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BALLEANS AND G-SPACES

БОЛЕАНИ ТА G-ПРОСТОРИ

We show that every ballean (equivalently, coarse structure) on a set X can be determined by some group G of permutations of X and some group ideal $\mathcal{I}$ on G. We refine this characterization for some basic classes of balleans: metrizable, cellular, graph, locally finite, and uniformly locally finite. Then we show that a free ultrafilter $\mathcal{U}$ on ω is a T-point with respect to the class of all metrizable locally finite balleans on ω if and only if $\mathcal{U}$ is a Q-point. The paper is concluded with a list of open questions.

Доведено, що кожен болеан (еквівалентно, груба структура) на множині X може бути визначений деякою групою підстановок G множини X та деяким груповим ідеалом $\mathcal{I}$ на G. Цю характеризацію уточнено для деяких основних класів болеанів: метризовних, стільникових, графових, локально скінченних, рівномірно локально скінченних. Далі ми доводимо, що вільний ультрафільтр $\mathcal{U}$ на ω є T-точкою відносно класу метризовних локально скінченних болеанів на ω тоді і тільки тоді, коли $\mathcal{U}$ є Q-точкою. Насамкінець наведено список відкритих проблем.

Following [5, 6], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the set of *radii*.

Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X \colon x \in B(y, \alpha)\}, \qquad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \qquad B^*(x, \beta) \subseteq B(x, \beta');$$

for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

A ballean $\mathcal{B}$ on X can also be determined in terms of entourages of the diagonal Δ_X of $X \times X$, in this case it is called a coarse structure [7] (Definition 2.3). Let $\mathcal{E}$ be a family of subsets of $X \times X$. The pair $(X, \mathcal{E})$ is a *coarse structure* if

$\Delta_X \subset E$ for each $E \in \mathcal{E}$;

if $E \in \mathcal{E}$ and $\Delta_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;

if $E_1, E_2 \in \mathcal{E}$ then $E_1 \cup E_2 \in \mathcal{E}$;

if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ where $E^{-1} = \{(y, x) \colon (x, y) \in \mathcal{E}\}$;

if $E_1, E_2 \in \mathcal{E}$ then $E_1 \circ E_2 \in \mathcal{E}$ where $E_1 \circ E_2 = \{(x, y) \colon (x, z) \in E_1, (z, y) \in E_2$ for some $z \in X\}$.

Each ballean $\mathcal{B} = (X, P, B)$ defines a coarse structure $(X, \mathcal{E})$ where the family $\mathcal{E}$ of entourages is defined by the rule:

$$E \in \mathcal{E} \Leftrightarrow \exists \alpha \in P \quad \forall x \in X \quad \forall y \in X\colon y \in B(x,\alpha) \Rightarrow (x,y) \in E.$$

On the other hand, each coarse structure $(X,\mathcal{E})$ determines the ballean $(X,\mathcal{E},B)$, where $B(x,E) = \{y \in X\colon (x,y) \in E\}$.

Let $\mathcal{B}_1 = (X_1,P_1,B_1)$, $\mathcal{B}_2 = (X_2,P_2,B_2)$ be balleans. A mapping $f\colon X_1 \to X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$, $f(B_1(x,\alpha)) \subseteq$ $\subseteq B_2(f(x),\beta)$. If there exists a bijection $f\colon X_1 \to X_2$ such that f and f^{-1} are $\prec$-mappings, then $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *asymorphic*.

Let G be a group, $\mathcal{I}$ be an ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G, i.e., if $A, B \in \mathcal{I}$ and $A' \subseteq A$ then $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a *group ideal* if, for all $A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$.

Now let X be a G-space with the action $G \times X \to X$, $(g,x) \mapsto gx$, and let $\mathcal{I}$ be a group ideal on G. We define a ballean $\mathcal{B}(G,X,\mathcal{I})$ as triple $(X,\mathcal{I},B)$ where $B(x,A) = Ax \cup \{x\}$ for all $x \in X$, $A \in \mathcal{I}$.

In Section 1 we show that every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G,X,\mathcal{I})$ for some group G of permutations of X and some group ideal $\mathcal{I}$ on G. Then we refine this statement to some basic classes of balleans: metrizable, cellular, graph, locally finite and uniformly locally finite.

Let $\mathcal{B} = (X,P,B)$ be a ballean. A subset $F \subseteq X$ is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $F \subseteq B(x,\alpha)$. A subset $T \subseteq X$ is *thin* if, for every $\alpha \in P$, there exists a bounded subset F such that $|B(x,\alpha) \cap T| \leqslant 1$ for each $x \in X \setminus F$.

Given a class $\mathcal{K}$ of balleans on $\omega = \{0,1,\ldots\}$, a free ultrafilter $\mathcal{U}$ on ω is said to be a T-*point with respect to* $\mathcal{K}$ if, for every ballean $\mathcal{B} \in \mathcal{K}$, there exists $U \in \mathcal{U}$ such that U is thin in $\mathcal{B}$. By Theorem 6, a T-point in ω^* defined in [3] is exactly a T-point with respect to the class of all metrizable uniformly locally finite balleans on ω. By [3] (Theorem 3), an ultrafilter $\mathcal{U} \in \omega^*$ is a T-point with respect to the class of all metrizable balleans on ω if and only if $\mathcal{U}$ is selective.

In Section 2 we prove that an ultrafilter $\mathcal{U} \in \omega^*$ is a T-point with respect to the class of all metrizable uniformly locally finite balleans on ω if and only if $\mathcal{U}$ is a Q-point. We give also some "sequential" characterization of T-points with respect to the class of all metrizable uniformly locally finite balleans on ω.

We conclude the paper with some comments and open questions in Section 3.

1. Balleans and G-spaces.

Theorem 1. *Every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G,X,\mathcal{I})$ for some subgroup G of the group S_X of all permutations of X and some group ideal $\mathcal{I}$ of G.*

Proof. Let $\mathcal{E}$ be a family of entourages of the diagonal Δ_X of $X \times X$ which determines $\mathcal{B}$. For each pair $(x,y) \in X \times X$, let $\pi(x,y)$ denote the permutation of X swapping x, y and acting identically on $X \setminus \{x,y\}$. Given any $E \subseteq X \times X$, we put

$$A_E = \{e, \pi(x,y)\colon (x,y) \in E\},$$

where e is the identity permutation. We denote by G the subgroup of S_X generated by $\cup\{A_E\colon E \in$ $\in \mathcal{E}\}$. To construct the ideal $\mathcal{I}$, we put $\mathcal{F}_0 = \{A_E\colon E \in \mathcal{E}\}$ and, for each $n \in \omega$,

$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup \mathcal{F}_n^{-1} \cup \{FF'\colon F, F' \in \mathcal{F}_n\}.$$

Then $\cup_{n\in\omega}\mathcal{F}_n$ is a base for $\mathcal{I}$, i.e., $\mathcal{I} = \{A \subseteq G \colon \exists F \in \cup_{n\in\omega}\mathcal{F}_n$ such that $A \subseteq F\}$.

Given any $E, E' \in \mathcal{E}$, we have

(1) $E = \{(x, y) \colon x \in X, y \in A_E x\}$;

(2) $E \circ E' = \{(x, y) \colon x \in X, y \in A_{E\circ E'} x\}$.

By (1), the identity mapping $id \colon X \to X$ is a $\prec$-mapping from $\mathcal{B}$ to $\mathcal{B}(G, X, \mathcal{I})$. Using (2) and inductive argument, we see that, for each $A \in \mathcal{F}_n$, $\{(x, y) \colon x \in X, y \in Ax\} \in \mathcal{E}$. Hence, id is a $\prec$-mapping from $\mathcal{B}(G, X, \mathcal{I})$ to $\mathcal{B}$, so $\mathcal{B}$ and $\mathcal{B}(G, X, \mathcal{I})$ are asymorphic.

Theorem 1 is proved.

A ballean $\mathcal{B} = (X, P, B)$ is called *connected* if, for all $x, y \in X$, there is $\alpha \in P$ such that $y \in B(x, \alpha)$. We observe that a ballean $\mathcal{B}(G, X, \mathcal{I})$ is connected if and only if, for all $x, y \in X$ there is $A \in \mathcal{I}$ such that $x \in Ay$. In particular, if $\mathcal{B}(G, X, \mathcal{I})$ is connected then G acts transitively on X.

Each metric space (X, d) determines the ballean $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$ where $\mathbb{R}^+ = \{r \in \in \mathbb{R} \colon r \geqslant 0\}$, $B_d(x, r) = \{y \in XG \colon d(x, y) \leqslant r\}$. A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for an appropriate metric space (X, d). By [6] (Theorem 2.1.1), a ballean $\mathcal{B}$ is metrizable if and only if $\mathcal{B}$ is connected and its set of radii P has a countable subset cofinal in the natural preordering of P.

Repeating arguments proving Theorem 1, we get the following theorem.

Theorem 2. *Every metrizable ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ with countable base such that, for all $x, y \in X$, there is $A \in \mathcal{I}$ such that $y \in Ax$.*

A ballean $\mathcal{B} = (X, \mathcal{E})$ is called *cellular* [6, p. 42] if, for each entourage $E' \in \mathcal{E}$ there is an entourage E such that $E' \subset E$, $E = E^{-1}$ and $E \circ E = E$. By [6] (Theorem 3.1.3), $\mathcal{B}$ is cellular if and only if asdim $\mathcal{B} = 0$.

Theorem 3. *Every cellular ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ on G which has a base consisting of subgroups.*

Proof. To apply arguments proving Theorem 1, it suffices to note that if $E \in \mathcal{E}$, $E = E^{-1}$ and $E \circ E = E$ then

$$\{(x, y)G \colon x \in X, y \in A_E x\} = \{(x, y)G \colon x \in X, y \in \langle A_E\rangle x\},$$

where $\langle A_E\rangle$ is a subgroup of S_X generated by A_E.

Theorem 3 is proved.

Every connected graph Γ can be considered as a metric space (V_Γ, d_Γ) where V_Γ is the set of vertices of Γ, d_Γ is the path metric on V_Γ. A ballean $\mathcal{B}$ is called a *graph ballean* [6, p. 79] if $\mathcal{B}$ is asymorphic to the ballean $\mathcal{B}(V_\Gamma, d_\Gamma)$ for an appropriate connected graph Γ. By [6] (Theorem 5.1.1), a ballean $\mathcal{B} = (X, \mathcal{E})$ is a graph ballean if and only if $\mathcal{B}$ is connected and there exists $E \in \mathcal{E}$ such that $E = E^{-1}$ and, for every $E' \in \mathcal{E}$, there is $n \in \omega$ such that $E' = E^n$ where E^n is a product of n copies of E.

Theorem 4. *Every graph ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ having a member $A \in \mathcal{I}$ such that $\{A^n G \colon n \in \in \omega\}$ is a base for $\mathcal{I}$ and, for all $x, y \in X$, there is $n \in \omega$ such that $y \in A^n x$.*

A ballean $\mathcal{B} = (X, P, B)$ is called *locally finite* if each ball $B(x, \alpha)$, $x \in X$, $\alpha \in P$ is finite. The next theorem was suggested by Taras Banakh.

Theorem 5. *Every locally finite ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some group ideal $\mathcal{I}$ on G with a base consisting of subsets compact in the topology of pointwise convergence on S_X.*

Proof. It suffices to observe that a subset $F \subset S_X$ is compact if and only if F is closed and the orbit Fx of each point $x \in X$ is finite, and the subset A_E defined in the proof of Theorem 1 is compact.

A ballean $\mathcal{B} = (X, P, B)$ is called *uniformly locally finite* if, for each $\alpha \in P$, there exists $n \in \omega$ such that $|B(x, \alpha)| \leqslant n$ for every $x \in X$.

Theorem 6. *Every uniformly locally finite ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{F}_G)$ for some subgroup G of S_X, $\mathcal{F}_G$ is the ideal of all finite subsets of G.*

Proof. [4] (Theorem 1).

2. Around *T*-points. Let $\mathcal{K}$ be a class of balleans with the support $\omega = \{0, 1, \ldots\}$, ω^* be the space of all ultrafilters on ω. We say that an ultrafilter $\mathcal{U} \in \omega^*$ is a *T-point with respect to $\mathcal{K}$* if, for every ballean $\mathcal{B} = (\omega, P, B)$ from $\mathcal{K}$, $\mathcal{U}$ has a member $U \in \mathcal{U}$ which is thin in $\mathcal{B}$, i.e., for every $\alpha \in P$, there exists a bounded subset $V \subset \omega$ such that $|B(x, \alpha) \cap U| \leqslant 1$ for each $x \in U \setminus V$.

We recall that an ultrafilter $\mathcal{U} \in \omega^*$ is

selective if, for every partition $\mathcal{P}$ of ω, either some block of $\mathcal{P}$ is a member of $\mathcal{U}$, or there is $U \in \mathcal{U}$ such that $|U \cap P| \leqslant 1$ for each $P \in \mathcal{P}$;

P-point if, for every partition $\mathcal{P}$ of ω, either some block of $\mathcal{P}$ is a member of $\mathcal{U}$, or there is $U \in \mathcal{U}$ such that $U \cap P$ is finite for each $P \in \mathcal{P}$;

Q-point if, for every partition $\mathcal{P}$ of ω into finite subsets, there is $U \in \mathcal{U}$ such that $|U \cap P| \leqslant 1$ for each $P \in \mathcal{P}$.

Theorem 7. *An ultrafilter $\mathcal{U} \in \omega^*$ is a T-point with respect to the class of all metrizable balleans on ω if and only if $\mathcal{U}$ is selective.*

Proof. [3] (Theorem 3).

Theorem 8. *An ultrafilter $\mathcal{U} \in \omega^*$ is a T-point with respect to the class of all metrizable locally finite balleans on ω if and only if $\mathcal{U}$ is a Q-point.*

Proof. Let $\mathcal{U}$ be a Q-point, d be a locally finite metric on ω. We fix $x_0 \in \omega$, and put

$$X_0 = \{x_0\}, \qquad X_{n+1} = B_d(x_0, (n+1)^2) \setminus B_d(x_0, n^2), \quad n \in \omega,$$

$$Y_0 = \bigcup_{n \in \omega} X_{2n}, \qquad Y_1 = \bigcup_{n \in \omega} X_{2n+1}.$$

Since $\mathcal{U}$ is a Q-point and each subset X_n is finite, there exist $U \in \mathcal{U}$ and $i \in \{0, 1\}$ such that $U \subseteq Y_i$ and $|U \cap X_n| \leqslant 1$ for each $n \in \omega$. Then for each $r > 0$, the set $\{x \in \mathcal{U}G \colon |B_\alpha(x, r) \cap \mathcal{U}| > 1\}$ is finite, so U is thin in $\mathcal{B}(\omega, d)$.

Now let $\mathcal{U}$ be a T-point in the class of all metrizable locally finite balleans on ω, $\{P_n G \colon n \in \omega\}$ be a partition of ω into finite subsets. We define a metric d on ω by the rule:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y, \quad x, y \in P_n; \\ \max\{n, m\}, & \text{if } x \in P_n, \quad y \in P_m, \quad n \neq m. \end{cases}$$

Since (ω, d) is locally finite, we can choose a subset $U \in \mathcal{U}$ which is thin in (ω, d). By the definition of thin subset, there exists a bounded subset Y of (ω, d) such that $B_d(x, 1) \cap B_d(x', 1) = \varnothing$ for all distinct $x, x' \in U \setminus Y$. Since Y is finite, we conclude that $\mathcal{U}$ is a Q-point.

Theorem 8 is proved.

A situation with T-points with respect to the class of all metrizable uniformly locally finite balleans on ω is much more delicate. It is proved in [3] that all P-points and all Q-points are T-points, but it is unknown whether there exists a T-point in ZFC without additional set-theoretical assumptions. Now we give some characterization of T-points based on sequences of coverings of ω.

Let us say that a covering $\mathcal{F}$ of ω is *uniformly bounded* if there exists a natural number n such that, for each $x \in \omega$,

$$|st(x, \mathcal{F})| \leqslant n,$$

where $st(x, \mathcal{F}) = \bigcup\{F \in \mathcal{F}G \colon x \in F\}$.

Theorem 9. *An ultrafilter $\mathcal{U} \in \omega^*$ is a T-point with respect to the class $\mathcal{K}$ of all metrizable uniformly locally finite balleans on ω if and only if, for every sequence $(\mathcal{F}_n)_{n \in \omega}$ of uniformly bounded coverings of ω, there exists $U \in \mathcal{U}$ such that, for each $n \in \omega$, $|F \cap U| \leqslant 1$ for all but finitely many $F \in \mathcal{F}_n$.*

Proof. We assume that $\mathcal{U}$ is a T-point with respect to $\mathcal{K}$ and fix a sequence $(\mathcal{F}_n)_{n \in \omega}$ of uniformly bounded coverings of ω. We consider a ball structure $\mathcal{B} = (\omega, \omega, B)$, where $B(x, n) = st(x, \mathcal{F}_n)$, and take the ballean envelope $\operatorname{env} \mathcal{B} = (\omega, P, B')$ of $\mathcal{B}$, the smallest ballean on ω such that $\mathcal{B} \prec \operatorname{env} \mathcal{B}$ (see [6, p. 191]). By the description, $\operatorname{env} \mathcal{B}$ is uniformly locally finite and P is countable. Joining the elements $0, n$ to the first member of $\mathcal{F}_n$, we may suppose that $\operatorname{env} \mathcal{B}$ is connected. By [6] (Theorem 2.1.1), $\operatorname{env} \mathcal{B}$ is metrizable. Since $\mathcal{U}$ is a T-point with respect to $\mathcal{K}$ and $\operatorname{env} \mathcal{B} \in \mathcal{K}$, there is a subset $U \in \mathcal{U}$ which is thin in $\operatorname{env} \mathcal{B}$. We note that $\omega \subseteq P$ and $B(x, n) = B'(x, n)$. Hence, for each $n \in \omega$, there exists a finite subset V of ω such that $|U \cap st(x, \mathcal{F}_n)| \leqslant 1$ for each $x \in U \setminus V$. It follows that $|U \cap F| \leqslant 1$ for all but finitely many $F \in \mathcal{F}_n$.

To prove the converse statement, we take an arbitrary uniformly locally finite metric d on ω, and put

$$\mathcal{F}_n = (B_d(m, n))_{m \in \omega}, \quad n \in \omega.$$

Since each covering $\mathcal{F}_n$ is uniformly bounded, there is $U \in \mathcal{U}$ such that, for each $n \in \omega$, $|U \cap \cap B_d(m, n)| \leqslant 1$ for all but finitely many $m \in \omega$. This means that U is thin in $\mathcal{B}(\omega, d)$.

Theorem 9 is proved.

We conclude this section with the following observation suggested by Sergiy Slobodianiuk.

Theorem 10. *An ultrafilter $\mathcal{U}$ on ω is a P-point if and only if, for every metric d on ω, either some member of $\mathcal{U}$ is bounded in (X, d) or there is $U \in \mathcal{U}$ such that (U, d) is locally finite.*

Proof. Assume that $\mathcal{U}$ is a P-point, fix $x_0 \in \omega$ and write ω as a union $\omega = \bigcup_{n\in\omega} B_d(x_0, n)$. It follows from a definition of a P-point that there is $U \in \mathcal{U}$ such that U is contained in some ball $B_d(x_0, n)$ or U meets each ball in finite number of points. Thus, U is bounded in the first case and (U, d) is locally finite in the second case.

To prove the converse statement, we use the metric d defined by the partition $\{P_nG\colon n \in \omega\}$ of ω in Theorem 8. Let U be a subset of X. If U is bounded in (X, d) then U is contained in the union of finitely many blocks. If (U, d) is locally finite then U meets each block in finite number of points.

Theorem 10 is proved.

3. Comments and open questions. We say that an ideal $\mathcal{I}$ on a group G is *invariant* if, for each $A \in \mathcal{I} \bigcup_{g\in G} g^{-1}Ag \in \mathcal{I}$. If X is a left-regular G-space and $\mathcal{I}$ is an invariant group ideal on G then, the mapping

$$\mathcal{B}(G, X, \mathcal{I}) \times \mathcal{B}(G, X, \mathcal{I}) \to \mathcal{B}(G, X, \mathcal{I}),$$

$(g, x) \mapsto gx$ is a $\prec$-mapping.

Question 1. *Given a ballean $\mathcal{B}$ on the set X, how to detect whether $\mathcal{B}$ is asymorphic to $\mathcal{B}(G, X, \mathcal{I})$ for some subgroup G of S_X and some* **invariant** *group ideal $\mathcal{I}$ on G?*

A subset A of a topological group G is called bounded if, for every neighbourhood $\mathcal{U}$ of the identity, there exists a finite subset F of G such that $A \subseteq FU$, $A \subseteq UF$. The family $\mathcal{I}$ of all bounded subsets of G forms an invariant group ideal. For balleans on topological groups determined by these ideals see [1].

Question 2. *Given an invariant group ideal $\mathcal{I}$ on a group G, how to detect whether there is a group topology τ on G such that $\mathcal{I}$ is an ideal of bounded subsets of the topological group (G, τ)?*

We say that a filter φ on a group G is a *group filter* if, for any $B \in \varphi$, there exists $A \in \varphi$ such that $A = A^{-1}$ and $AA \subseteq B$. A filter φ is *invariant* if, for any $B \in \varphi$ and $g \in G$, there is $A \in \varphi$ such that $g^{-1}Ag \subseteq B$. Clearly, each invariant group filter is a base at identity for some group topology on G.

Let X be a G-space, φ be a group filter on G. The triple (G, X, φ) determines a uniformity $\mathcal{U}$ on X with a base of entourages of the diagonal Δ_X consisting of all subsets of the form $\{(x, y)G\colon y \in \in Ax\}$, $A \in \varphi$. If φ is invariant then, for each $g \in G$, the mapping $X \to X$, $x \mapsto gx$ is uniformly continuous in $\mathcal{U}$.

Question 3. *Given a uniform space $(X,\mathcal{U})$, how to detect whether there exist a subgroup G of S_X and a group ideal (an invariant group ideal) $\mathcal{I}$ such that the triple (G, X, φ) determines $(X,\mathcal{U})$?*

In the following questions “T-point” means “T-point in the class of all metrizable uniformly locally finite balleans on ω”.

An ultrafilter $\mathcal{U} \in \omega^*$ is a P-point if and only if, for every Hausdorff topology τ on ω, some member $U \in \mathcal{U}$ has at most one limit point in (ω, τ), in particular, some member of $\mathcal{U}$ is discrete in (ω, τ).

Question 4 (T. Banakh). *Let $\mathcal{U}$ be a free ultrafilter on ω such that, for each metrizable topology τ on ω, some member of $\mathcal{U}$ is discrete. Is $\mathcal{U}$ a T-point?*

We say that an ultrafilter $\mathcal{U} \in \omega^*$ is *sequentially selective* if, for any sequence $(\mathcal{P}_n)_{n\in\omega}$ of uniformly bounded partitions of ω, there is $U \in \mathcal{U}$ such that, for each $n \in \omega$, $|P \cap U| \leqslant 1$ for all but finitely many $P \in \mathcal{P}_n$. By Theorem 9, each T-point is sequentially selective.

Question 5. *Is every sequentially selective ultrafilter $\mathcal{U} \in \omega^*$ a T-point? Does there exist a sequentially cell selective ultrafilter in ZFC?*

An ultrafilter $\mathcal{U} \in \omega^*$ is called *rapid* if, for any partition $\{P_n G\colon n \in \omega\}$ of ω into finite subsets, there exists $U \in \mathcal{U}$ such that $|U \cap P_n| \leqslant n$ for each $n \in \omega$.

Jana Flašková noticed that a rapid ultrafilter needs not to be a T-point. Her arguments: a square of rapid ultrafilters is rapid but a product of two free ultrafilters could not be a T-point. If $\mathcal{U}$, $\mathcal{V}$ are ultrafilters on ω then $\mathcal{U}\mathcal{V}$ is an ultrafilter on $\omega \times \omega$ with the base of subsets of the form $\bigcup_{x\in U}(x, V_x)$, $V_x \in \mathcal{V}, U \in \mathcal{U}$. To see that $\mathcal{U}\mathcal{V}$ is not a T-point, we can either apply Theorem 8 or endow ω with the structure of an arbitrary group and use the definition of a T-point.

Each countable group G of permutations of ω is contained in some 2-generated subgroup of S_ω (see [2]), so in the definition of a T-point given in [3] we can use only 2-generated subgroup of S_ω. We say that an ultrafilter $\mathcal{U} \in \omega^*$ is a *cyclic T-point* if, for each infinite cyclic subgroup G of S_ω, there exists $U \in \mathcal{U}$ thin in the ballean $\mathcal{B}(G, \omega, \mathcal{F}_g)$.

Question 6. *Is every cyclic T-point a T-point?*

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset L of X is called *large* if there exists $\alpha \in P$ such that $X = B(x, \alpha)$. A subset S of X is called *small* if $(X \setminus S) \cap L$ is large for each large subset L.

Let G be a group of all permutations of ω with finite support ($\operatorname{supp} g = \{x \in \omega G\colon gx \neq x\}$). A subset A of ω is small in $\mathcal{B}(G, \omega, \mathcal{F}_G)$ if and only if A is finite, but each subset of ω is thin. On the other hand, if X is left regular G-space then each thin subset is small.

Question 7. *Does there exist a ZFC-example of a free ultrafilter on ω such that, for every countable group G and every left regular action of G on ω, there is a member of $\mathcal{U}$ small in the ballean $\mathcal{B}(G, \omega, \mathcal{F}_G)$? Is a weak P-point such an ultrafilter?*

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