. *Suppose that Assumptions* $H[\psi]$ *and* $H[F]$ *are satisfied and*

$$\varphi\in C^{1,L}[E_0,s]\quad \textit{with}\quad r_0=0.$$

Then there exist

$$(d_0,d_1,d_2)\in R_+^3,\quad c\in(0,a],\quad \lambda_0,\lambda_1\in L([0,c],R_+),$$

and a function $v\in C^{1,L}_{\varphi,c}[d,\lambda]$ *such that* v *is a weak solution of* (45), (2) *with* $r_0=0$.

Moreover, if $\bar{\varphi}\in C^{1,L}[E_0,s]$ *and* $\bar{v}\in C^{1,L}_{\bar{\varphi},c}[d,\lambda]$ *is a solution of* (45), (36) *then there is* $\bar{\kappa}\in C([0,c],R_+)$ *such that estimate* (37) *holds.*

This theorem is a consequence of Theorem 1.

***Remark* 7.** Note that if we apply Theorem 3 from [20] to problem (45), (2) then we need the following additional assumption on ψ': $-r \le \psi'(t, x) - x \le r$ on E. Results of the paper [18] are not applicable to problem (45), (2).

1. *Brandi P., Ceppitelli R.* Existence, uniqueness and continuous dependence for a first order nonlinear partial differential equations in a hereditary structure // Ann. pol. math. – 1986. – **47**. – P. 121 – 136.
2. *Kamont Z.* Existence of solutions of first order partial differential functional equations // Ann. Soc. math. pol., Comm. math. – 1985. – **25**. – P. 249 – 263.
3. *Szarski J.* Generalized Cauchy problem for differential functional equations with first order partial derivatives // Bull. Acad. pol. sci., Sér. sci. math., astron. et phys. – 1976. – **24**. – P. 575 – 580.
4. *Ważewski T.* Sur le probléme de Cauchy relatif á un systéme d'équations aux dériveés partielles // Ann. Soc. pol. math. – 1936. – **15**. – P. 101 – 127.
5. *Umanaliev M. I., Vied' J. A.* On integral differential equations with first order partial derivatives (Russian) // Differents. Uravneniya. – 1989. – **25**. – P. 465 – 477.
6. *Czlapiński T., Kamont Z.* Generalized solutions for quasilinear hyperbolic systems of partial differential functional equations // J. Math. Anal. and Appl. – 1993. – **172**. – P. 353 – 370.
7. *Besala P.* Observations on quasi-linear partial differential equations // Ann. pol. math. – 1991. – **53**. – P. 267 – 283.
8. *Czlapiński T.* On the existence of generalized solutions of nonlinear first order partial differential functional equations in two independent variables // Czesh. Math. J. – 1991. – **41**. – P. 490 – 506.
9. *Kamont Z., Zacharek S.* On the existence of weak solutions of nonlinear first order partial differential equations in two independent variables // Boll. Unione mat. ital. – 1986. – **6**, № 5-B. – P. 851 – 879.
10. *Brandi P., Kamont Z., Salvadori A.* Existence of weak solutions for partial differential functional equations (Preprint).
11. *Kamont Z.* Hyperbolic functional differential inequalities and applications. – Dordrecht etc.: Kluwer Acad. Publ., 1999.
12. *Cinquini Cibrario M.* Nuove ricerche sui sistemi di equazioni non lineari a derivate parziali in piú variabili indipendenti // Rend. Semin. mat. e fis. Univ. Milano. – 1982. – **52**. – P. 531 – 550.
13. *Cinquini Cibrario M.* Sopra una classe di sistemi di equazioni non lineari a derivate parziali in piú variabili indipendenti // Ann. math. pura ed appl. – 1985. – **140**. – P. 223 – 253.
14. *Brzychczy S.* Approximate iterative method and the existence of solutions of nonlinear parabolic differential functional equations // Ann. pol. math. – 1983. – **42**. – P. 37 – 43.
15. *Brzychczy S.* Chaplygin method for a system of nonlinear parabolic differential functional equations (Russian) // Differents. Uravneniya. – 1986. – **22**. – P. 705 – 708.
16. *Brzychczy S.* Existence of solutions for nonlinear systems of differential functional equations of parabolic type in arbitrary domains // Ann. pol. math. – 1987. – **47**. – P. 309 – 317.
17. *Kamont Z.* Initial value problems for hyperbolic differential functional systems // Boll. Unione mat. Ital. – 1994. – **7**, № 8-B. – P. 965 – 984.
18. *Szarski J.* Cauchy problem for an infinite system of differential functional equations with first order partial derivatives // Ann. Soc. math. pol., Comm. math. – 1978. – **Spec. I**. – P. 293 – 300.
19. *Szarski J.* Comparison theorems for infinite systems of differential functional equations and strongly coupled infinite systems of first order partial differential equations // Rocky Mountain J. Math. – 1980. – **10**. – P. 237 – 246.
20. *Jaruszewska-Walczak D.* Generalized solutions of the Cauchy problem for infinite systems of functional differential equations // Funct. Different. Equat. – 1999. – **6**, № 3-4. – P. 305 – 326.
21. *Brzychczy S.* Existence of solutions and monotone iterative method for infinite systems of parabolic differential functional equations // Ann. pol. math. (to appear).

Received 19.02.2002