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HEDGING OF OPTIONS UNDER MEAN-SQUARE CRITERION AND SEMI-MARKOV VOLATILITY

ХЕДЖУВАННЯ ОПЦІОНУ ЗА УМОВ СЕРЕДНЬОКВАДРАТИЧНОГО КРИТЕРІЮ ТА ПІВМАРКОВСЬКИХ МІНЛИВОСТЕЙ

We consider a problem of hedging of the European call option for a model such that appreciation rate and volatility are functions of a semi-Markov process. In such a model, the market is incomplete.

Розглядається задача хеджування Європейського опціону купівлі для моделі з нормою повернення та коефіцієнтом мінливості, що залежать від півмарковського процесу. В такій моделі ринок є неповним.

1. Introduction. In famous Black – Scholes model, which is used for evaluation of option prices, it is supposed that the dynamic of stocks prices is set by the linear stochastic differential equation

$$dS_t = aS_t dt + \sigma S_t dW_t,$$

where a and σ are deterministic functions (in the simplest case – constants).

We suppose that, in our model, the coefficients a and σ, appreciation rate and volatility respectively, are depended on a semi-Markov process X_t, which doesn't depend on standard Wiener process W_t. We consider the hedging problem of the European call option with terminal payment $H = f(S_T)$.

Since the additional source of randomness exists (the semi-Markov process X_t) in addition to the Wiener process W_t, the market is incomplete and perfect hedging is not possible. We find a strategy, which locally minimizes the risk.

2. Description of the model and preliminary notions. Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), \mathcal{P})$ be a probability space. We suppose that risk assets (stock) exist and their price evolution is given by the following stochastic differential equation [5]:

$$dS_t = a(X_t)S_t dt + \sigma(X_t)S_t dW_t, \tag{1}$$

where W_t is a Wiener process, X_t is some observed variable, which is described by a semi-Markov process [4, 5] with the phase space $(\mathbf{X}, \mathcal{X})$, $X_t := X_{\nu(t)}$, $\nu(t) := \max\{n : \tau_n \le t\}$, $\tau_n := \sum_{k=1}^{n} \theta_k$, $(X_n, \theta_n : n \ge 0)$ is a Markov renewal process, $\mathcal{P}\{\omega : X_{n+1} \in A, \ \theta_{n+1} \le t / X_n = x\} = P(x, A) \cdot G_x(t)$, $x \in \mathbf{X}$, $A \in \mathcal{X}$, $t \ge 0$. We suppose that $G_x(t)$ is a differentiable function of t and $g_x(t) := dG_x(t)/dt$, $\forall x \in \mathbf{X}$. Coefficients a and σ are measurable functions on $\mathbf{X}$, $\sigma > 0$, processes W_t and X_t are independent, and filtration $\mathbf{F}$ is generated by X_t and W_t.

We solve the problem of hedging of the European call option which is sold at the moment $t = 0$, with terminal payment $H = f(S_T)$ at a cancellation moment T. It's considered that $EH^{2+\varepsilon} < +\infty$, $\varepsilon > 0$.

Besides the risk assets, we have nonrisk assets (bond or bank account), and we suppose that its price is a constant and is equal to one (without of loss generality) at all moments (i.e., percentage rate is equal to zero).

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Stock exchange strategy (SES) π is a pair (γ, β), where $\gamma = (\gamma_t)$ is such a predictable process that

$$E\int_0^T \gamma_t^2\sigma^2(X_t)S_t^2 dt + E\left(\int_0^T |\gamma_t|\,|a(X_t)|S_t dt\right)^2 < +\infty, \qquad (2)$$

$\beta = (\beta_t)$ is a coordinated process, $E\beta_t^2 < +\infty$, $\forall t \le T$.

SES defines a portfolio with a number of units of risk assets γ_t (which the holder has at the moment t) and with a number of means that were invested in bonds at the moment t.

The value process $V(\pi)$ of the portfolio with respect to the strategy π is defined as

$$V_t(\pi) = \gamma_t S_t + \beta_t, \qquad (3)$$

and the *cost process*

$$C_t(\pi) = V_t(\pi) - \int_0^t \gamma_t\, dS_t \qquad (4)$$

SES π is said to be *H-admissible* if $V_T(\pi) = H$; SES π is said to be *self-financing* (or mean-value self-financing) if the cost process $C_t(\pi)$ is a constantine time (or martingale).

The *residual risk* is defined by the formula

$$\mathcal{R}_t(\pi) := E\{|C_T(\pi) - C_t(\pi)|^2 / \mathcal{F}_t\}. \qquad (5)$$

The SES H-admissible strategy π^* is called a *risk-minimizing* if, for any H-admissible SES π and for any t,

$$\mathcal{R}_t(\pi^*) \le \mathcal{R}_t(\pi)$$

The purpose of this article is to find a risk-minimizing H-admissible SES.

It was stated in [2] that existence of a risk-minimizing H-admissible SES is equivalent to the existence of the expansion of the terminal payment H in the form

$$H = H_0 + \int_0^T \gamma_r^H dS_r + L_T^H \quad (\mathcal{P}-\text{a.s.}), \qquad (6)$$

where $H_0 \in L^2(\mathcal{F}_0, \mathcal{P})$, γ_H satisfies (2), and L_T^H is a square-integrable martingale, which is orthogonal to the martingale component S. Then the γ-component of a risk-minimizing strategy π is $\gamma = \gamma^H$ and $C_\pi = H_0 + L^H$.

To obtain expansion (6), it is introduce a *minimal martingale measure* $\tilde{\mathcal{P}}$ [2].

For our model (1), the minimal martingale measure is $\tilde{\mathcal{P}} = \rho_T \mathcal{P}$, where the density ρ_T is defined by the equality

$$\rho_T = \exp\left\{-\int_0^T \frac{a(X_r)}{\sigma(X_r)}\, dW_r - \frac{1}{2}\int_0^T \frac{a^2(X_r)}{\sigma^2(X_r)}\, dr\right\}. \qquad (7)$$

In our situation, when the process S_t has continuous paths, the process $a(X)/\sigma(X)$ is bounded and $H \in L^{2+\varepsilon}(\mathcal{P})$, $\varepsilon > 0$, and the desired expansion (6) can be obtained from Kunita – Watanabe's expansion (as $t = T$) with respect to the measure $\tilde{\mathcal{P}}$:

$$\tilde{E}(H/\mathcal{F}_t) = \tilde{E}H + \int_0^t \tilde{\gamma}_r^H dS_r + \tilde{L}_t^H. \tag{8}$$

In such a way, it is need to find the Kunita – Watanabe's expansion.

Let's introduce some notations.

A jump measure for X_t has the following form:

$$\mu([0,t]\times A) = \sum_{n\ge 0} I(X_n \in A,\ \tau_n \le t), \quad A \in \mathcal{X}, \quad t \ge 0. \tag{9}$$

It's known [4], that **F**-dual predictable projection for μ has the form:

$$\nu(dt, dy) = \sum_{n\ge 0} I(\tau_n < t \le \tau_{n+1}) \frac{P(X_n, dy) g_{X_n}(t)}{\overline{G}_{X_n}(t)} dt. \tag{10}$$

where $\overline{G}_x(t) := 1 - G_x(t)$, $g_x(t) := dG_x(t)/dt$, $\forall x \in \mathbf{X}$, $t \ge 0$.

For given $H \in L_2(\tilde{\mathcal{P}})$, we find the expansion

$$\tilde{E}(H/\mathcal{F}_t) = \tilde{E}H + \int_0^t \tilde{\gamma}_r^H dS_r + \int_0^t \int_{\mathbf{X}} \tilde{\psi}^H(r, y)(\mu - \nu)(ds, dy). \tag{11}$$

We note that the last integral in (11) is $\tilde{\mathcal{P}}$-orthogonal to S (it's an $(\mathbf{F}, \tilde{\mathcal{P}})$-martingale), and the uniqueness of the Kunita – Watanabe's expansion is a guarantee that (11) is the desired expansion of (8) for our case.

Finally, a risk-minimizing H-admissible strategy π^* is defined by $\pi^* = (\gamma^*, \beta^*)$, where $\gamma^* = \tilde{\gamma}^H$, and β^* is such that $V_t(\pi^*) = \tilde{E}(H/\mathcal{F}_t)$, i.e.,

$$\beta_t^* = \tilde{E}(H/\mathcal{F}_t) - \tilde{\gamma}_t^H S_t. \tag{12}$$

In the next section we will obtain an exact representation for $\pi^* = (\gamma^*, \beta^*)$.

3. The result. Let $f(z)$ be a function such that $|f(z)| \le c\cdot(1+|z|)^m$ for some $m \ge 0$. Let's consider a function $u(t, z, x)$ on $[0, T]\times \mathbf{R}_+ \times \mathbf{X}$ such that it is a solution of the Cauchy problem:

$$\begin{cases} u_t(t,z,x) + \dfrac{1}{2}\sigma^2(x)\cdot z^2 \cdot u_{zz}(t,z,x) + Au(t,z,x) = 0, \\ u(T,z,x) = f(z), \end{cases} \tag{13}$$

where

$$Au(t,z,x) := \frac{g_x(t)}{\overline{G}_x(t)} \int_{\mathbf{X}} P(x, dy)[u(t,z,y) - u(t,z,x)]. \tag{14}$$

Theorem 1. *The risk-minimizing H-admissible stock exchange strategy $\pi^* = (\gamma^*, \beta^*)$ is given by the following formula:*

$$\gamma_t^* = u_z(t, S_t, X_t),$$
$$\beta_t^* = V_t(\pi^*) - \gamma_t^* \cdot S_t, \tag{15}$$

where

$$V_t(\pi^*) = \tilde{E} f(S_T) + \int_0^t u_z(r, S_r, X_r) dS_r + \int_0^t \int_{\mathbf{X}} \psi(r, y)(\mu - \nu)(dr, dy), \qquad (16)$$

$$\psi(r, y) = u(r, S_r, y) - u(r, S_r, X_{r-}).$$

The residual risk process has the form

$$\mathcal{R}_t(\pi^*) = E\left(\int_t^T [A u^2(r, S_r, X_r) - 2u(r, S_r, X_r) A u(r, S_r, X_r)] \, ds / \mathcal{F}_t \right).$$

In particular, the residual risk at the moment $t = 0$ *is equal to*

$$\mathcal{R}_0(\pi^*) = E\left(\int_0^T [A u^2(r, S_r, X_r) - 2u(r, S_r, X_r) A u(r, S_r, X_r)] \, ds \right), \qquad (17)$$

where the operator A *was defined in* (14).

4. Proof. By applying Ito's formula to the solution of (13) we obtain:

$$f(S_T) = u(T, S_T, X_T) = u(0, z, x) + \int_0^T u_z(r, S_r, X_r) dS_r +$$

$$+ \int_0^T \left[u_t(r, S_r, X_r) + \frac{1}{2} \sigma^2(X_r)(S_r)^2 u_{zz}(r, S_r, X_r) \right] dr +$$

$$+ \sum_{r \le T} [u(r, S_r, X_r) - u(r-, S_{r-}, X_{r-})]. \qquad (18)$$

We note that, for any function h on $[0, T] \times \mathbf{X}$, right-continuous and left-limit of t, we have

$$\sum_{r \le T} [h(r, X_r) - h(r-, X_{r-})] = \int_0^t \int_{\mathbf{X}} [h(r, y) - h(r-, X_{r-})] \mu(dr, dy) =$$

$$= \int_0^t \int_{\mathbf{X}} [h(r, y) - h(r-, X_{r-})] (\mu - \nu)(dr, dy) +$$

$$+ \int_0^t \int_{\mathbf{X}} [h(r, y) - h(r-, X_{r-})] \nu(dr, dy) =$$

$$= \int_0^t \int_{\mathbf{X}} [h(r, y) - h(r-, X_{r-})] (\mu - \nu)(dr, dy) +$$

$$+ \int_0^t \int_{\mathbf{X}} \frac{g_{X_{r-}}(r)}{\overline{G}_{X_{r-}}(r)} P(X_{r-}, dy) [h(r, y) - h(r-, X_{r-})] \, dr =$$

$$= \int_0^t \int_{\mathbf{X}} [h(r, y) - h(r-, X_{r-})] (\mu - \nu)(dr, dy) + \int_0^t A h(r-, X_{r-}) \, dr \qquad (19)$$

(see, (10) and (14)).

Hence, from (13), (18), and (19) we obtain

$$f(S_T) = u(T, S_T, X_T) = u(0, z, x) + \int_0^T u_z(r, S_r, X_r)\,dS_r +$$

$$+ \int_0^T \int_{\mathbf{X}} [\, u(r, S_r, X_r) - u(r, S_r, X_{r-})](\mu - \nu)(dr, dy), \tag{20}$$

and relations (15), (16) are valid.

Residual risk process can be expressed in the following was (see (11)):

$$\mathcal{R}_t(\pi^*) = E\left(\left[\int_t^T \int_{\mathbf{X}} \psi(r, y)(\mu - \nu)(dr, dy) \right]^2 / \mathcal{F}_t \right) =$$

$$= E\left(\int_t^T \int_{\mathbf{X}} [\, u(r, S_r, y) - u(r, S_r, x_r)]^2 \nu(dr, dy) / \mathcal{F}_t \right) =$$

$$= E\left(\int_t^T \int_{\mathbf{X}} \frac{g_{X_r}(r)}{\overline{G}_{X_r}(r)} P(X_{r-}, dy) [\, u(r, S_r, y) - u(r, S_r, X_{r-})]^2 / \mathcal{F}_t \right) =$$

$$= E\left(\int_t^T [\, A u^2(r, S_r, X_r) - 2u(r, S_r, X_r) A u(r, S_r, X_r)]\, dr / \mathcal{F}_t \right),$$

and the theorem is proved.

We must only prove that the solution of Cauchy problem (13) exists. We do it in the next section.

5. Random evolution approach. Let $\tilde{S}_t$ be a solution of the stochastic differential equation

$$d\tilde{S}_t = \sigma(X_t)\tilde{S}_t dW_t, \quad \tilde{S}_0 = z. \tag{21}$$

This solution has the form

$$\tilde{S}_t = z \exp\left\{ \int_0^t \sigma(X_r)\,dW_r - \frac{1}{2} \int_0^t \sigma^2(X_r)\,dr \right\}. \tag{22}$$

We note that $\tilde{S}_t$ is a continuous semi-Markov random evolution [5; 4, p. 77], $V^c(t)$:

$$V^c(t) f(z) := E[\, f(\tilde{S}_t) / X(s),\ 0 \le s \le t]. \tag{23}$$

i.e., random evolution underlying the semi-Markov process X_t. This evolution is generated by the following generating operators:

$$\Gamma(x) f(z) := \frac{1}{2} \sigma^2(x) z^2 f''(z) \quad \forall\, f(z) \in \mathbf{C}^2(\mathbf{R}_+). \tag{24}$$

Further, let's consider the following process $(X_t, t - \tau_{\nu(t)})$. It is a Markov process on $\mathbf{X} \times \mathbf{R}_+$ with the infinitesimal operator $\hat{A} = A + d/dt$, where A is defined in (14).

The expectation for the random evolution $V^c(t)$ of Markov process $(X_t, t - \tau_{\nu(t)})$ satisfies the following equation:

$$v(t, z, x) := E[\, V^c(t) f(z, X_t, t - \tau_{\nu(t)})].$$

$$\begin{cases} \dfrac{dv}{dt} = \Gamma(x)v + \hat{A}v, \\ v(0, z, x) = f(z, x, 0), \end{cases} \tag{25}$$

where $\Gamma(x)$ is defined in (24).

Let $(\tilde{S}_t^{z,x}, X_t^x, t - \tau_{\nu(t)})$ be a Markov process with the initial point $(z, x, 0)$ with the first component $\tilde{S}_t$ in (21). X_t^x be a semi-Markov process. From (21)–(25) we obtain the following result:

Lemma 1. *Function*

$$u(t, z, x) := E\, f(\tilde{S}_{T-t}^{z,x}) \tag{26}$$

is a solution of problem (13).

Proof. From (22) we have

$$u(T-t, z, x) = E\, f(\tilde{S}_t^{z,x}) = \int f(y) y^{-1} h(y; t, z, x)\, dy, \tag{27}$$

where

$$h(y; t, z, x) = \int \varphi\Big(\xi, \ln\frac{y}{z} + \frac{1}{2}\xi \Big) F_t^x(d\xi) = E\varphi\Big(z_t^x, \ln\frac{y}{z} + \frac{1}{2} z_t^x \Big), \tag{28}$$

$\varphi(t, x) = (2\pi t)^{-1/2} \exp\{-x^2/2t\}$. F_t^x is a distribution of the random variable

$$Z_t^x = \int_0^t \sigma^2(X_r^x)\, dr. \tag{29}$$

Let v be a solution of the equation

$$v_t(t, \eta, x) = \sigma^2(x) v_{\eta}(t, \eta, x) + \hat{A}v(t, \eta, x) \tag{30}$$

with the initial condition $v(0, \eta, x) = g(\eta)$. From Ito's formula we have that

$$v(t, \eta, x) = E g(\eta + Z_t^x). \tag{31}$$

By substituting in this formula $g(\eta) = \varphi(\eta - \ln(y/x), (\ln(y/x) + \eta)/2)$, we obtain that

$$h(y; t, z, x) = v\Big(\ln\frac{y}{z}, t, x \Big). \tag{32}$$

From (30) and (32) we have:

$$h_t(y; t, z, x) = \sigma^2(x) E\Big[\varphi_t\Big(Z_t^x, \ln\frac{y}{z} + \frac{1}{2} Z_z^x \Big) +$$

$$+ \frac{1}{2}\varphi_z\Big(Z_t^x, \ln\frac{y}{z} + \frac{1}{2} Z_z^x \Big) \Big] + \hat{A}h(y; t, z, x).$$

Differentiation of (28) gives the equality

$$h_{zz} = z^{-2} E[\varphi_{zz} + \varphi_z].$$

Since φ satisfies the heat equation $\varphi_t = 1/2\, \varphi_{zz}$,

$$h_t = \frac{1}{2}\sigma^2(x) z^2 h_{zz} + \hat{A}\, h.$$

Hence, the function u in (26) is a solution of (13).

***Remark* 1.** Let's define the following process:

$$m_t := f(\tilde{S}_t, X_t, t-\tau_{\nu(t)}) - f(z, x, 0) - \int_0^t \left(A + \frac{d}{dr} \right) f(\tilde{S}_r, X_r, r-\tau_{\nu(r)})\, dr. \quad (33)$$

It's an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t := \sigma\{X_s, W_s; 0 \le s \le t\}$. It's quadratic variation is equal to

$$\langle m_t \rangle = \int_0^t \left[\left(A + \frac{d}{dr} \right) f^2(\tilde{S}_r, X_r, r-\tau_{\nu(r)}) - \right.$$

$$\left. - 2f(\tilde{S}_r, X_r, r-\tau_{\nu(r)}) \left(A + \frac{d}{dr} \right) f(\tilde{S}_r, X_r, r-\tau_{\nu(r)}) \right] dr =$$

$$= \int_0^t [A f^2(\tilde{S}_r, X_r, r-\tau_{\nu(r)}) - 2f(\tilde{S}_r, X_r, r-\tau_{\nu(r)}) A f(\tilde{S}_r, X_r, r-\tau_{\nu(r)})]\, dr. \quad (34)$$

In such a way, from (17) and (34), it follows that $\mathcal{R}_0(\pi^*) = \langle m_T \rangle$ with the function u replacing f in (33).

***Remark* 2.** In the Markov case, the operator A in (14) has the following form:

$$A f(x) = \lambda(x) \int_{\mathbf{X}} P(x, dy)[f(y) - f(x)], \quad (35)$$

where $\lambda(x)$ are an intensities of jumps of the jump Markov process X_t; in this case

$$Q(x, A, t) = P(x, dy)(1 - e^{-\lambda(x)t}),$$

$$G_x(t) = 1 - e^{-\lambda(x)t},$$

$$\overline{G}_x(t) = e^{-\lambda(x)t},$$

$$g_x(t) = \lambda(x) e^{-\lambda(x)t},$$

and

$$\frac{g_x(t)}{\overline{G}_x(t)} = \lambda(x).$$

In this way, the operator A in (35) is an infinitesimal operator of the jump Markov process X_t.

Corollary 1. *Initial capital for hedging strategy in our model is defined by the formula:*

$$V_0(\pi) = \tilde{E} f(S_T) = \int \left(\int f(y) y^{-1} \varphi\left(\eta, \ln\frac{y}{z} + \frac{1}{2}\eta \right) dy \right) F_T^x(d\eta).$$

In particular, for the European call options $f(y) = (y-K)^+$, we have

$$V_0(\pi) = \int C_{BS}((z/T)^{1/2}, T) F_T^x(dz),$$

where $C_{BS}(\hat{\sigma}, T)$ is the Black – Scholes price for call option with volatility $\hat{\sigma}$.

i.e., $C_{BS}((z/T)^{1/2}, T) = S_0\Phi(d_+) - K\Phi(d_-)$, *where*

$$d_{\pm} = \left[\ln\frac{S_0}{K} \pm \frac{S_0}{2}\right] / \sqrt{S_0}.$$

***Corollary* 2.** *Let* $X = (1, 2)$, *and* $\nu(t)$ *be a counting process for* X_t, *then*

$$Z_T^1 = \int_0^T [\sigma^2(1)I(X_t = 1) + \sigma^2(2)I(X_t = 2)]\, dt = aT + bJ_T,$$

where

$$J_T = \int_0^T (-1)^{\nu(t)} dt,$$

$$a = \frac{1}{2}(\sigma^2(1) + \sigma^2(2)),$$

$$b = \frac{1}{2}(\sigma^2(1) - \sigma^2(2)).$$

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