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COCONVEX POINTWISE APPROXIMATION

КООПУКЛЕ ПОТОЧКОВЕ НАБЛИЖЕННЯ

Let a function $f \in C[-1,1]$ change its convexity at the finite collection $Y := \{y_1, \ldots, y_s\}$ of s points $y_i \in (-1,1)$. For each $n > N(Y)$, we construct an algebraic polynomial P_n of degree $\le n$ which is coconvex with f, i. e., changes its convexity at the same points y_i as f and

$$|f(x) - P_n(x)| \le c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad x \in [-1,1],$$

where c is an absolute constant, $\omega_2(f,t)$ is the second modulus of smoothness of f, and if $s = 1$, then $N(Y) = 1$. We also give some counterexamples showing that this estimate cannot be extended for the greater smoothness.

Нехай функція $f \in C[-1,1]$ змінює свою опуклість у скінченному наборі $Y := \{y_1, \ldots, y_s\}$ s точок $y_i \in (-1,1)$. Для кожного $n > N(Y)$ будується алгебраїчний многочлен P_n степеня $\le n$, який є коопуклим з f, тобто змінює свою опуклість в тих самих точках y_i, що й f, а також

$$|f(x) - P_n(x)| \le c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad x \in [-1,1],$$

де c — абсолютна стала, $\omega_2(f,t)$ — другий модуль неперервності f, і якщо $s = 1$, то $N(Y) = 1$. Наведено також контрприклади, що показують, зокрема, неможливість поширення цієї оцінки для більшої гладкості.

1. Introduction and stating the results. Recall the classic estimate of approximation of a function $f \in C[-1,1]$, by algebraic polynomials P_n of degree $\le n$: *for every $n \ge k-1$, $k \in \mathbb{N}$, there exists P_n such that*

$$|f(x) - P_n(x)| \le c\omega_k(f, \rho_n(x)), \quad x \in [-1,1], \tag{1}$$

where c is an absolute constant,

$$\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n},$$

and $\omega_k(f,t)$ is the k-th order modulus of smoothness of f.

This estimate is called the *Nikolskii's type pointwise estimate*, and was proved by Timan ($k = 1$), Dzyadyk ($k = 2$), Freud ($k = 2$), Brudnyi ($k > 2$). See [1] for the details.

Telyakovskii [2] for $k = 1$ and DeVore [3] for $k = 2$ proved that in (1) the function ρ_n may be replaced by the function

$$\delta_n(x) := \frac{\sqrt{1-x^2}}{n},$$

that is for $k = 1$ and $k = 2$ there exist polynomials P_n such that

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$$|f(x) - P_n(x)| \le c\omega_k(f, \delta_n(x)), \quad x \in [-1, 1]. \tag{2}$$

Yu [4], Gonska, Leviatan, Shevchuk, Wenz [5] proved that in contrast with (1) the estimate (2) generally speaking is invalid for $k > 2$.

Starting from the papers by Lorentz, Zeller, DeVore, Newman, Shvedov, Leviatan, Yu and others the problem of approximation of monotone and piecewise monotone functions by comonotone with the functions polynomials was investigated. See the survey by Leviatan [6] and our paper [7] for details.

Let $\Delta^{(2)}$ be the set of convex in $I := [-1, 1]$ functions $f \in C := C[-1, 1]$. Denote by $\mathcal{P}_n$ the set of algebraic polynomials of degree $\le n$. Everywhere below c stand either for different positive absolute constants which may differ even in the same line or for positive constants which may depend only on a number s.

Leviatan [8] for every function $f \in \Delta^{(2)}$ and each $n \ge 1$, proved the existence of a polynomial $P_n \in \mathcal{P}_n \cap \Delta^{(2)}$, such that the estimate (2) holds with $k \le 2$. It was mentioned above that it is impossible to have inequality (2) with $k > 2$, anyway for $k = 3$ Kopotun [9] proved the validity of (1) for $f \in \Delta^{(2)}$ and $P_n \in \Delta^{(2)}$. If $k > 3$, then even the estimate (1) fails to hold for convex approximation. Namely, recently Yushchenko [10] constructed a function $f \in \Delta^{(2)}$, such that for each sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(2)} \cap \mathcal{P}_n$, we have

$$\limsup_{n \to \infty} \left\| \frac{f - P_n}{\omega_4(f, \rho_n)} \right\| = \infty. \tag{3}$$

Here and below

$$\|f\| := \max_{x \in I} |f(x)|.$$

Earlier Wu and Zhou [11] proved (3) with ω_k, $k \ge 5$, instead of ω_4.

As to the estimates of piecewise convex approximation, the authors know three papers [12 – 14] on this subject only. We will use the methods of these papers. In [12, 13] the uniform estimates for coconvex approximation involving $\omega_k(f, 1/n)$, $k \le 3$, were proved. In [14] the pointwise estimate (1) for $k = 3$ was proved but for so called nearly coconvex approximation.

The main result of our paper is Theorem 1 where for the first time an estimate for pointwise coconvex approximation is obtained. Moreover in Theorems 2 – 4 we show that this estimate is the final one in some sense.

We need some notations. Denote by $\mathcal{Y}_s$, $s \in \mathbb{N}$, the set of all collections $Y :=$ $:= \{y_i\}_{i=1}^{s}$ of points y_i:

$$-1 < y_s < \ldots < y_1 < 1.$$

Let $\Delta^{(2)}(Y)$ denote the set of all functions $f \in C$, that change convexity at the points y_i, and are convex in $[y_1, 1]$. That is $f \in \Delta^{(2)}(Y)$ iff f is convex in $[y_1, 1]$, concave in $[y_2, y_1]$, convex in $[y_3, y_2]$, and so on. Recall that if a function f is twice differentiable in I, then $f \in \Delta^{(2)}(Y)$ means that

$$f''(x)\Pi(x) \ge 0, \qquad x \in I,$$

where

$$\Pi(x) := \prod_{i=1}^{s} (x - y_i).$$

Theorem 1. *If $Y \in \mathcal{Y}_s$ and $f \in \Delta^{(2)}(Y)$, then for every $n \geq N(Y)$ there exists a polynomial $P_n \in \mathcal{P}_n$, such that*

$$P_n \in \Delta^{(2)}(Y), \tag{4}$$

and

$$|f(x) - P_n(x)| \leq c\omega_2(f, \delta_n(x)) \tag{5}$$

where $N(Y)$ is the constant depending only on

$$\min_{i=1,\ldots,s-1}(y_i - y_{i+1}),$$

if $s > 1$, and

$$N(Y) = 1, \quad \text{if} \quad s = 1.$$

Theorem 1 readily implies Corollaries 1 – 4.

***Corollary* 1.** *Under the conditions of Theorem 1, for every $n \geq N(Y)$ we have*

$$|f(x) - P_n(x)| \leq c\omega_2(f, \rho_n(x)), \quad x \in I.$$

Denote by W^r, $r \in \mathbb{N}$, the class of functions $f \in C$, that have absolutely continuous derivative $f^{(r-1)}$ in I, and such that $|f^{(r)}(x)| \leq 1$ almost everywhere in I.

***Corollary* 2.** *Let $r = 1$ or $r = 2$, and $Y \in \mathcal{Y}_s$. If $f \in \Delta^{(2)}(Y)$, then for every $n \geq N(Y)$, there exists a polynomial $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$, such that*

$$\left\| \frac{f - P_n}{\delta_n^r} \right\| \leq c, \tag{6}$$

whence

$$\left\| \frac{f - P_n}{\rho_n^r} \right\| \leq c. \tag{6'}$$

The following Theorems 2 and 3 show that if $s = 1$ then Corollaries 1 and 2 (and hence Theorem 1) cannot be had for smoothness larger than 2.

Theorem 2. *For each $r > 2$, $Y \in \mathcal{Y}_1$ and $n \in \mathbb{N}$ there exists a function $f \in \Delta^{(2)}(Y) \cap W^r$, such that for every polynomial $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$ the inequality*

$$\left\| \frac{f - P_n}{\rho_n^r} \right\| > C(Y, r)n^{r-2}$$

holds, where $C(Y, r) = \text{const}$, depends only on Y and r.

Theorem 2 implies Theorem 3.

Denote by C^r, $r \in \mathbb{N}$, the set of r times differentiable in I functions $f \in C$, $C^0 := C$.

Theorem 3. *For no $Y \in \mathcal{Y}_1$, $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, such that $k + r > 2$, there exist constants $C = C(k, r, Y)$ and $N = N(k, r, Y)$ independent of f and n, such that for each function $f \in \Delta^{(2)}(Y) \cap C^r$ and every $n \geq N$ there exists a polynomial $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$, satisfying*

$$|f(x) - P_n(x)| \leq C\rho_n^r(x)\omega_k(f, \rho_n(x)), \quad x \in I. \tag{7}$$

Remark 1. If $s \neq 1$, then we conject that Theorems 2 and 3 fail to hold, that is (7) and (6′) hold with $C = C(k, r, Y)$ and $N = N(k, r, Y)$ for all k's and r's, except the known negative cases. These negative cases are: $(r = 0,\ k \geq 4)$, $(r = 1,\ k \geq 3)$, see Wu and Zhou [11], and $(r = 2,\ k > 3)$, see Gilewicz and Yushchenko [15].

Corollary 3. *If $Y \in \mathcal{Y}_s$ and $f \in \Delta^{(2)}(Y) \cap C^1$, then for every $n \geq N(Y)$, there exists a polynomial $P_n \in \mathcal{P}_n$, such that*

$$P_n \in \Delta^{(2)}(Y),$$

$$|f(x) - P_n(x)| \leq c\delta_n(x)\omega_1(f', \delta_n(x)), \quad x \in I,$$

and

$$|f'(x) - P_n'(x)| \leq c\omega_1(f', \rho_n(x)), \quad x \in I, \tag{8}$$

where $\omega_1(f', t)$ is the (first) modulus of continuity of f'.

Remark 2. If $s > 1$, then in Theorem 1 and in all its corollaries one cannot replace the constant $N(Y)$ by a constant independent of Y, see say [13].

Remark 3. If in (8) one replace ρ_n with δ_n, then, for $s = 1$, the statement of Corollary 3 is, generally speaking, invalid, see Theorem 4.

Theorem 4. *For every $Y \in \mathcal{Y}_1$, $n \in \mathbb{N}$ and $A > 0$, there exists the function $f \in \Delta^{(2)}(Y) \cap C^1$, such that for any polynomial $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$, satisfying*

$$P_n(-1) = f(-1), \qquad P_n(1) = f(1), \qquad P_n'(-1) = f'(-1), \qquad P_n'(1) = f'(1), \tag{9}$$

there is a point $x \in I$, for which

$$|f'(x) - P_n'(x)| \geq A\|f'\| \geq \frac{1}{2}A\,\omega_1(f', \delta_n(x)).$$

To formulate the last Corollary 4, we denote by $\mathrm{Lip}^*\alpha$, $0 < \alpha \leq 2$, the set of functions $f \in C$, such that

$$\omega_2(f, t) = O(t^\alpha), \quad t \to 0.$$

Corollary 4. *Let $0 < \alpha < 2$ and $Y \in \mathcal{Y}_s$. Then $f \in \Delta^{(2)}(Y) \cap \mathrm{Lip}^*\alpha$ iff there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$, such that*

$$\left\| \frac{f - P_n}{\delta_n^\alpha} \right\| = O(1), \quad n \to \infty.$$

We prove Theorems 2 and 4 in Section 4. Theorem 1 is proved in Section 3. To prove Theorem 1 we need Theorem 5, which is of separate interest. We prove Theorem 5 in Section 2. To formulate Theorem 5 we need some notations.

Let $x_j := x_{j,n} := \cos(j\pi/n)$, $j = 0, \ldots, n$, be the Chebyshev partition of I. For a fixed $n \in \mathbb{N}$ and $Y = \{y_i\}_{i=1}^s \in \mathcal{Y}_s$, denote

$$O_i := O_i(n, Y) := (x_{j+2}, x_{j-2}), \quad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

where $x_{n+1} := -1$, $x_{n+2} := -1$ and $x_{-1} := 1$. Let

$$O = O(n, Y) := \bigcup_{i=1}^{s} O_i.$$

We will write $j \in H$, if $x_j \in (-1, 1) \setminus O$.

Theorem 5. *Let $Y \in \mathcal{Y}_s$. If $f \in \Delta^{(2)}(Y)$ then for every $n \ge N(Y)$, there exists a polygon L having the knots at x_j's, with $j \in H$ only, such that*

$$L \in \Delta^{(2)}(Y)$$

and

$$|f(x) - L(x)| \le c\omega_2(f, \delta_n(x)), \quad x \in I, \tag{10}$$

where $N(Y)$ is a constant depending only on Y. If $s = 1$, then $N(Y) = 1$.

One should emphasize, that the polygon L in Theorem 5 is not allowed to have knots at x_j's, if $x_j \in O$.

Below for each $j = 1, \dots, n$, we denote by $I_j := I_{j,n} := [x_j, x_{j-1}]$. For any interval E, let $|E|$ be it's length in particular $|I_j| = x_{j-1} - x_j =: h_j$. We will use the following well-known inequalities without special references

$$h_{j\pm1} < 3h_j,$$

$$\rho_n(x) < h_j < 5\rho_n(x), \quad x \in I_j,$$

$$\rho_n(x) < 2\delta_n(x), \quad x \in I \setminus (I_1 \cup I_n),$$

and Whitney inequality

$$|g(x) - l_1(x)| \le \omega_2\left(g, \frac{b-a}{2}, [a,b]\right), \quad x \in [a,b],$$

where l_1 is the linear function that interpolates g at the points a and b, and $\omega_2(g, t, [a,b])$ is the second modulus of smoothness taken over the interval $[a,b]$.

2. Proof of Theorem 5. Set

$$\omega(t) := \omega_2(f, t).$$

If $s = 1$ then put $N(Y) = 1$, if $s > 1$ then choose $N(Y)$ satisfying

$$O_i(n, Y) \cap O_{i+1}(n, Y) = \varnothing,$$

for all $n \ge N(Y)$ and $i = 1, \dots, s-1$. Fix $n \ge N(Y)$. Denote

$$O_i = O_i(n, Y) =: (\bar{y}_i, \underline{y}_i), \quad i = 1, \dots, s,$$

that is $\bar{y}_i$ and $\underline{y}_i$ are respectively the left and the right ends of the interval O_i.

The following four cases are possible: a) $\underline{y}_1 = 1$, $\bar{y}_s \ne -1$; b) $\underline{y}_1 = 1$, $\bar{y}_s = -1$; c) $\underline{y}_1 \ne 1$, $\bar{y}_s = -1$; d) $\underline{y}_1 \ne 1$, $\bar{y}_s \ne -1$. Let us consider the case a) only. All other cases are similar. That is everywhere below in the proof of Theorem 5

$$\underline{y}_1 = 1, \quad \bar{y}_s \ne -1.$$

We are going to define the functions L_i for each $i = 1, \dots, s$. If $i \ne 1$ then denote by $\bar{L}_i$ the polygon, consisted of three intervals, such that $\bar{L}_i(-1) = 0$, $\bar{L}_i(\bar{y}_i) = 1$, $\bar{L}_i(y_i) = \bar{L}_i(1) = 0$.

Similarly, if $i \ne 1$ then denote by $\underline{L}_i$ the polygon, consisted of three intervals, such that $\underline{L}_i(-1) = \underline{L}_i(y_i) = 0$, $\underline{L}_i(\underline{y}_i) = -1$, $\underline{L}_i(1) = 0$.

Remark that if i is odd then $\bar{L}_i \in \Delta^{(2)}(Y)$ and $\underline{L}_i \in \Delta^{(2)}(Y)$. If i is even then $-\bar{L}_i \in \Delta^{(2)}(Y)$ and $-\underline{L}_i \in \Delta^{(2)}(Y)$.

For every $i = 1, \dots, s$ denote by $\bar{l}_i$ the linear function that interpolates the function f at the points $\bar{y}_i$ and y_i; denote by $\underline{l}_i$ the linear function that interpolates the function f at the points y_i and $\underline{y}_i$.

Put

$$L_1(x) := \frac{1}{2}\left(f(1) - \bar{l}_1(1)\right)(x+1).$$

For every $i \neq 1$, put

$$L_i(x) := \begin{cases} \left(\underline{l}_i(\underline{y}_i) - \bar{l}_i(\underline{y}_i)\right)\underline{L}_i(x), & \text{if } \ (-1)^i\left(\underline{l}_i(\bar{y}_i) - \bar{l}_i(\bar{y}_i)\right) \geq 0; \quad (11) \\ \left(\underline{l}_i(\bar{y}_i) - \bar{l}_i(\bar{y}_i)\right)\overline{L}_i(x), & \text{otherwise.} \quad (12) \end{cases}$$

Evidently,

$$L_i \in \Delta^{(2)}(Y), \quad i = 1, \dots, s. \tag{13}$$

For $i \neq 1$ let

$$l_i^* := l_i^*(x) := \begin{cases} \bar{l}_i(x), & \text{if } \ x \in [\bar{y}_i, y_i]; \\ \underline{l}_i(x), & \text{if } \ x \in [y_i, \underline{y}_i]. \end{cases}$$

Then in the case (11) we have

$$l_i^*(x) + L_i(x) \equiv \bar{l}_i(x), \quad x \in O_i, \tag{14}$$

and, in the case (12),

$$l_i^*(x) + L_i(x) \equiv \underline{l}_i(x), \quad x \in O_i. \tag{15}$$

Now, for $i \neq 1$, we prove the inequality

$$|L_i(x)| \leq c\omega(\rho_n(x)), \quad x \in I. \tag{16}$$

To this end, we consider, say, the case (11). In this case, if $x \in [-1, \bar{y}_i]$, then $L_i(x) \equiv 0$, and (16) is trivial. If $x \in O_i$, then Whitney inequality implies

$$\left|f(x) - \bar{l}_i(x)\right| \leq c\omega(|O_i|),$$

and

$$\left|f(x) - \underline{l}_i(x)\right| \leq c\omega(|O_i|),$$

hence

$$\left|\bar{l}_i(x) - \underline{l}_i(x)\right| \leq c\omega(|O_i|), \quad x \in O_i.$$

Therefore (16) holds for $x \in O_i$. In particular

$$\left|L_i(\underline{y}_i)\right| \leq c\omega(\rho_n(y_i)) \leq c\omega\left(\frac{1}{n}\right).$$

Now, if $x \geq \left|\underline{y}_i\right|$, then

$$|L_i(x)| \leq c\omega\left(\frac{1}{n}\right)(1-x) \leq c\omega\left(\frac{1}{n}\right)(1-x^2) \leq$$
$$\leq c\omega(\delta_n(x)) \leq c\omega(\rho_n(x)).$$

If $\underline{y}_i \leq x < \left|\underline{y}_i\right|$, then $\rho_n(y_i) \leq \rho_n(x)$, whence

$$|L_i(x)| \le \left|L_i(\underline{y}_i)\right| \le c\omega(\rho_n(y_i)) \le c\omega(\rho_n(x)).$$

Thus (16) is proved.

For $i = 1$, the similar arguments yield

$$|L_1(x)| \le c\omega\left(\frac{1}{n^2}\right), \quad x \in I_1,$$

whence

$$|L_1(x)| \le c\omega\left(\frac{1}{n^2}\right)(1+x) \le c\omega\left(\frac{1}{n^2}\right) \le$$

$$\le c\omega(\rho_n(x)), \quad x \in I. \tag{17}$$

Denote by $L^* := L^*(x)$ the polygon with the knots at points x_j's, $j \in H$, and points y_i, $i = 1, \dots, s$, that interpolates the function f at these knots and also at points -1 and y_1 (that is, generally speaking, $L^*(1) \ne f(1)$). Evidently

$$L^* \in \Delta^{(2)}(Y). \tag{18}$$

Whitney inequality yields

$$\left|f(x) - L^*(x)\right| \le c\omega(\rho_n(x)), \quad x \in I. \tag{19}$$

Finally we show that the polygon

$$L := L(x) := L^*(x) + \sum_{i=1}^{s} L_i(x)$$

is a required one. Indeed, (18) and (13) readily imply $L \in \Delta^{(2)}(Y)$. Relations (14) and (15) mean that the polygon L does not have any knots, except x_j with $j \in H$. Inequalities (16), (17) and (19) yield the estimate (10) for $x \in I \setminus (I_1 \cup I_n)$.

So we left with (10) for $x \in I_1$ and $x \in I_n$. By its construction, L is a linear function in I_1 and in I_n, and $L(-1) = f(-1)$, $L(1) = f(1)$. We put $g(x) := f(x) - L(x)$. Then we have $g(-1) = 0$, $g(1) = 0$, $\omega_2(f, t, I_1) = \omega_2(g, t, I_1) \le \omega(t)$ and $\omega_2(g, t, I_n) \le \omega(t)$, where $\omega_2(g, t, [a,b])$ is the second modulus of smoothness taken over the interval $[a,b]$. Besides, inequalities (16), (17) and (19) imply

$$\|g\|_{I_1} \le c\omega\left(\frac{1}{n^2}\right) \quad \text{and} \quad \|g\|_{I_n} \le c\omega\left(\frac{1}{n^2}\right).$$

Then, say, for $x \in I_1$, we apply Marchaud inequality and get

$$|f(x) - L(x)| = |g(x)| = |g(x) - g(1)| \le c(1-x)\int_{1-x}^{|I_1|} \frac{\omega(u)}{u^2}du + \frac{1-x}{|I_1|}\omega(|I_1|) \le$$

$$\le c(1-x)\int_{1-x}^{\delta_n(x)} \frac{\omega(u)}{u^2}du + c(1-x)\int_{\delta_n(x)}^{|I_1|} \frac{\omega(u)}{u^2}du + c\omega(\delta_n(x)) \le$$

$$\le c(1-x)\omega(\delta_n(x))\int_{1-x}^{\infty} \frac{du}{u^2} + c(1-x^2)|I_1|\frac{\omega(\delta_n(x))}{\delta_n^2(x)} + c\omega(\delta_n(x)) \le$$

$$\le c\omega(\delta_n(x)) + cn^2|I_1|\omega(\delta_n(x)) \le c\omega(\delta_n(x)).$$

Similarly one checks (10) for $x \in I_n$. Theorem 5 is proved.

Corollary of Theorem 5. *If L is the polygon guaranteed by Theorem 5, then*

$$\Pi(x_j)\left[x_{j-1},x_j,x_{j+1};L\right] \geq 0, \quad j \in H, \tag{20}$$

$$\left\|\left[x_{j-1},x_j,x_{j+1};L\right]\right\| \leq c\frac{\omega(h_j)}{h_j^2}, \quad j = 1,\ldots,n-1, \tag{21}$$

and

$$\left[x_{j-1},x_j,x_{j+1};L\right] = 0, \quad j \notin H, \tag{22}$$

where $\left[x_{j-1},x_j,x_{j+1};L\right]$ are the second divided differences of L.

3. Proof of Theorem 1. Following [16], we put

$$t_j(x) := t_{j,n}(x) := \frac{\cos^2 2n\arccos x}{\left(x-x_j^0\right)^2} + \frac{\sin^2 2n\arccos x}{\left(x-\bar{x}_j\right)^2},$$

where $\bar{x}_j = \cos\frac{(j-1/2)\pi}{n}$ and $x_j^0 = \cos\beta_j^0$ with $\beta_j^0 = \frac{(j-1/4)\pi}{n}$, $j \leq n/2$, and $\beta_j^0 = \frac{(j-3/4)\pi}{n}$, $j > n/2$. Note that $\bar{x}_j$ and x_j^0 are zeros of the respective numerators which are contained strictly in the interior of I_j, and that the t_j are algebraic polynomials of degree $4n-2$ with the following property

$$t_j(x) \leq \frac{c}{\left(|x-x_j|+h_j\right)^2} \leq ct_j(x), \quad x \in I.$$

Following [7, 9, 13, 14] we consider two polynomials of degree $\leq cn$

$$T_j(x) = T_{j,n}(x;Y) := \frac{1}{d_j}\int_{-1}^{x} t_j^{6s}(u)\Pi(u)\,du, \quad j \in H$$

$\left(T_{j\pm1} := T_{2j\pm1,2n},\ \text{if}\ j\pm1 \notin H\right)$, where

$$d_j = \int_{-1}^{1} t_j^{6s}(u)\Pi(u)\,du,$$

and

$$\tau_j(x) := \alpha\int_{-1}^{x} T_{j-1}(u)\,du + (1-\alpha)\int_{-1}^{x} T_{j+1}(u)\,du, \quad j \in H,$$

where $0 \leq \alpha \leq 1$ is chosen from the condition

$$\tau_j(1) = 1 - x_j.$$

Denote

$$\chi(x,a) := \begin{cases} 0, & \text{if}\quad x \leq a; \\ 1, & x > a, \end{cases} \quad a \in I, \qquad \chi_j(x) := \chi(x,x_j),$$

$$(x-x_j)_+ := (x-x_j)\chi_j(x),$$

$$\Gamma_j(x) := \frac{h_j}{|x-x_j|+h_j},$$

and recall the estimate

$$h_j \Gamma_j(x) \le c\rho_n(x), \quad x \in I. \tag{23}$$

We have the following lemma.

Lemma 1 [7, 13, 14]. *For each* $j \in H$, *the polynomial* τ_j *satisfies*

$$\tau_j''(x)\Pi(x)\Pi(x_j) \ge 0, \quad x \in I, \tag{24}$$

$$\tau_j'(\pm 1) = \chi_j(\pm 1), \quad \tau_j(-1) = 0, \quad \tau_j(1) = (1 - x_j), \tag{25}$$

$$\left|(x - x_j)_+ - \tau_j(x)\right| \le c h_j \Gamma_j^6(x), \quad x \in I, \tag{26}$$

$$\left|\chi_j(x) - \tau_j'(x)\right| \le c\Gamma_j^4(x), \quad x \in I. \tag{27}$$

Lemma 1 implies Lemma 2.

Lemma 2. *For each* $j \in H$, *the polynomial* τ_j *satisfies*

$$\left|(x - x_j)_+ - \tau_j(x)\right| \le c(1 - x^2)\Gamma_j^4(x), \quad x \in I. \tag{28}$$

Proof. If $x \notin (I_1 \cup I_n)$ the (23) implies

$$h_j \Gamma_j^2(x) \le c n^2 \rho_n^2(x) \le c(1 - x^2),$$

whence (28) follows from (26). If $x \in I_1$, then by (25) and (27)

$$\left|(x - x_j)_+ - \tau_j(x)\right| = \left|\left((x - x_j)_+ - \tau_j(x)\right) - \left((1 - x_j)_+ - \tau_j(1)\right)\right| =$$

$$= \left|\int_x^1 \left(\chi_j(u) - \tau_j'(u)\right)du\right| \le \int_x^1 \left|\left(\chi_j(u) - \tau_j'(u)\right)\right|du \le$$

$$\le c(1 - x)\max_{t \in I_1} \Gamma_j^4(t) = c(1 - x)\Gamma_j^4(1) \le c(1 - x)\Gamma_j^4(x) \le c(1 - x^2)\Gamma_j^4(x).$$

Similarly one proves (28) for $x \in I_n$. Lemma 2 is proved.

Proof of Theorem 1. Let L be the polygon guaranteed by Theorem 5. We represent L in the form

$$L(x) \equiv l(x) + \sum_{j=1}^{n-1} \left[x_{j+1}, x_j, x_{j-1}; L\right](x_{j-1} - x_{j+1})(x - x_j)_+ \equiv$$

$$\equiv l(x) + \sum_{j \in H} \left[x_{j+1}, x_j, x_{j-1}; L\right](x_{j-1} - x_{j+1})(x - x_j)_+,$$

where $l(x) := \left[x_n, x_{n-1}; L\right](x + 1) + L(-1)$, and where we used (22).

Put

$$P_n(x) := l(x) + \sum_{j \in H} \left[x_{j+1}, x_j, x_{j-1}; L\right](x_{j-1} - x_{j+1})\tau_j(x).$$

The inequalities (20) and (24) imply

$$\left[x_{j+1}, x_j, x_{j-1}; L\right](x_{j-1} - x_{j+1})\tau_j''(x)\Pi(x) =$$

$$= \frac{1}{\Pi^2(x_j)}\Pi(x_j)\left[x_{j+1}, x_j, x_{j-1}; L\right](x_{j-1} - x_{j+1})\tau_j''(x)\Pi(x) \ge 0,$$

$$x \in I, \quad j \in H,$$

that yields (4). To prove (5) we represent the difference $f - P_n$ in the form

$$f(x) - P_n(x) = f(x) - L(x) + L(x) - P_n(x) = f(x) - L(x) +$$

$$+ \sum_{j \in H} [x_{j+1}, x_j, x_{j-1}; L](x_{j-1} - x_{j+1})((x - x_j)_+ - \tau_j(x)) =$$

$$=: f(x) - L(x) + \sum_{j \in H} \alpha_j(x).$$

By virtue of (10),

$$|f(x) - L(x)| \le c\omega(\delta_n(x)), \quad x \in I.$$

To estimate $\alpha_j(x)$ we use the inequalities (26), (28) and (21). If $x \notin I_1$ and $x \notin I_n$, then

$$|\alpha_j(x)| \le c\frac{\omega(h_j)}{h_j^2} h_j h_j \Gamma_j^4(x) = c\omega(h_j)\Gamma_j^4(x) \le$$

$$\le c\omega(\rho_n(x))\left(1 + \frac{h_j^2}{\rho_n^2(x)}\right)\Gamma_j^4(x) \le c\omega(\rho_n(x))\Gamma_j^2(x) \le c\omega(\delta_n(x))\Gamma_j^2(x),$$

where we used (23). If $x \in I_1$, then

$$|\alpha_j(x)| \le c\frac{\omega(h_j)}{h_j^2} h_j (1 - x^2)\Gamma_j^4(x) \le c\omega(\delta_n(x))\Gamma_j^2(x),$$

where we again used (23). Therefore taking into account that (23) implies $\left\| \sum_{j=1}^n \Gamma_j^2 \right\| \le c$, we get

$$\left| \sum_{j \in H} \alpha_j(x) \right| \le c\omega(\delta_n(x)) \sum_{j \in H} \Gamma_j^2(x) \le c\omega(\delta_n(x)) \left\| \sum_{j=1}^n \Gamma_j^2 \right\| \le c\omega(\delta_n(x)), \quad x \in I.$$

Theorem 1 is proved.

4. Proofs of Theorems 2 and 4. Let us recall some well-known facts. Estimate (1), Dzyadyk [1] inequality $\|\rho_n^{1-r} P_n'\| \le c \|\rho_n^{-r} P_n\|$ and Trigub [17] estimates of simultaneous approximation imply the following. *If* $f \in W^r$ *and* $P_n \in \mathcal{P}_n$, $n \ge r - 1$, *then*

$$\left\| \frac{f' - P_n'}{\rho_n^{r-1}} \right\| \le c_1 \left\| \frac{f - P_n}{\rho_n^r} \right\| + c_2 \| f^{(r)} \|, \tag{29}$$

where one may assume that $c_1 \ge 1$.

Let $y_1 \in (-1, 1)$. Then Bernstein inequality provides the existence of a constant n_0, such that for each $n \ge n_0$ and $P_n \in \mathcal{P}_n$, we have

$$|P_n''(x)| \le \frac{2}{\delta_n^2(y_1)} \| P_n \|, \quad x \in [y_1 - \delta_n(y_1), y_1]. \tag{30}$$

Denote by

$$S(x) := \frac{\int_{-1}^{x} u^{r-2}(u+1)^{r-2}\, du}{\int_0^{-1} u^{r-2}(u+1)^{r-2}\, du},$$

$$c_3 := \max_{x \in [-1, 0]} |S^{(r-1)}(x)|,$$

and remark that

$$S(-1) = 0, \quad S(0) = -1, \quad S^{(j)}(-1) = S^{(j)}(0) = 0, \quad j = 1, \dots, r-2,$$
$$S'(x) \le 0, \quad x \in [-1, 0],$$

and

$$\int_{-1}^{0} S(x)\,dx = -\frac{1}{2}.$$

***Proof of Theorem* 2.** We fix $y_1 \in (-1, 1)$, put $Y := \{y_1\}$ and recall that $\Delta^{(2)}(Y)$ is a set of continuous in I functions, which are convex in $[y_1, 1]$ and concave in $[-1, y_1]$. Then we fix an arbitrary large $n \ge n_0$, satisfying $y_1 - \delta_n(y_1) > -1$, and $c_2 c_3 \le 0{,}1\, n^{2r-2} \delta_n^r(y_1)$. Put

$$\delta := \delta_n(y_1) = \frac{\sqrt{1-y_1^2}}{n},$$

$$S_\delta(x) := \begin{cases} 0, & \text{if} \quad x < y_1 - \delta; \\ S\left(\frac{x - y_1}{\delta}\right), & \text{if} \quad y_1 - \delta \le x \le y_1; \\ -1, & \text{if} \quad x > y_1, \end{cases}$$

$$f(x) := \int_{-1}^{x} S_\delta(u)\,du.$$

Evidently,

$$f \in \Delta^{(2)}(Y),$$
$$f(-1) - f(1) = -f(1) = 1 - y_1 + 0{,}5\delta, \tag{31}$$

and

$$\left\| f^{(r)} \right\| = \left\| S_\delta^{(r-1)} \right\| = c_3 \delta^{1-r}.$$

Now Theorem 2 follows from the inequality

$$\left\| \frac{f - P_n}{\rho_n^r} \right\| \ge \frac{1}{20 c_1} \delta n^{2r-2}, \tag{32}$$

for each polynomial $P_n \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$. To prove (32) we assume to the contrary that, for some polynomial $P \in \Delta^{(2)}(Y) \cap \mathcal{P}_n$,

$$\left\| \frac{f - P}{\rho_n^r} \right\| < \frac{1}{20 c_1} \delta n^{2r-2}. \tag{33}$$

Then (29) and the choice of n imply

$$\left\| \frac{f' - P'}{\rho_n^{r-1}} \right\| \le \frac{1}{20} \delta n^{2r-2} + c_2 c_3 \delta^{1-r} \le 0{,}1 \delta n^{2r-2}.$$

In particular

$$\left| f'(\pm 1) - P'(\pm 1) \right| \le 0{,}1\delta.$$

Since $(x - y_1) P'(x) \ge 0$, $x \in I$, then

$$P(1) - P(y_1) = \int_{y_1}^{1} P'(x)dx \le P'(1)(1-y_1) \le (0{,}1\delta - 1)(1-y_1),$$

$$P(y_1-\delta) - P(-1) \le P'(-1)(1+y_1-\delta) \le 0{,}1\delta(1+y_1-\delta),$$

and

$$-\|P'\| = P'(y_1) \le P'(1) \le 0{,}1\delta - 1.$$

Therefore (30) implies

$$P(y_1) - P(y_1-\delta) \le \delta P'(y_1) + \frac{\delta^3}{3!}P'''(y_1 - \Theta\delta) \le$$

$$\le \delta P'(y_1) + 0{,}2\delta\|P'\| = 0{,}8\delta P'(y_1) \le 0{,}8\delta(0{,}1\delta - 1).$$

Hence

$$P(1) - P(-1) \le (0{,}1\delta-1)(1-y_1) + 0{,}1\delta(1+y_1-\delta) + 0{,}8\delta(0{,}1\delta-1) <$$

$$< -0{,}6\delta - 1 + y_1.$$

Thus, by (31)

$$f(1) - f(-1) - P(1) + P(-1) > 0{,}1\delta,$$

that contradicts to (33), since (33) implies

$$|f(1) - f(-1) - P(1) + P(-1)| \le |f(1) - P(1)| + |f(-1) - P(-1)| \le$$

$$\le \frac{2}{20c_1}\delta n^{2r-2}\frac{1}{n^{2r}} \le 0{,}1\delta\frac{1}{n^2} < 0{,}1\delta.$$

Theorem 2 is proved.

***Proof of Theorem* 4.** For the simplicity let $y_1 = 0$. Choose a number $b > 0$ from the condition

$$bn^2(A+1) < 1.$$

Put

$$f'(x) := \begin{cases} 0, & \text{if } x \in [0,1]; \\ -\dfrac{x}{b}, & \text{if } x \in [-b, 0]; \\ 1, & \text{if } x \in [-1, -b], \end{cases}$$

$$f(x) = \int_{-1}^{x} f'(u)du.$$

Let us show that this function is a required one. Indeed, assume to the contrary that there is $P_n \in \Delta^{(2)}(\{0\})$ satisfying (9) and

$$|f'(x) - P_n'(x)| \le A\|f'\| = A, \qquad x \in I.$$

Then $\|P_n'\| \le A+1$ and by Markov inequality

$$\|P_n''\| \le n^2(A+1).$$

Since $P_n' \in \Delta^{(1)}(\{0\})$, $P_n(-1) = f(-1)$ and $P_n(1) = f(1)$, then $P_n'(x) \le f'(x)$,

$x \in [-1, -b] \cup [0, 1]$. Show that $P_n'(x) < f'(x)$, $x \in (-b, 0)$. Indeed, otherwise the graph of P_n' would intersect the graph of f' at least at two points in $(-b, 0)$, and hence a point $\Theta \in (-b, 0)$ would exist, such that $P_n''(\Theta) = f''(\Theta) = -1/b$. Therefore $1/b \le \|P_n''\| \le n^2(A+1)$, that contradicts to the choice of the number b. Thus we have, $P_n(\pm 1) = f(\pm 1)$, $P_n'(x) \le f'(x)$, $x \in I$, and $P_n'(x) < f'(x)$, $x \in (-b, 0)$, which is impossible. Theorem 4 is proved.

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