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ON EQUALITIES INVOLVING INTEGRALS OF THE LOGARITHM OF THE RIEMANN ζ-FUNCTION AND EQUIVALENT TO THE RIEMANN HYPOTHESIS

ПРО РІВНОСТІ, ЩО МІСТЯТЬ ІНТЕГРАЛИ ВІД ЛОГАРИФМА ζ-ФУНКЦІЇ РІМАНА І ЕКВІВАЛЕНТНІ ГІПОТЕЗІ РІМАНА

Using the generalized Littlewood theorem about a contour integral involving the logarithm of an analytical function, we show how an infinite number of integral equalities involving integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis can be established and present some of them as an example. It is shown that all earlier known equalities of this type, viz., the Wang equality, Volchkov equality, Balazard–Saias–Yor equality, and an equality established by one of the authors, are certain particular cases of our general approach.

Показано як за допомогою узагальненої теореми Літтлвуда про контурний інтеграл, що містить логарифм аналітичної функції, можна отримати нескінченну кількість інтегральних рівностей, що містять інтеграли від логарифма ζ-функції Рімана і є еквівалентними гіпотезі Рімана, і наведено кілька таких рівностей у якості прикладу. Показано, що деякі відомі рівності такого типу, а саме, рівності Ванга, Волчкова, Балазарда – Сайаса – Йора та рівність, що встановлена одним із авторів, є частинними випадками нашого загального підходу.

1. Introduction. Among numerous statements shown to be equivalent to the Riemann hypothesis (RH; see e.g. [1] for details and general discussion of the Riemann ζ-function and Riemann hypothesis) there are a few which state that certain integrals involving the logarithm of the Riemann ζ-function must have some well defined value. Actually, we know four of them established by Wang [2], Volchkov [3], Balazard, Saias, Yor [4], and one of us [5]. As it stands today, these equalities look as isolated and not-connected with each other, and the methods used to obtain them were quite different. In this paper, which is based on our hitherto unpublished notes posted in ArXiv [6], we, using generalized Littlewood theorem about contour integrals over the logarithm of an analytical function, introduce a general recipe how infinite number of integral equalities of the aforementioned type can be obtained. This recipe is illustrated by presenting a number of such equalities. In particular, all earlier known integral equalities equivalent to the Riemann hypothesis are obtained as certain particular cases of our general approach.

2. Generalized Littlewood theorem. Littlewood theorem concerning contour integrals of the logarithm of an analytical function [7] is well known and widely used in the theory of functions in general and Riemann-function research in particular, see e.g. [1]. The following generalization of the theorem is quite straightforward and was used actually already by Wang [2].

Throughout the paper we use the notation $z = x + iy$ and/or $z = \sigma + it$ on the equal footing. Let us consider functions $f(z)$, $g(z)$ and let $F(z) = \ln(f(z))$. Let us introduce and analyze an integral $\int_C F(z)g(z)dz$ along the contour C which is the rectangle formed by the broken line composed by four straight line segments connecting the vertices with the coordinates X_1+iY_1, X_1+iY_2, X_2+iY_2, $X_2 + iY_1$; we suppose that $X_1 < X_2$, $Y_1 < Y_2$ and that $f(z)$ is analytic and non-zero on C and meromorphic inside it, and that $g(z)$ is analytic on C and meromorphic inside it. Assume first that the poles and zeroes of the functions $F(z)$, $g(z)$ do not coincide and that the function $f(z)$ has no poles

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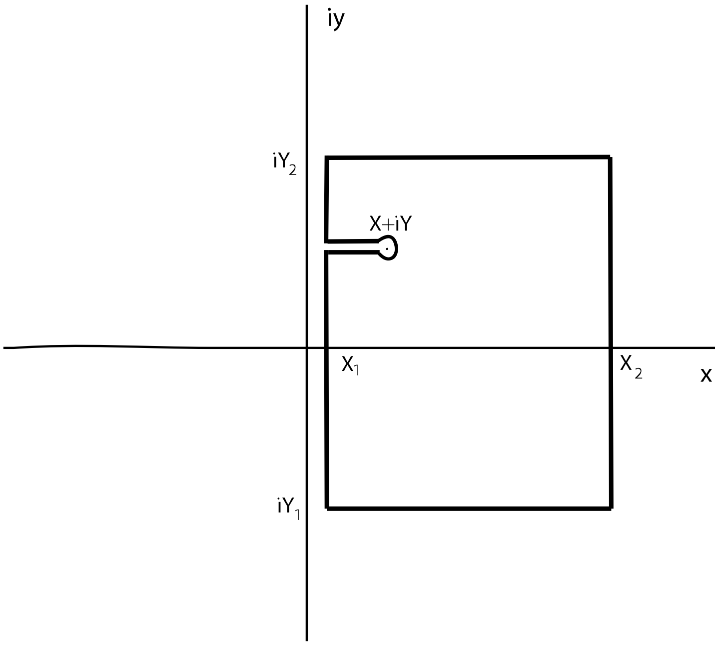

Contour C using during the proof of generalized Littlewood theorem

and only one simple zero located at the point $X + iY$ in the interior of the contour. Then let us do the cut along the straight line segment $X_1 + iY$, $X + iY$ and consider a new contour C' which is the initial contour C plus the cut indenting the point $X+iY$ (see Figure). Cauchy residue theorem can be applied to the contour C' which means that the value of $\int_{C'} F(z)g(z)dz$ is $2\pi i \sum_\rho F(\rho)\,\mathrm{res}(g(\rho))$, where the sum is over all simple poles ρ of the function $g(z)$ lying inside the contour.

An appropriate choice of the branches of the logarithm function assures that the difference between two branches of the logarithm function appearing after the integration path indents the point $X+iY$ is $2\pi i$ (see below for an exact definition), which means that the value of an integral along the initial contour C is $2\pi i \left(\sum_\rho F(\rho)\,\mathrm{res}(g(\rho)) - \int_{X_1+iY}^{X+iY} g(z)dz \right)$. Remind, that the integral is taken here along the straight segment parallel to the real axis hence for this integral $\mathrm{Im}(z) = \mathrm{const} = Y$ and $dz = dx$. Analogously, a single pole of $f(z)$ function occurring at (other) point $X + iY$ contributes $2\pi i \int_{X_1+iY}^{X+iY} g(z)dz$ to the contour integral value. Thus by summing all terms arising from different zeroes and poles of the function $f(z)$ one obtains the following generalization of the Littlewood theorem (modification for the case when poles/zeroes of $f(z)$ with an order $m > 1$ are present is evident):

Theorem 1. *Let C denotes the rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, $y = Y_2$ where $X_1 < X_2$, $Y_1 < Y_2$ and let $f(z)$ be analytic and non-zero on C and meromorphic inside it, let also $g(z)$ is analytic on C and meromorphic inside it. Let $F(z) = \ln(f(z))$, the logarithm being defined as follows: we start with a particular determination on $x = X_2$, and obtain the value at other points by continuous variation along $y = \mathrm{const}$ from $\ln(X_2 + iy)$. If, however, this path would cross a zero or pole of $f(z)$, we take $F(z)$ to be $F(z \pm i0)$ according as we approach the path* *from above or below. Let also the poles and zeroes of the functions* $f(z)$, $g(z)$ *do not coincide. Then*

$$\int\limits_C F(z)g(z)dz = 2\pi i\left(\sum_{\rho_g}\operatorname{res}(g(\rho_g)F(\rho_g)) - \sum_{\rho_f^0}\int\limits_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum_{\rho_f^{pol}}\int\limits_{X_1+iY_\rho^{pol}}^{X_\rho^{pol}+iY_\rho^{pol}} g(z)dz\right)$$

where the sum is over all ρ_g *which are simple poles of the function* $g(z)$ *lying inside C, all* $\rho_f^0 = X_\rho^0 + iY_\rho^0$ *which are zeroes of the function* $f(z)$ *counted taking into account their multiplicities* (*that is the corresponding term is multiplied by m for a zero of the order m*) *and which lye inside C, and all* $\rho_f^{\mathrm{pol}} = X_\rho^{\mathrm{pol}} + iY_\rho^{\mathrm{pol}}$ *which are poles of the function* $f(z)$ *counted taking into account their multiplicities and which lye inside C. For this is true all relevant integrals in the right-hand side of the equality should exist.*

Of course, exact nature of the contour is actually irrelevant for the theorem as it is for the Littlewood theorem. As for the last remark concerning the existence of integrals taken along the segments $[X_1 + iY_\rho, X_\rho + iY_\rho]$ parallel to the real axis (the inexistence might happen e.g. if a pole of the $g(z)$ function occurs on the segment), the corresponding integration path often can be modified in such a manner that one will be able to handle these integrals. The case of the coincidence of poles and zeroes of the functions $f(z)$, $g(z)$ often does not pose real problems and can be easily considered. We will deal with a few such cases below.

This is also clear that the theorem remains true if there are poles or zeroes of $f(z)$ on the left border of the contour $x = X_1$: they contribute nothing to the contour integral value but the integral over the line $x = X_1$ is to be understood as a principal value. Occurrence of poles/zeroes of $f(z)$ on the right border of the contour $x = X_2$ can be treated similarly.

3. Application of the generalized Littlewood theorem to establish equalities equivalent to the Riemann hypothesis. All integrals involving the logarithm of the Riemann ζ-function below are understood in the sense of a generalized Littlewood theorem.

3.1. Power functions. The most "natural" starting point to apply the generalized Littlewood theorem to establish an equality equivalent to the Riemann hypothesis is to take in the conditions of the aforementioned theorem $f(z) = \zeta(z)$, $g(z) = \dfrac{1}{z^2}$, $X_1 = b$, $X_2 = b + T$, $Y_1 = -T$, $Y_2 = T$ with real $b \geq 1/2$, $b \neq 1$, $T \to \infty$ and consider the contour integral $\int_C \ln(f(z))g(z)dz$. Known asymptotic properties of the Riemann function for large argument values assure that the value of the contour integral taken along the "external" three straight lines containing at least one of the points $b + T + iT$, $b + T - iT$ tends to zero when $T \to \infty$ (this has been shown for a similar $O(1/z^2)\, g(z)$ function already by Wang [2], and his consideration can be one-to-one repeated for any $O(1/z^\alpha)$, $\operatorname{Re}(\alpha) > 1$, function), and thus we obtain that the value of the contour integral tends to $-i\int_{-\infty}^{\infty}\dfrac{\ln(\zeta(b+it))}{(b+it)^2}dt$ $\Big($minus sign comes from the necessity to describe the contour in the counterclockwise direction; integral is taken along the line $z = b + it$ where $dz = idt$ and $g(z) = \dfrac{1}{(b+it)^2}\Big)$. Suppose that RH holds and there are no zeroes inside the contour. Then by Littlewood theorem the value of the integral is equal to the contribution of the simple pole of the Riemann function occurring at $z = 1$:

$$-i\int\limits_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=2\pi i\int\limits_{b}^{1}\frac{dx}{x^2}=-2\pi i(1-1/b)$$

and thus $\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=2\pi(1-1/b)$ if $b<1$ and $\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=0$ if $b>1$. If there is a zero $\rho=\sigma_k+it_k$ with the real part $\sigma_k>b$, it contributes $-2\pi i\int_{b+it}^{\sigma+it}\frac{dx}{x^2}$; correspondingly the joint contribution of two complex conjugate zeroes is equal to $4\pi i\left(\frac{1}{\sigma^2+t^2}-\frac{1}{b^2+t^2}\right)$ and we have the equality

$$\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\frac{\ln(\zeta(z))}{z^2}dz=1-1/b-2\sum_{\rho:\sigma_k>b,t_k>0}\left(\frac{1}{\sigma_k^2+t_k^2}-\frac{1}{b^2+t_k^2}\right).$$

All terms under the sum sign are definitely negative and thus we have established our first equality equivalent to the Riemann hypothesis:

Theorem 2. *An equality*

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=1-\frac{1}{b}, \tag{1}$$

where $1>b\geq 1/2$, holds true for some b if and only if there are no Riemann function zeroes with $\sigma>b$. For $b=1/2$ this equality is equivalent to the Riemann hypothesis.

The case $b=1/2$, viz. $\int_{-\infty}^{\infty}\frac{\ln(\zeta(1/2+it))}{(1/2+it)^2}dt=-2\pi$, has been tested numerically.

The same theorem takes more elegant form if we apply the generalized Littlewood theorem to the function $f(z)=\zeta(z)(z-1)$ thus killing the contribution of the simple pole:

Theorem 2a. *An equality* $\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\frac{\ln(\zeta(z)(z-1))}{z^2}dz=0$, *where $1>b\geq 1/2$, holds true for some b if and only if there are no Riemann function zeroes with $\sigma>b$. For $b=1/2$ this equality is equivalent to the Riemann hypothesis.*

***Remark* 1.** The use of inverse square function $g(z)=1/z^2$ is not necessary for conditions of Theorem 2 and 2a, very broad set of power functions $g(z)=1/z^\alpha$, where α is real and greater than 1, although not excessively large, can be used instead. From numerical calculations performed by Wedeniwski as cited by Ramaré and Saouter [8], it follows that there are no "incorrect" Riemann zeroes with $\sigma>1/2$ if $t<t_{\min}=3.3\cdot 10^9$. (There is more recent calculations of Gourdon and Demichel where it is reported that the first 10^{13} Riemann zeroes are located on the critical line [9], but we were unable to get an exact value of *T* from their paper.) This means that the argument of the "incorrect" zeroes lies in the very narrow intervals $(\pi/2,\cong\pi/2-1/t_{\min})$ for positive *t* and $(-\pi/2,$ $\cong-\pi/2+1/t_{\min})$ for negative *t*. Correspondingly, the argument of the integrand in the contribution of such Riemann zero to the contour integral values, $\int_{b+it_k}^{\sigma_k+it_k}\frac{1}{z^\alpha}dz$, lies in the intervals $(\pm\alpha\pi/2,$ $\cong\pm\alpha\pi/2\mp\alpha/t_{\min})$ and so the same sign of the real part of the sum of two paired contributions of complex conjugate zeroes is certain apart from the case when α takes the form $\alpha=2n+1+\delta$

where *n* is an integer and δ is positive and very small, or when α is very large. For example, for α close to 1 it suffices to take α "a bit larger" than $1 + \dfrac{2}{\pi t_{\min}} \cong 1 + 2 \cdot 10^{-10}$ to ensure the same signs of all "incorrect" Riemann zeroes contributions. Similarly, from above α is limited by $\alpha < \dfrac{\pi t_{\min}}{2} \cong 5.18 \cdot 10^{9}$. All the necessary details can be easily precisely elaborated but we will not do this here and limit ourselves just with a few examples of other equalities equivalent to the Riemann hypothesis:

$$\int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^{1+3\cdot 10^{-10}}} dz = 0,$$

$$\int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^{3}} dz = 0,$$

$$\int_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^{5\cdot 10^{9}}} dz = 0,$$

etc.

Despite their very simple form, above equalities have a drawback that they are "non-symmetrical" with respect to real and imaginary parts and can not be recast in the form containing integrals of only an argument or logarithm of the module. This drawback can be eliminated by taken "more symmetric" functions $g(z)$, of which $g(z) = \dfrac{1}{a^2 - (z-b)^2}$ is the most natural first example. Here *a, b* are arbitrary real positive numbers, $1 > b \geq 1/2$, $a + b \neq 1$, and the contour integral $\displaystyle\int_C \ln(f(z))g(z)dz$ is reduced to the form $I = -i \displaystyle\int_{-\infty}^{\infty} \frac{\ln(\zeta(b+it))}{a^2+t^2} dt$. Now inside the contour we have a simple pole at $z = b + a$ which gives the contribution $-\dfrac{i\pi}{a}\left(\ln\left(\zeta(a+b)\right)\right)$ to the integral value. Contribution of the Riemann function zero $\rho = \sigma_k + it_k$ with the real part $\sigma_k > b$ is given by an elementary integral $\displaystyle\int_0^{\sigma_k - b} \frac{dp}{a^2 - (p+it_k)^2}$ which real part is equal to $\displaystyle\int_0^{\sigma_k - b} \frac{a^2 - p^2 + t_k^2}{(a^2 - p^2 + t_k^2)^2 + 4p^2 t_k^2} dp$. Because this is well known that possible values of t_k for the Riemann function zeroes with $\sigma_k > 1/2$, if they exist, are very large while $0 < p < 1/2$, the expression $a^2 - p^2 + t_k^2$ is always positive and hence all contributions of such Riemann zeroes are also positive. Calculation of the contribution of the simple pole at $z = 1$ is trivial, and we get the following theorem.

Theorem 3. *An equality*

$$\frac{a}{\pi} \int\limits_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{a^2+t^2} dt = \ln \left| \frac{\zeta(a+b)\,(a+b-1)}{a-b+1} \right|, \tag{2}$$

where a, b are arbitrary real positive numbers and $1 > b \geq 1/2$, $a + b \neq 1$, *holds true for some b if and only if there are no Riemann function zeroes with* $\sigma > b$. *For* $b = 1/2$ *this equality is equivalent to the Riemann hypothesis.*

Note that here an integral is taken over the logarithm of the module; corresponding equality for the integrals involving an argument is trivially fulfilled due to the parity of the functions involved. Similarly as above, by considering $f(z) = \zeta(z)(z-1)$ it can be recast in the form $\dfrac{a}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\ln|\zeta(b+it)(b-1+it)|}{a^2+t^2}dt = \ln|\zeta(a+b)(a+b-1)|$. This same hint always can be applied and we will not mention this more. At this stage this is also worthwhile to note that we have tested numerically not only an equality (2) with $a=1$, $b=1/2$, but also more general case $\dfrac{a}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\ln|\zeta(b+it)|}{a^2+t^2}dt = \ln\left|\dfrac{\zeta(a+b)(a+b-1)}{a-b+1}\right| + \sum_{\sigma_k>b}\ln\left|\dfrac{a+\sigma_k-b+it_k}{a-\sigma_k+b-it_k}\right|$ with $a=1$, $b=1/4$ where first 649 Riemann function zeroes were taken into account [6].

The case $a+b=1$ does not pose problems being the transparent limit of above equalities. Putting $a=1-b+\delta$ and considering the limit $\delta\to 0$ we have $\dfrac{1-b}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt =$ $= \ln|\zeta(1+\delta)\,\delta| - \ln|2-2b|$. Using the known Laurent expansion of the Riemann zeta-function in the vicinity of $z=1$, $\zeta(1+\delta) = \dfrac{1}{\delta}+\ldots$ we see that simply $\dfrac{1-b}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt = -\ln|2-2b|$ and thus we have the following theorem.

Theorem 3a. *An equality*

$$\frac{1-b}{\pi}\int_{-\infty}^{\infty}\frac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt = -\ln(2-2b), \tag{3}$$

where b is an arbitrary positive number such that $1 > b \geq 1/2$, holds true for some b if and only if there are no Riemann function zeroes with $\sigma > b$. For $b = 1/2$ this equality, that is $\displaystyle\int_{-\infty}^{\infty}\frac{\ln|\zeta(1/2+it)|}{1/4+t^2}dt = 0$, is equivalent to the Riemann hypothesis.

The last equality here is of course Balazard, Saias, Yor's one [4].

Analogously, by considering "impair" functions $g(z)$, of which the most straightforward example is $g(z) = -i\dfrac{z-b}{(c^2-(z-b)^2)(d^2-(z-b)^2)}$ that takes the form $g(t) = \dfrac{t}{(c^2+t^2)(d^2+t^2)}$ along the line $z=b+it$, we can get integral equalities involving integrals over an argument. In such a way we obtain the following theorem.

Theorem 4. *An equality*

$$\int_{0}^{\infty}\frac{t\arg(\zeta(b+it))}{(c^2+t^2)(d^2+t^2)}dt = \frac{\pi}{2(d^2-c^2)}\ln\left|\frac{\zeta(b+d)}{\zeta(b+c)}\right| + \frac{\pi}{2(d^2-c^2)}\ln\left|\frac{(d^2-(1-b)^2)c^2}{(c^2-(1-b)^2)d^2}\right|, \tag{4}$$

where b, c, d are real positive numbers such that $c\cdot d \leq 3.2\cdot 10^{19}$, $c\neq d$, $b+c\neq 1$, $b+d\neq 1$ and $1 > b \geq 1/2$, holds true for some b if and only if there are no Riemann function zeroes with $\sigma > b$. For $b = 1/2$ this equality is equivalent to the Riemann hypothesis.

Details of the proof are straightforward and thus omitted here; they can be found in our ArXive submissions [6]. The appearance of the condition $c\cdot d \leq 3.2\cdot 10^{19}$ is due to the circumstance that the contribution of the Riemann function zero $\rho = \sigma_k + it_k$ with $\sigma_k > b$ is given by $I_\rho =$

$= -2\pi \int_0^{\sigma_k - b} \dfrac{q + it_k}{(c^2 - q^2 + t_k^2 - 2iqt_k)(d^2 - q^2 + t_k^2 - 2iqt_k)} dq$ from which it immediately follows that the real part of this integral is an integral of the expression

$$-q \frac{-3t_k^4 - 2t_k^2(q^2 + c^2 + d^2) - q^2(c^2 + d^2) + c^2 d^2 + q^4}{((c^2 - q^2 + t_k^2)^2 + 4q^2 t_k^2)((d^2 - q^2 + t_k^2)^2 + 4q^2 t_k^2)}.$$

By our choice of b we have $0 \leq q < 1/2$ (all Riemann zeroes lye in the critical strip) while $t < t_{\min} = 3.3 \cdot 10^9$, see Remark 1. Hence, provided the product cd is not very large, the sign of this expression is certainly positive: it suffices that $c \cdot d \leq 3t_{\min}^2$ whence the theorem condition.

As an example one can take $b = 1/2$, $c = 3/2$, $d = 7/2$ and then in view of $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ obtain the following rather elegant criterion equivalent to RH: $\int_0^\infty \dfrac{t\arg(\zeta(1/2 + it))}{(9/4 + t^2)(49/4 + t^2)} dt = \dfrac{\pi}{20} \ln(18\pi^2/245)$. This condition, as some others, has been tested numerically [6].

Again, the case $b + c = 1$ or $b + d = 1$ does not create any problem because the corresponding limit is quite transparent. Proceeding as above, we get the following theorem.

Theorem 4a. *An equality*

$$\int_0^\infty \frac{t\arg(\zeta(b + it))}{(c^2 + t^2)((1 - b)^2 + t^2)} dt = \frac{\pi}{2(c^2 - (1 - b)^2)} \ln \left| \frac{\zeta(b + c)(c^2 - (1 - b)^2)(1 - b)}{2c^2} \right|, \tag{5}$$

where b, c *are real positive numbers such that* $c(1 - b) \leq 3.2 \cdot 10^{19}$, $b + c \neq 1$ *and* $1 > b \geq 1/2$, *holds true for some* b *if and only if there are no Riemann function zeroes with* $\sigma > b$. *For* $b = 1/2$ *this equality is equivalent to the Riemann hypothesis.*

As an illustration, we can take, for example, $b = 1/2$, $c = 3/2$ and then in view of $\zeta(2) = \pi^2/6$ obtain another rather elegant equality equivalent to RH (this also has been tested numerically): $\int_0^\infty \dfrac{t\arg(\zeta(1/2 + it))}{(9/4 + t^2)(1/4 + t^2)} dt = \dfrac{\pi}{4} \ln\left(\dfrac{\pi^2}{27}\right)$.

The case of the integrals $\int_0^\infty \dfrac{t\arg(\zeta(b + it))}{(a^2 + t^2)^2} dt$ is considered similarly introducing the function $g(z) = -i \dfrac{z - b}{(a^2 - (z - b)^2)^2}$ and analyzing the contour integral $\int_C \ln(\zeta(z)) g(z) dz$ taken round the same contour C. Now in the interior of the contour we have the pole of the second order at $z = a + b$ which, according to the residue theorem, contributes $2\pi i \dfrac{d}{dz} \left(-i \dfrac{z - b}{(a + z - b)^2} \ln(\zeta(z)) \right) \Big|_{z = b + a} = \dfrac{\pi}{2a} \dfrac{\zeta'(a + b)}{\zeta(a + b)}$ to the contour integral value. Proceeding as above, we have the following theorem.

Theorem 5. *An equality*

$$\int_0^\infty \frac{t\arg(\zeta(b + it))}{(a^2 + t^2)^2} dt = \frac{\pi}{4a} \frac{\zeta'(a + b)}{\zeta(a + b)} + \frac{\pi}{2} \left(\frac{1}{a^2 - (1 - b)^2} - \frac{1}{a^2} \right), \tag{6}$$

where a, b *are real positive,* $a + b \neq 1$, $a \leq 5.1 \cdot 10^9$ *and* $1 > b \geq 1/2$, *holds true for some* b *if and only if there are no Riemann function zeroes with* $\sigma > b$. *For* $b = 1/2$ *this equality is equivalent to the Riemann hypothesis.*

Again, the case of $a+b=1$ is a transparent limit of above expressions for $a+b=1+\delta$, $\delta\to 0$ (see [6] for details):

Theorem 5a. *An equality*

$$\int\limits_0^\infty \frac{t\arg(\zeta(b+it))}{((1-b)^2+t^2)^2}dt = \frac{\pi}{4(1-b)}\left(\gamma - \frac{3}{2(1-b)}\right), \tag{7}$$

where b is real $1/2\le b<1$ and γ is the Euler-Mascheroni constant, holds true for some b if and only if there are no Riemann function zeroes with $\sigma>b$. For $b=1/2$ this equality is equivalent to the Riemann hypothesis.

For $b=1/2$ Theorem 5a gives

$$\frac{1}{\pi}\int\limits_0^\infty \frac{2t\arg(\zeta(1/2+it))}{(1/4+t^2)^2}dt = \gamma - 3 \tag{8}$$

and this is nothing else than Volchkov criterion [3].

***Remark* 2.** Volchkov paper more or less explicitly contains this relation but actually he gave another form of his criterion. After obtaining (8), he considered the function $S_1(x)=\int_0^x \arg(\zeta(1/2+ + it))dt$. In view of $S_1(x)=O(\ln x)$ [1, p. 214], one can integrate by parts to obtain

$$\int\limits_0^\infty \arg(\zeta(1/2+ix))g(x)dx = -\int\limits_0^\infty S_1(x)\frac{d}{dx}g(x)dx.$$

Then Volchkov introduced the relation

$$S_1(x)=\int\limits_{1/2}^\infty (\ln|\zeta(\sigma+ix)|-\ln|\zeta(\sigma)|)d\sigma$$

and, noting that $\int_{1/2}^\infty \ln|\zeta(\sigma)|d\sigma=\text{const}$ which contributes nothing to the $\int_0^\infty S(x)g_1(x)dx$, obtained

$$\frac{32}{\pi}\int\limits_0^\infty \frac{1-12t^2}{(1+4t^2)^3}dt\int\limits_{1/2}^\infty \ln|\zeta(\sigma+it)|d\sigma = 3-\gamma;$$

exactly in this form his criterion was published.

***Remark* 3.** Our above equalities involving integrals containing the function $\arg(\zeta(b+it))$ as a factor under the integral sign, may be interpreted as certain sums over Riemann function zeroes. For example, for $b=1/2$ we have a factor $\arg(\zeta(1/2+it))$ and this same factor is exactly the "non-trivial" part of an expression describing the number of the Riemann function zeroes lying in the critical strip and having the imaginary part $0<t<x$:

$$N(x)=1-\frac{x\ln\pi}{2\pi}+\frac{1}{\pi}\mathrm{Im}\Big(\ln\Gamma(1/4)+\frac{ix}{2}\Big)+\frac{1}{\pi}\arg(\zeta(1/2+ix)), \tag{9}$$

see e.g. [1, p. 212] for details. The integral $\int_0^\infty G(x)dN(x)$, provided, of course, that it exists, is equal to the sum $\sum_{\rho,t_k>0} G(t_k)$ taken over all Riemann zeroes. In practice, this calculation is interesting if the function $G(x)$ is such that the integration by parts can be used:

$$\int_0^\infty G(x)dN(x) = G(x)N(x)|_0^\infty - \int_0^\infty g(x)N(x)dx;$$

here, evidently, $g(x) = dG(x)/dx$. Exactly in this way Volchkov established his criterion.

Of course, an infinite number of other integral equalities equivalent to the Riemann hypothesis can be obtained in the same way. Note also that by considering the "semi-contour" C with $X_1 = b$, $X_2 = b + T$, $Y_1 = 0$, $Y_2 = T$, a number of interesting equalities relating with each other integrals of the logarithm of the Riemann function taken along the real axis and the line $(b, b + i\infty)$ can be obtained. In some cases the consideration of the "square root type" functions like $g(z) = \dfrac{1}{(a^2 - (z-b)^2)^{3/2}}$ might be interesting as well; for details see our contributions [6].

3.2. Exponential functions. The case of exponential functions $g(z)$, viz. $g(z) = \dfrac{i}{\cos(a(z-b))}$, $g(z) = \dfrac{-i(z-b)}{\cos^2(a(z-b))}$, etc. can be considered along the same lines. Here we will limit ourselves with the first aforementioned function; for the treatment of the second see [6].

Let us take the same contour as above. For $z - b = x + iy$ where x, y are real, we have that for large y $|\cos(a(z-b))|^{-1} = O(e^{-a|y|})$ provided $\arg(z-b) \neq 0$ and $\arg(z-b) \neq \pi$. This asymptotic, together with the known asymptotic of $\ln(\zeta(z))$, guaranties the disappearance of the integral taken along the external lines of the contour (i.e., along lines other than its left border): the problems might appear only for real positive $z - b$, but for such a case we have $\ln(\zeta(z)) \cong 2^{-z}$, so it is enough to avoid such values of X for which $g(z)$ has poles when $X \to \infty$.

In the interior of the contour we have simple poles at the points $z = b + \pi/(2a) + \pi n/a$ where n is an integer or zero, and, if $b < 1$, also a simple pole of the Riemann function at $z = 1$. If $b < 1$, in the interior of the contour we also can have a number of zeroes of the Riemann function, and we definitely have an infinite number of them if $b < 1/2$. The contribution of the poles of $g(z)$ to the contour integral value is, by a residue theorem, equal to $2\pi i \left(i \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{a} \ln(\zeta(b + \pi/(2a) + \pi n/a) \right)$. To find the corresponding contribution of the pole of the Riemann function we apply the generalized Littlewood theorem: for $b < 1$ it is equal to

$$2\pi i \int_b^1 \frac{i}{\cos(a(z-b))} dz = -2\pi \int_0^{1-b} \frac{1}{\cos(ax)} dx = \frac{\pi}{a} \ln \frac{1 - \sin(a(1-b))}{1 + \sin(a(1-b))};$$

we take $a(1-b) < \pi/2$ to avoid the problems with this integral. Analogously, the order l_k zero of the Riemann function $\rho = \sigma_k + it_k$, for $b < \sigma_k$, contributes

$$-2\pi i \int_{b+it_k}^{\sigma_k+it_k} \frac{il_k}{\cos(a(z-b))} dz = 2\pi \int_0^{\sigma_k-b} \frac{l_k}{\cos(ax)\cosh(at_k) - i\sin(ax)\sinh(at_k)} dx$$

to the contour integral value. Pairing the complex conjugate zeroes we see that due to the symmetry of the distribution of the Riemann function zeroes the imaginary part of these contributions vanishes and thus, collecting everything together, we have the following equality: for $-2 \le b < 1$

$$\int_0^\infty \frac{\ln|\zeta(b+it)|}{\cosh(at)}dt = \pi \sum_{\rho, t_k > 0} 2l_k \int_0^{\sigma_k - b} \frac{\cos(ax)\cosh(at_k)}{\cos^2(ax)\cosh^2(at_k) + \sin^2(ax)\sinh^2(at_k)}dx +$$

$$+\frac{\pi}{2a}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \pi\sum_{n=0}^\infty \frac{(-1)^n}{a}\ln(\zeta(b+\pi/(2a)+\pi n/a)). \qquad (10)$$

As usual, the sum here is taken over all zeroes $\rho = \sigma_k + it_k$ with $b < \sigma_k$ taken into account their multiplicities; when obtaining (10) we paired the contributions of the complex conjugate zeroes $\rho = \sigma_k \pm it_k$ hence $t_k > 0$.

This is easy to see that if we take $a(1-b) < \pi/2$ than the integrand in

$$\int_0^{\sigma_k - b} \frac{\cos(ax)\cosh(at_k)}{\cos^2(ax)\cosh^2(at_k) + \sin^2(ax)\sinh^2(at_k)}dx$$

is always positive (the value $\sigma_k - b$ can not exceed $1-b$) and thus we have proven the following theorem.

Theorem 6. *An equality*

$$\frac{a}{\pi}\int_0^\infty \frac{\ln|\zeta(b+it)|}{\cosh(at)}dt = \frac{1}{2}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \sum_{n=0}^\infty (-1)^n \ln(\zeta(b+\pi/(2a)+\pi n/a)), \qquad (11)$$

where b, a are real positive numbers such that $1 > b \ge 1/2$, $a(1-b) < \pi/2$ holds true for some b if and only if there are no Riemann function zeroes with $\sigma > b$. For $b = 1/2$ this equality is equivalent to the Riemann hypothesis.

It is interesting to take $b = 1/2$ and consider the limit $a \to \pi$. There are no Riemann function zeroes with $\operatorname{Re} s = 1$ hence the positivity of the contributions of Riemann function zeroes nonlying on the critical line is still certain and it remains only to consider the limit

$$\lim_{a\to\pi}\left(\frac{1}{2}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \ln\left(\zeta\left(1/2+\frac{\pi}{2a}\right)\right)\right).$$

This can be done without problems and we obtain the following equality equivalent to the Riemann hypothesis

$$\int_0^\infty \frac{\ln|\zeta(1/2+it)|}{\cosh(\pi t)}dt = \ln\frac{\pi}{2} + \sum_{n=2}^\infty (-1)^{n+1}\ln(\zeta(n)). \qquad (12)$$

Here those logarithms of the Riemann function which correspond to even n can be expressed via Bernoulli numbers B_{2m}, $m = 1, 2, 3, \dots$: $\zeta(2m) = (-1)^{m+1}(2\pi)^{2m}\dfrac{B_{2m}}{2(2m)!}$, and the following remarkable property of the Riemann ζ-function $\prod_{n=2}^\infty \zeta(n) = C = 2.29485\dots$, that is

$\sum_{n=2}^{\infty} \ln(\zeta(n)) = \ln C = 0.8306\ldots$can be also used for calculations, see [10, p. 16]. Here C is the residue of the pole at $s = 1$ of the Dirichlet series whose coefficients $g(n)$ are the numbers of non-isomorphic Abel's group of order n. Eq. (12) also has been tested numerically.

Conclusions. In this paper we have established a number of criteria involving the integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis. Our results include all earlier known criteria of this kind [3, 6, 9] which are certain particular cases of the general approach proposed. An infinite number of other criteria of the type "$\int_{b-i\infty}^{b+i\infty} g(z)\ln(\zeta(z))dz = f(b)$ is equivalent to RH" can be constructed following the lines of the present paper, viz. selecting an appropriate function $g(z)$ and calculating the value of a contour integral exploiting the generalized Littlewood theorem.

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