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B. Schneider (Izmir Univ. Econ., Turkey)

SOME PROPERTIES OF THE CAUCHY-TYPE INTEGRAL FOR THE MOISIL – THEODORESCO SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

ДЕЯКІ ВЛАСТИВОСТІ ІНТЕГРАЛІВ ТИПУ КОШІ ДЛЯ СИСТЕМ МОІСІЛ – ТЕОДОРЕСКО ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ

Our main interest is the analog of the Cauchy-type integral for the theory of Moisil – Theodoresco system of differential equations in the case of a piecewise Liapunov surface of integration. The topics of the paper concern theorems which cover basic properties of that Cauchy-type integral: the Sokhotski – Plemelj theorem for it as well as the necessary and sufficient condition for the possibility to extend a given Hölder function from such a surface up to a solution of Moisil – Theodoresco system of partial differential equations in a domain. A formula for the square of the singular Cauchy-type integral is given. The proofs of all these facts are based on intimate relations between the theory of Moisil – Theodoresco system of partial diferential equations and some versions of quaternionic analysis.

Роботу в основному присвячено вивченню аналога інтеграла типу Коші для теорії систем Моісіл – Теодореско диференціальних рівнянь у випадку кускової поверхні інтегрування Ляпунова. Розглядаються теореми, що охоплюють базові властивості цього інтеграла типу Коші, а саме теорема Сохоцького – Племель для нього, а також необхідна і достатня умова продовжуваності заданої функції Гельдера з названої вище поверхні до розв'язку системи Моісіл – Теодореско диференціальних рівнянь з частинними похідними в області. Наведено формулу квадрата сингулярного інтеграла типу Коші. Доведення всіх цих фактів базується на близьких зв'язках між теорією систем Моісіл – Теодореско диференціальних рівнянь з частинними похідними і деякими версіями кватерніонного аналізу.

1. Introduction. As is well known, the role of the Cauchy-type integral in holomorphic function theory of one complex variable is very important. In this article, we investigate the properties of the Cauchy-type integral for the first order elliptic system in $\mathbb{R}^3$. Let Ω be a domain in $\mathbb{R}^3$. Suppose that $f = f_0 + \vec{f} \in C^1(\Omega, \mathbb{R}^4)$. The homogeneous system

$$\begin{gathered} \mathrm{div} f = 0, \\ \mathrm{grad} f_0 + \mathrm{rot} \vec{f} = 0 \end{gathered}$$

is called Moisil – Theodoresco system and is the simplest analog of the Cauchy – Riemann system in the three-dimensional case. Thus, the theory of solutions of the Moisil – Theodoresco system of differential equations reduces, in some degenerate cases, to that of complex holomorphic functions. Hence, one may consider the former to be a generalization of the latter.

Note that if $f_0 = 0$, we have

$$\begin{gathered} \mathrm{div} f = 0, \\ \mathrm{rot} f = 0. \end{gathered} \tag{1}$$

Solutions to system (1) are called solenoidal and irrotational vector fields (cf. [1], where some applications to geophysics are given. It is known that solutions of (1) satisfy the Laplace equation and are sometimes called Laplacian or harmonic vector fields. In [2], we studied some properties of the Cauchy-type integral for the Laplace vector fields theory, also.

In the present paper, we follow the approach presented in paper [3] in which we studied the analog of the Cauchy-type integral for the theory of time-harmonic solutions of the relativistic Dirac equation in the case of a piecewise Liapunov surface of integration. The paper is organized as follows. In Section 2, we formulate a series of theorems which cover basic properties of the Cauchy-type integral for the theory of Moisil – Theodoresco system of differential equations in the case of a piecewise Liapunov surface of integration. The proofs of all of them one can find in Section 4 in the form of more or less direct corollaries of the corresponding facts valid for hyperholomorphic function theory, which is developed in Section 3 and [4].

2. Moisil – Theodoresco system of partial differential equations and the Cauchy – Moisil – Theodoresco integral. ***2.1.*** Let Ω denote a domain in $\mathbb{R}^3$ and let $\Gamma := \partial\Omega$ be its boundary. For $\Omega \subset \mathbb{R}^3$ consider an $\mathbb{R}^4$-valued function $f = (f_0, f_1, f_2, f_3)$, which satisfies the following system of partial differential equations:

$$\begin{aligned} 0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} &= 0, \\ \frac{\partial f_0}{\partial x_1} + 0 - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0, \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + 0 - \frac{\partial f_3}{\partial x_1} &= 0, \\ \frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + 0 &= 0. \end{aligned}$$

It is usually called a *Moisil – Theodoresco system.* Let $V_{st} := \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a = (\delta_j^k)_{j,k=1}^3$ (δ_j^k is the Kronecker symbol), $x = (0, x_1, x_2, x_3)^T$, and $d\hat{x} = (0, dx_{[1]}, -dx_{[2]}, dx_{[3]})^T$, where $dx_{[k]}$ denotes, as usual, the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. The integral

$$^{V_{st}}K_\Gamma[f](x) := \frac{1}{4\pi} \int_\Gamma \frac{1}{|\tau - x|^3} B_l(V_{st}^T \cdot (\tau - x)) B_l(V_{st}^T \cdot d\hat{\tau}) f(\tau), \quad x \notin \Gamma,$$

plays the role of an analog of the Cauchy-type integral in the theoryof the Moisil – Theodoresco system of partial differential equations with $f : \Gamma \to \mathbb{R}^4$ (see [5]). We shall call it the *Cauchy – Moisil – Theodoresco-type integral.*

2.2. For reader's convenience, we collect here some definitions which we use in the sequel. Let $H_\mu(\Gamma, \mathbb{R}^4)$ denote the class of functions satisfying the Hölder condition $\{f \in \mathbb{R}^4 |\ |f(t_1) - f(t_2)| \le L_f |t_1 - t_2|^\mu\ \forall\{t_1, t_2\} \subset \Gamma,\ L_f = \text{const}\}$ with the exponent $0 < \mu \le 1$. Here, $|f|$ means the Euclidean norm in $\mathbb{R}^4$ while $|t|$ is the Euclidean norm in $\mathbb{R}^3$. We say (see, e.g., [6]) that the surface Γ in $\mathbb{R}^3$ is a Liapunov surface if the following conditions are satisfied:

1. At each point $t \in \Gamma$, there is the tangential hyperplane.

2. There exists a constant number $R > 0$ such that for any point $t \in \Gamma$, the set $\Gamma \cap \mathbb{B}^3(t, R)$ is connected and lines, that are parallel to the normal $\vec{n}(t)$ to the surface Γ at the point t, intersect $\Gamma \cap \mathbb{B}^3(t, R)$ at not more than one point. Here, $\mathbb{B}^3(t, R)$ is an open ball in $\mathbb{R}^3$ centered at the point t and with radius R.

3. The normal vector field $\vec{n} : \Gamma \to \mathbb{R}^3$ satisfies the Hölder condition.

A conical surface in $\mathbb{R}^3$ is a surface generated by a straight line (the generator), which passes through a fixed point (the vertex or conical point) and moves along a fixed curve (the directing curve). A solid angle in $\mathbb{R}^3$ is a part of the space $\mathbb{R}^3$ bounded by some conical surface. A tangential conical surface to Γ at the point t_0 is the conical surface generated by straight tangent lines to surface Γ at point t_0 (the conical point of tangential conical surface). In particular, for a smooth point, the tangential conical surface is its tangential plane. The measure of a solid angle in $\mathbb{R}^3$ is the surface area cut out by the solid angle from the unit sphere having its center in the vertex; the value of the measure is defined in accordance with the orientation of the conical surface.

Let $\mathbf{l}$ be a smooth, closed, and simple curve on the surface $\Gamma \subset \mathbb{R}^3$ such that $\Gamma \setminus \mathbf{l}$ is a Liapunov surface. Then the curve $\mathbf{l}$ is called an edge of the surface Γ and Γ is called a Liapunov surface with edge.

For $\mathbf{l}$ as above, let $t_0 \in \mathbf{l}$. Then the normal plane to the curve at the point t_0 intersects the surface Γ by the curve l_{t_0}. The curve l_{t_0} is a smooth curve except, possibly, t_0. Assume that the curve l_{t_0} has both one-sided tangents P_1 and P_2 at t_0. Let $\mathbf{p}$ be a tangent line to the curve $\mathbf{l}$ itself at point t_0. Then the plane T_1, passing through P_1 and $\mathbf{p}$, and the plane T_2, passing through P_2 and $\mathbf{p}$, generate a dihedral angle which is called tangential dihedral angle.

A linear measure of the tangential dihedral angle is the value of the angle formed by the one-sided tangents P_1 and P_2. Denote it by $\eta(t)$. In the sequel, we take $\eta(t_0) = \mathrm{const}$ on $\mathbf{l}$, the constant being different from 0 and 2π. If $\eta(t) = \pi$ on $\mathbf{l}$, then Γ is a smooth surface. In particular, for a smooth surface, any closed, smooth, and simple curve is an edge.

A solid measure of the tangential dihedral angle is the surface area cut out by the planes T_1 and T_2 from the unit sphere having its center at the point $t_0 \in \mathbf{l}$; the value of the measure is defined in accordance with the orientation of the surface with edge.

Let Γ be a surface in $\mathbb{R}^3$ which contains a finite number of conical points and a finite number of nonintersecting edges such that none of the edges contain any of conical points. If the complement (in Γ) of the union of conical points and edges is a Liapunov surface, then we shall refer to Γ as a piecewise Liapunov surface in $\mathbb{R}^3$.

2.3. **Theorem** (Sokhotski – Plemelj formulas for the Cauchy – Moisil – Theodoresco-type integral with the piecewise Liapunov surface of integration). *Let Ω be a bounded domain in $\mathbb{R}^3$ with the piecewise Liapunov boundary. Let $f \in H_\mu(\Gamma, \mathbb{R}^4)$. Then the following limits exist*:

$$\lim_{\Omega^\pm \ni x \to t \in \Gamma} {}^{V_{st}}K_\Gamma[f](x) =: {}^{V_{st}}K_\Gamma[f]^\pm(t);$$

moreover, the following identities hold:

$${}^{V_{st}}K_\Gamma[f]^+(t) = \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + {}^{V_{st}}K_\Gamma[f](t) := \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + \frac{1}{2}\, {}^{V_{st}}S_\Gamma[f](t),$$

$${}^{V_{st}}K_\Gamma[f]^-(t) = -\frac{\gamma(t)}{4\pi} f(t) + {}^{V_{st}}K_\Gamma[f](t) := -\frac{\gamma(t)}{4\pi} f(t) + \frac{1}{2}\, {}^{V_{st}}S_\Gamma[f](t)$$

for all $t \in \Gamma$, where

$${}^{V_{st}}S_\Gamma[f](t) := 2\, {}^{V_{st}}K_\Gamma[f](t),$$

the integrals being understood in the sense of the Cauchy principal value, $\gamma(t)$ is the measure of a solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t.

2.4. We shall call the operator ${}^{V_{st}}S_\Gamma$ *the singular Cauchy – Moisil – Theodoresco integral operator*. It's appeared that many properties which are of interest for us, can be expressed better in terms of another operator

$$ {}^{V_{st}}\breve{S}_\Gamma[f](t) := \frac{2\pi - \gamma(t)}{2\pi} f(t) + {}^{V_{st}}S_\Gamma[f](t) $$

for any $t \in \Gamma$. We shall call ${}^{V_{st}}\breve{S}_\Gamma$ *the modified singular Cauchy – Moisil – Theodoresco integral operator*.

2.5. **Theorem** (Plemelj – Privalov's-type theorem for the Moisil – Theodoresco system of partial differential equations theory). *Let Ω be a bounded domain in $\mathbb{R}^3$ with piecewise Liapunov boundary. Then*

$$ f \in H_\mu(\Gamma, \mathbb{R}^4) \Rightarrow {}^{V_{st}}\breve{S}_\Gamma[f](t) \in H_\mu(\Gamma, \mathbb{R}^4) \tag{2} $$

for $0 < \mu < 1$.

2.6. **Theorem** (extension of a Hölder function given on Γ up to solution of the Moisil – Theodoresco system of partial differential equations). *Let Ω be a bounded domain in $\mathbb{R}^3$ with piecewise Liapunov boundary.*

1. *In order that a function $f \in H_\mu(\Gamma, \mathbb{R}^4)$ be a boundary value of a function $\tilde{f}$ which satisfies a Moisil – Theodoresco system of partial differential equations in Ω^+ and is continuous in $\overline{\Omega^+}$, it is necessary and sufficient that*

$$ f(t) = {}^{V_{st}}\breve{S}_\Gamma[f](t) \quad \forall t \in \Gamma. $$

2. *In order that a function $f \in H_\mu(\Gamma, \mathbb{R}^4)$ be a boundary value of a function $\tilde{f}$ which satisfies a Moisil – Theodoresco system of partial differential equations in Ω^- and is continuous in $\overline{\Omega^-}$ and vanishes at infinity, it is necessary and sufficient that*

$$ f(t) = -{}^{V_{st}}\breve{S}_\Gamma[f](t) \quad \forall t \in \Gamma. $$

2.7. **Theorem** (on the square of the operators ${}^{V_{st}}S_\Gamma$ and ${}^{V_{st}}\breve{S}_\Gamma$). *If Γ is a piecewise Liapunov surface, then we have the following formulas for $f \in H_\mu(\Gamma, \mathbb{R}^4)$, $0 < \mu < 1$:*

$$ {}^{V_{st}}S^2_\Gamma[f](t) = a_1(t)f(t) + a_2(t){}^{V_{st}}S_\Gamma[f](t) + {}^{V_{st}}S_\Gamma[a_3 f](t), \tag{3} $$

$$ {}^{V_{st}}\breve{S}^2_\Gamma[f](t) = f(t) \tag{4} $$

for all $t \in \Gamma$, i.e., the modified singular Cauchy – Moisil – Theodoresco integral operator ${}^{V_{st}}\breve{S}_\Gamma$ is an involution on $H_\mu(\Gamma, \mathbb{R}^4)$, $0 < \mu < 1$,

$$ {}^{V_{st}}\breve{S}^2_\Gamma = I, $$

where

$$ a_1(t) := \frac{\gamma(t)}{\pi} - \frac{\gamma^2(t)}{4\pi^2}, \qquad a_2(t) := \frac{\gamma(t)}{2\pi} - 2, \qquad a_3(t) := \frac{\gamma(t)}{2\pi}. $$

The proofs of these theorems can be found in Section 4.

3. Hyperholomorphic function theory: general information. In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [7 – 9].

3.1. We consider the skew-field of real quaternions $\mathbb{H}$:

$$ \mathbb{H} := \{x = x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3;\ (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4\}, $$

where i_0 is the unit, and i_1, i_2, i_3 are the quaternionic imaginary units with the properties:

$$i_0^2 = i_0 = -i_k^2, \quad i_0 i_k = i_k i_0 = i_k, \quad k \in \mathbb{N}_3;$$

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2.$$

Let $x = \sum_{k=0}^{3} x_k i_k \in \mathbb{H}$. Then

$$x_0 =: \mathrm{Sc}(x) \quad \text{and} \quad \vec{x} := \sum_{k=1}^{3} x_k \cdot i_k =: \mathrm{Vect}(x)$$

are called, respectively, the scalar and the vector part of a quaternion. We can write

$$x = x_0 + \vec{x}.$$

In vector terms, the multiplication of two arbitrary real quaternions x, y can be rewritten as follows:

$$x \cdot y = \big(x_0 + \vec{x}\big) \cdot \big(y_0 + \vec{y}\big) = x_0 \cdot y_0 - \langle \vec{x}, \vec{y} \rangle + x_0 \vec{y} + y_0 \vec{x} + [\vec{x}, \vec{y}],$$

where $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ denote the usual scalar and vector products of three-dimensional vectors. In particular, if $x_0 = y_0 = 0$, then we have

$$x \cdot y = -\langle \vec{x}, \vec{y} \rangle + [\vec{x}, \vec{y}].$$

The quaternionic conjugation of $x = x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is given by

$$\bar{x} := x_0 i_0 - x_1 i_1 - x_2 i_2 - x_3 i_3.$$

We use the Euclidean norm $|x|$ in $\mathbb{H}$, defined by

$$|x| := \sqrt{x\bar{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

An important property is that

$$|xy| = |x| \cdot |y|.$$

3.2. Let the matrix

$$B_l(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \tag{5}$$

be the left regular representation of real quaternion b, and, respectively, let the matrix

$$B_r(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{pmatrix}$$

be the right regular representation of real quaternion b. Then $\mathbb{H}$ can be identified as a skew-field with $\mathcal{B}_l := \{B_l(b) \mid b \in \mathbb{H}\}$. The same holds for $\mathcal{B}_r := \{B_r(b) \mid b \in \in \mathbb{H}\}$ and $\mathbb{H}$. Moreover, the left-multiplication by the real quaternion b corresponds to the multiplication by the matrix $B_l(b)$, i.e.,

$$b \cdot x \leftrightarrow B_l(b) \cdot (x_0, x_1, x_2, x_3)^T,$$

where $(x_0, x_1, x_2, x_3)^T := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$

3.3. We shall consider functions ranged in $\mathbb{H}$ and defined in a domain $\Omega \subset \mathbb{R}^3$. Notations $C^p(\Omega, \mathbb{H})$, $p \in \mathbb{N} \cup \{0\}$, have the usual componentwise meaning. A function f is called left-hyperholomorphic if

$$D[f] := \sum_{k=1}^{3} i_k \frac{\partial f}{\partial x_k} =: \sum_{k=1}^{3} i_k \partial_k[f] = 0$$

holds in Ω. Let $\theta = -\frac{1}{4\pi}\frac{1}{|x|}$ be the fundamental solution of the Laplace operator. Then the fundamental solution to the operator D, $\mathcal{K}$, is given by the formula (see [9])

$$\mathcal{K}(x) := -D[\theta](x) = \frac{1}{4\pi}\sum_{k=1}^{3} \bar{i}_k \frac{x_k}{|x|^3} = \frac{1}{4\pi}\frac{1}{|x|^3} B_l(V_{st}^T \cdot x), \tag{6}$$

where $st := \{i_1, i_2, i_3\}$. Set

$$\sigma_x := i_1 dx_{[1]} - i_2 dx_{[2]} + i_3 dx_{[3]},$$

where $dx_{[k]}$ denotes, as usual, the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. Note that if Γ is a piecewise smooth surface in $\mathbb{R}^3$ and if $\vec{n}(\tau) = (n_1(\tau), n_2(\tau), n_3(\tau))$ is the outward unit normal to surface Γ at τ, then

$$\sigma \mid_\Gamma = \vec{n}(\tau) ds_\tau =: \sum_{k=1}^{3} n_k(\tau) i_k ds_\tau,$$

where ds is the differential form of the two-dimensional surface Γ in $\mathbb{R}^3$. Let $\Omega = \Omega^+$ be a domain in $\mathbb{R}^3$ with the boundary Γ which is assumed to be a piecewise Liapunov surface; denote $\Omega^- := \mathbb{R}^3 \setminus (\Omega^+ \cup \Gamma)$. If f is a Hölder function, then its left-hyperholomorphic Cauchy-type integral is defined as follows:

$$K_\Gamma[f](x) := \int_\Gamma \mathcal{K}(\tau - x) \cdot \sigma_\tau \cdot f(\tau), \quad x \in \Omega^\pm.$$

For more information about hyperholomorphic functions, we refer to [7–10] (see also [11]).

4. Proofs of the theorems from Section 2. In this section, we prove all theorems from Section 2 using the relations between the Moisil–Theodoresco system of partial differential equations theory and the theory of hyperholomorphic functions.

4.1. We start this section with a brief description of the relations between the Moisil–Theodoresco system of partial differential equations theory and the theory of hyperholomorphic functions.

On the set $C^1(\Omega, \mathbb{H})$, the well-known Moisil–Theodoresco operator is defined by the formula

$$D := \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}.$$

Using matrix (5), the equality $D[f] = 0$ (the Moisil–Theodoresco system) can be also rewritten as

$$B_l\left(\sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}\right) f^T = 0$$

with

$$B_l\left(\sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}\right) = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}.$$

Thus,

$$D[f] = 0 \Longleftrightarrow \begin{pmatrix} 0 & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0,$$

i.e., one can identify the class of the solutions of the elliptic system of the partial differential equations with the constant coefficients by the set of hyperholomorphic functions. By the equality (6) for $\mathbb{R}^4$-valued function f, we have

$$K_\Gamma[f](x) := \frac{1}{4\pi}\int\limits_\Gamma \frac{1}{|\tau - x|^3} B_l(V_{st}^T \cdot (\tau - x)) B_l(V_{st}^T \cdot d\hat{\tau}) f(\tau), \quad x \notin \Gamma.$$

So, the integral $K_\Gamma[f](x)$ coincides with ${}^{V_{st}}K_\Gamma[f](x)$. In the same way,

$$S_\Gamma[f](t) := 2K_\Gamma[f](t) =$$

$$= \frac{1}{2\pi}\int\limits_\Gamma \frac{1}{|\tau - t|^3} B_l(V_{st}^T \cdot (\tau - t)) B_l(V_{st}^T \cdot d\hat{\tau}) f(\tau) \quad \forall t \in \Gamma,$$

so, the integral S_Γ for $f \in H_\mu(\Gamma, \mathbb{R}^4)$ coincides with ${}^{V_{st}}S_\Gamma[f]$.

***4.2.* Proof of Theorem 2.3.** Let $f \in H_\mu(\Gamma, \mathbb{R}^4)$. Consider ${}^{V_{st}}K_\Gamma[f](x)$. It was proved that

$${}^{V_{st}}K_\Gamma[f](x) = K_\Gamma[f](x).$$

By [4] (Theorem 2.1 for $\alpha = 0$), see also [7, 8], there exists $K_\Gamma[f]^\pm(t)$ and

$$K_\Gamma[f]^+(t) = \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + K_\Gamma[f](t) =: \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + \frac{1}{2} S_\Gamma[f](t),$$

$$K_\Gamma[f]^-(t) = -\frac{\gamma(t)}{4\pi} f(t) + K_\Gamma[f](t) =: -\frac{\gamma(t)}{4\pi} f(t) + \frac{1}{2} S_\Gamma[f](t).$$

Hence, there exists ${}^{V_{st}}K_\Gamma[f]^\pm(t)$ and, after not complicated computation, we obtain the required result. Set

$$\breve{S}_\Gamma[f](t) := \frac{2\pi - \gamma(t)}{2\pi} f(t) + S_\Gamma[f](t)$$

for any $t \in \Gamma$.

***4.3. Proof of Theorem* 2.5.** Let $f \in H_\mu(\Gamma, \mathbb{R}^4)$, consider ${}^{V_{st}}K_\Gamma[f](x)$. By Theorem 2.3, there exists ${}^{V_{st}}K_\Gamma[f]^\pm(t)$ and

$$ {}^{V_{st}}K_\Gamma[f]^+(t) = \frac{1}{2}[f(t) + {}^{V_{st}}\breve{S}_\Gamma[f](t)], $$

$$ {}^{V_{st}}K_\Gamma[f]^-(t) = \frac{1}{2}[-f(t) + {}^{V_{st}}\breve{S}_\Gamma[f](t)], $$

where ${}^{V_{st}}\breve{S}_\Gamma$ was defined in Subsection 2.4. By Subsection 4.1, $f \in H_\mu(\Gamma, \mathbb{R}^4)$, hence, on Γ, ${}^{V_{st}}\breve{S}_\Gamma[f] = \breve{S}_\Gamma[f]$. In [4] (Subsection 2.2 for $\alpha = 0$), it was proved that $\breve{S}_\Gamma$ satisfy the Hölder condition. So, recalling the relationship between the operators $\breve{S}_\Gamma$ and ${}^{V_{st}}\breve{S}_\Gamma$, we have that ${}^{V_{st}}\breve{S}_\Gamma[f] \in H_\mu(\Gamma, \mathbb{R}^4)$.

***4.4. Proof of Theorem* 2.6.** This proof follows from [4] (Theorem 2.3 for $\alpha = 0$) taking into account the above relation between the class of solutions of the Moisil – Theodoresco system of partial differential equations and the set of hyperholomorphic functions.

***4.5. Proof of Theorem* 2.7.** Let $f \in H_\mu(\Gamma, \mathbb{R}^3)$. Consider ${}^{V_{st}}K_\Gamma[f]$. In Subsection 4.1, it was proved that

$$ f \in H_\mu(\Gamma, \mathbb{R}^4) \Longrightarrow f \in H_\mu(\Gamma, \mathbb{R}^4). $$

So, we obtain (3) after taking into account [4] (Theorem 2.4 for $\alpha = 0$), see also [8], combined with a straightforward calculation. Using the definition of the modified singular operator ${}^{V_{st}}S_\Gamma$, we obtain (4).

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