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SOME PROBLEMS IN NONCLASSICAL ALGEBRAIC GEOMETRY

ДЕЯКІ ПРОБЛЕМИ В НЕКЛАСИЧНІЙ АЛГЕБРАЇЧНІЙ ГЕОМЕТРІЇ

We describe the general approach to a nonstandard geometry with the emphasis on associative algebras.

Описано загальний підхід до нестандартної алгебраїчної геометрії з акцентом на асоціативних алгебрах.

1. Introduction. Fix an infinite field P. Consider two varieties of algebras.

Com-P – the variety of all commutative and associative algebras with the unity over P.

Ass-P – the variety of all associative, but not necessarily commutative algebras with the unity over P.

Let Θ be an arbitrary variety of algebras. In every such Θ one can consider its algebraic geometry (algebraic geometry in Θ) [1–4].

The classical algebraic geometry over the field P is algebraic geometry associated with variety Com-P. Correspondingly, we define nonclassical geometry as an algebraic geometry in the variety of Ass-P and in various subvarieties of this variety.

One can consider algebraic geometry over H for every Θ and every algebra $H \in \Theta$. The principal question is as follows: *for which H_1 and H_2 do the corresponding algebraic geometries coincide?*

Let us pose this question in more detailed way. For every $H \in \Theta$ we consider two categories

$K_\Theta(H)$ — the category of algebraic sets over H,

$\tilde{K}_\Theta(H)$ — the category of algebraic varieties over H.

Since algebraic variety is regarded as an algebraic set considered up to an isomorphism in the category $K_\Theta(H)$, the category $\tilde{K}_\Theta(H)$ is a skeleton of the category $K_\Theta(H)$.

All precise definitions will be given below.

Both categories $K_\Theta(H)$ and $\tilde{K}_\Theta(H)$ are geometric invariants of the algebra H, and in many senses they are responsible for the geometry in H.

Now we can formulate the main question in two more precise ways:

When are the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ isomorphic?

When are the categories $\tilde{K}_\Theta(H_1)$ and $\tilde{K}_\Theta(H_2)$ isomorphic?

Category theory says that two categories C_1 and C_2 are equivalent if and only if their skeletons $\tilde{C}_1$ and $\tilde{C}_2$ are isomorphic [5]. Hence, the second question may be replaced by the following one:

When are the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ equivalent?

It is clear that if 1) holds for H_1 and H_2, then 2) also holds, but not vice versa.

In the classical situation both question are solved to some extent [4, 6], and the main part here plays the notion of geometrical equivalence of two algebras. There are reasons to believe that a similar picture exists for the variety Ass-P.

We need to make the following remark.

In the classical situation if the field P is infinite, then it generates the whole variety Com-P. Therefore every algebra of this variety generates the whole variety. In the situation of Ass-P this is far from being so, and we will proceed from the assumption that the algebras H_1 and H_2 generate the whole variety Ass-P.

Before we pass to the definitions, we would like to make another remark. Along with the categories $K_\Theta(H)$, one can consider also a category K_Θ [3]. This is the category of algebraic sets over different $H \in \Theta$. Correspondingly, we have a category of algebraic varieties $\tilde{K}_\Theta$. Also here arise the problems of isomorphism and equivalence of the categories K_{Θ_1} and K_{Θ_2} from different Θ_1 and Θ_2, where Θ_1 and Θ_2 are subvarieties of the variety Ass-P.

Our plan is as follows:

To give the definitions of necessary notions.

The main focus is on geometrical equivalence of two algebras.

To consider the main problem for the variety Ass-P.

2. Definitions. Recall first of all that a variety of algebra is a class of algebras determined by identities. If $\mathfrak{X}$ is an arbitrary class of algebras in Θ, then the variety generated by $\mathfrak{X}$ is the minimal variety in Θ containing $\mathfrak{X}$. This variety is determined by the identities of algebras from $\mathfrak{X}$ and is denoted by $\Theta = \mathrm{Var}(\mathfrak{X})$. It is known that $\mathrm{Var}(\mathfrak{X}) = QSC(\mathfrak{X})$. The language of operators on classes is used here: $C(\mathfrak{X})$ is the class of cartesian products of algebras from $\mathfrak{X}$, $S(\mathfrak{X})$ is the class of all subalgebras of algebras from $\mathfrak{X}$, and $Q(\mathfrak{X})$ is the class of all homomorphic images of algebras from $\mathfrak{X}$. Further we will use such a language.

In every variety Θ there are free algebras. Denote by Θ^0 the category of all free in Θ algebras $W = W(X)$ with finite X. We assume that all these X are subsets of some infinite fixed universum X^0. Then the category Θ^0 is a small category whose objects constitute a set.

Fix now an algebra $H \in \Theta$. For every object $W = W(X)$ of the category Θ^0, consider the set of homomorphisms $\mathrm{Hom}(W, H)$. If $X = \{x_1, \ldots, x_n\}$, then we have a bijection $\alpha_X : \mathrm{Hom}(W, H) \to H^{(n)}$. For every $\nu : W \to H$ we have $\alpha_X(\nu) = (\nu(x_1), \ldots, \nu(x_n))$. Now we consider $\mathrm{Hom}(W, H)$ as an *affine space*. Its points are homomorphisms $\nu : W \to H$. We can also consider the category of affine spaces. In this category morphisms have the form

$$\tilde{s} : \mathrm{Hom}(W(X), H) \to \mathrm{Hom}(W(Y), H),$$

where $s : W(Y) \to W(X)$ is a morphism in Θ^0 and $\tilde{s}(\nu) = \nu s : W(Y) \to H$ for every point $\nu : W(X) \to H$.

Further, for the sake of simplicity, we consider only associative algebras. If $w \in W$, then the point ν is a root of this w if $w^\nu = 0$, $w \in \mathrm{Ker}\nu$.

Let now T be a subset in W and A a subset in $\mathrm{Hom}(W, H)$. We are interested in the following Galois correspondence:

$$T'_H = A = \{\nu : W \to H \mid T \subset \mathrm{Ker}\nu\},$$

$$A' = T = \bigcap_{\nu \in A} \mathrm{Ker}\nu.$$

We call every A of the form $A = T'_H$ a closed (algebraic) set. For each A we have a closure $A'' = (A')'_H$. Every T of the form $T = A'$ is an ideal in W, and we call it an H-closed ideal. For an arbitrary T we have a closure T''_H. An ideal T is H-closed if and only if there is an injection $W/T \to H^I$ for some I. This is equivalent to $W/T \in SC(H)$.

The class of algebras $SC(H)$ is closed under operators S and C. We call every such class a prevariety. For an arbitrary wet T, its closure T''_H is an intersection of all ideals T_α containing T and such that $W/T_\alpha \in SC(H)$.

We give the main definition.

Definition 1. *Algebras H_1 and H_2 in Θ are called geometrically equivalent* *if* $T''_{H_1} = T''_{H_2}$ *for every* $W = W(X)$ *and every* T *in* W.

3. Geometrically equivalent algebras. It is proved [7] that the algebras H_1 and H_2 are geometrically equivalent if and only if

$$LSC(H_1) = LSC(H_2).$$

Here L is an operator, determined by the condition: $H \in L\mathfrak{X}$, if every finitely generated subalgebra in H belongs to the class $\mathfrak{X}$. The class $LSC(\mathfrak{X})$ is a locally closed prevariety over the class $\mathfrak{X}$. We will also consider quasivarieties, i. e., classes of algebras determined by quasiidentities (formulas) of the form

$$u_1 \equiv 0 \wedge \ldots \wedge u_n \equiv 0 \Rightarrow v \equiv 0.$$

Denote by $q\,\mathrm{Var}(\mathfrak{X})$ the quasivariety over $\mathfrak{X}$. Then the embedding $LSC(\mathfrak{X}) \subset$ $\subset q\,\mathrm{Var}(\mathfrak{X}) \subset \mathrm{Var}(X)$ always holds true.

Let us return to the closure T''_H. It is easy to understand that the inclusion $v \in T''_H$ takes place if and only if the formula

$$\left(\bigwedge_{u\in T} u \equiv 0\right) \Rightarrow v \equiv 0$$

holds in the algebra H.

We call such a formula a generalized (infinite) quasiidentity. Now we can say that the algebras H_1 and H_2 are geometrically equivalent if and only if every generalized quasiidentity of the algebra H_1 holds also in H_2, and vice versa.

From this follows that if H_1 and H_2 are geometrically equivalent, then $q\,\mathrm{Var}(H_1) = q\,\mathrm{Var}(H_2)$. Is the opposite true? We consider this problem in more detail.

Long ago, A. I. Malcev [8, 9] proved that the prevariety $SC(\mathfrak{X})$ is an axiomatized class if and only if it is a quasivariety. This motivated the problem: for which $\mathfrak{X}$ does the equality

$$SC(\mathfrak{X}) = q\,\mathrm{Var}(\mathfrak{X})$$

hold? The problem was solved by V. A. Gorbunov [10].

We are interested in the conditions when

$$LSC(H) = q\,\mathrm{Var}(H).$$

Assume that this is true for the algebras H_1 and H_2. Then

$$q\,\mathrm{Var}(H_1) = LSC(H_1),$$
$$q\,\mathrm{Var}(H_2) = LSC(H_2),$$

and the algebras H_1 and H_2 are geometrically equivalent if and only if $q\,\mathrm{Var}(H_1) =$ $= q\,\mathrm{Var}(H_2)$, that is, H_1 and H_2 have the same quasiidentities.

Let us also remark that if the above equality holds for H, then for every W *and* $T \subset W$ the closure T''_H is the intersection of all ideals T_α in W, containing T and with the condition $W/T_\alpha \in q\,\mathrm{Var}(H)$.

This is a variant of Hilbert's Nullstallensatz theorem in the general situation.

We need the following definitions.

Definition 2. *The algebra* $H \in \Theta$ *is called geometrically noetherian if for every* W *and* $T \subset W$, *there is a finite part* T_0 *in* T *such that* $T''_H = T''_{0_H}$.

The algebra H is geometrically noetherian if and only if every ascending sequence of H-closed ideals is finite.

We call a variety noetherian if every algebra $W = W(X)$ with the finite X is noetherian. All noetherian subvarieties in Ass-P are described. The variety Com-P is one of them.

It is clear that if Θ is a noetherian variety, then every algebra $H \in \Theta$ is geometrically noetherian.

Let us now weaken the condition to be geometrically noetherian.

Definition 3. *The algebra H is called locally geometrically noetherian if for every W and $T \subset W$, and for every $v \in T''_H$ in T there is a finite part T_0 such that $v \in T''_{0_H}$.*

This is equivalent to the fact that every generalized quasiidentity

$$\left(\bigwedge_{u \in T} u \equiv 0\right) \Rightarrow v \equiv 0$$

can be reduced to an ordinary quasiidentity

$$\left(\bigwedge_{u \in T_0} u \equiv 0\right) \Rightarrow v \equiv 0.$$

The set T_0 depends, in general, on v. It is easy to prove that the algebra H is locally geometrically noetherian if and only if for every W the union of each increasing sequence of H-closed ideals is H-closed. In case of geometrically noetherian algebras every such chain is finite.

Now we state the following theorem.

Theorem 1. *The equality*

$$LSC(H) = q\,\mathrm{Var}(H)$$

holds if and only if the algebra H is locally geometrically noetherian.

This theorem for groups was proved by Myasnikov and Remeslennikov [11] (see also [12, 13]) as a development of ideals of Gorbunov [10]. But their proof is valid also for every variety Θ.

Let us apply this to the next theorem (compare [11]).

Theorem 2. *If the algebra H is not locally geometrically noetherian, then there exists its ultrapower H' which is not geometrically equivalent to H.*

Proof. It is known [14] that $q\,\mathrm{Var}(\mathfrak{X}) = SCC_{\mathrm{up}}(\mathfrak{X})$ is always true. Here $C_{\mathrm{up}}(\mathfrak{X})$ are ultraproducts of algebras from $\mathfrak{X}$.

Let now the algebra H be not locally geometrically noetherian. Then

$$LSC(H) \neq q\,\mathrm{Var}(H) = SCC_{\mathrm{up}}(H).$$

This means that some algebra H' in $C_{\mathrm{up}}(H)$ does not lie in $LSC(H)$. Then $LSC(H) \neq LSC(H')$, and H and H' are not geometrically equivalent.

The theorem is proved.

There arises a question about examples of nonlocally geometrical algebras. There are such examples for groups and associative algebras. First consider the case of groups (compare [15]).

Let a group G be a discrete direct product of finitely presented groups, i.e., the groups of the form $F(X)/U$, where $F = F(X)$ is the free group over a finite set X, and U is an invariant subgroup in F generated by a finite set of elements. It can be proven that the group G is not locally geometrically noetherian. The proof uses the following group theoretic fact [16]: there exists continuum of 2-generator simple groups. The similar fact for associative algebras has been proven recently by A. Lichtman (unpublished). This allows to construct the example of associative nonlocally geometrically noetherian algebra.

Let us point out the following Problems.

1. To investigate the wreath products of groups from the point of view to be locally geometrically noetherian.

2. To investigate from the same point of view relations between groups and group algebras.

3. Whether free associative or free Lie algebras are locally geometrically noetherian. Note [17] that a free group is geometrically noetherian and correspondingly, locally geometrically noetherian.

Let us note that these question can be solved relatively simply in the situation of algebras with big fixed algebra of constants.

4. Isomorphisms and equivalences of categories of algebraic sets. We now pass to the main problems, which were formulated above. Define first the category $K_\Theta(H)$. Objects in each such category are of the form (X, A), where A is an algebraic set in an affine space $\mathrm{Hom}(W(X), H)$. Now define morphisms:

$$(X, A) \to (Y, B).$$

We proceed from $s : W(Y) \to W(X)$ and have

$$\tilde{s} : \mathrm{Hom}(W(X), H) \to \mathrm{Hom}(W(Y), H).$$

Consider further s, such that $\tilde{s}(\nu) \in B \ \forall \nu \in A$. These are exactly those s, which induce a homomorphism

$$\bar{s} : W(Y)/B' \to W(X)/A',$$

$su \in A'$ if $u \in B'$. For such s we have a mapping $[s] : A \to B$, treated as a morphism $(X, A) \to (Y, B)$.

Simultaneously, we consider a category $C_\Theta(H)$. Its objects are algebras $W(X)/T$, where T is an H-closed ideal in $W(X)$, and morphisms are homomorphisms of algebras.

The transition $(X, A) \to W(X)/A'$ determines duality of categories $K_\Theta(H) \to \to C_\Theta(H)$. If the algebras H_1 and H_2 are geometrically equivalent, then the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$ coincide, and the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.

If $\mathrm{Var}(H) = \Theta$, then the category Θ^0 is a subcategory in $C_\Theta(H)$ and Θ^0, and the category of affine spaces are dual. The last is always a subcategory in $K_\Theta(H)$. Let, further, $\mathrm{Var}(H_1) = \mathrm{Var}(H_2) = \Theta$.

An isomorphism $F: K_\Theta(H_1) \to K_\Theta(H_2)$ is called correct, if the corresponding Φ : $C_\Theta(H_1) \to C_\Theta(H_2)$ induces an automorphism $\varphi : \Theta^0 \to \Theta^0$ and, besides, it is assumed that Φ takes a natural homomorphism of $C_\Theta(H_1)$ into natural homomorphisms of $C_\Theta(H_2)$.

For F this means that $F(\mathrm{Hom}(W, H_2)) = \mathrm{Hom}(\varphi(W), H_2)$. If A is an algebraic set in $\mathrm{Hom}(W(X), H_1)$, then $F(X, A) = (Y, B)$, where B is an algebraic set in $\mathrm{Hom}(W(Y), H_2)$, $W(Y) = \varphi(W(X))$.

Recall that the algebras H_1 and H_2 are semiisomorphic if there is a pair (σ, ν), where $\sigma \in \mathrm{Aut}P$, $\nu : H_1 \to H_2$ is an isomorphism of rings, and $\nu(\lambda a) = \lambda^\sigma \nu(a)$, $a \in H_1$, $\lambda \in P$.

H_1 and H_2 are antiisomorphic if there is an isomorphism of P-modules $\mu : H_1 \to \to H_2$ such that $\mu(ab) = \mu(b) \cdot \mu(a)$. Here μ is an antiisomorphism of algebras.

Now let us formulate the theorems for Com-P and a conjecture for Ass-P.

Theorem 3. *The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ in* Com-P *are correctly isomorphic if and only if there exists H such that H and H_1 are semiisomorphic and H and H_2 are geometrically equivalent.*

Here Θ = Com-P, and H is automatically built by H_1.

The proof of this theorem depends on investigations of automorphisms $\varphi : \Theta^0 \to \to \Theta^0$, where Θ = Com-P. It is proved [6] that every such φ is semiinner. In the same classical situation we consider a question on equivalence of categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$. It turned out that here we have the same result. Also here everything depends on investigations of autoequivalencies of the category Θ^0. All of them are also semiinner.

We now pass to Θ = Ass-P. For every finite X denote by $S(X)$ a free monoid over X and by $S_0(X)$ a free semigroup over X. $W(X) = W$ is a semigroup algebra, $W = PS(X)$. Each of its elements has the form

$$w = \lambda_0 + \lambda_1 u_1 + \ldots + \lambda_k u_k, \quad \lambda \in P, \quad u \in S_0(X).$$

Denote $\bar{u} = x_{i_n} \ldots x_{i_1}$ for every $u = x_{i_1} \ldots x_{i_n}$. By definition

$$\bar{w} = \lambda_0 + \lambda_1 \bar{u}_1 + \ldots + \lambda_k \bar{u}_k.$$

Consider further a mirror automorphism σ of the category Θ^0. It does not change objects, and for every $\nu: W(X) \to W(Y)$ we set $\delta(\nu): W(X) \to W(Y)$ determined by the rule

$$\delta(\nu)(x) = \overline{\nu(x)} \qquad \forall\, x \in X.$$

Conjecture. *Every automorphism* $\varphi: \Theta^0 \to \Theta^0$ *can be represented as* $\varphi = \varphi_0 \delta$, *where* φ_0 *is semiinner or already* φ *is semiinner or* $\varphi = \delta$.

If this question can be solved positively, then we can state that the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if there exists a sequence

$$H_1, H, H', H_2$$

such that H_1 and H_2 are antiisomorphic, H and H' are semiisomorphic, H' and H_2 are geometrically equivalent.

The algebras H and H' are easily constructed by H_1. The same result could also be obtained to the question about equivalence of the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ (see [18]).

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