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GENERALIZED DE RHAM – HODGE COMPLEXES, THE RELATED CHARACTERISTIC CHERN CLASSES AND SOME APPLICATIONS TO INTEGRABLE MULTIDIMENSIONAL DIFFERENTIAL SYSTEMS ON RIEMANNIAN MANIFOLDS

УЗАГАЛЬНЕНІ КОМПЛЕКСИ ДЕ РАМА – ХОДЖА, СПОРІДНЕНІ ХАРАКТЕРИСТИЧНІ КЛАСИ ЧЕРНА ТА ДЕЯКІ ЗАСТОСУВАННЯ ДО ІНТЕГРОВНИХ БАГАТОВИМІРНИХ ДИФЕРЕНЦІАЛЬНИХ СИСТЕМ НА РІМАНОВИХ МНОГОВИДАХ

The differential-geometric aspects of generalized de Rham – Hodge complexes naturally related with integrable multidimensional differential systems of M. Gromov type, as well as the geometric structure of Chern characteristic classes are studied. Special differential invariants of the Chern type are constructed, their importance for the integrability of multidimensional nonlinear differential systems on Riemannian manifolds is discussed. An example of the three-dimensional Davey – Stewartson type nonlinear integrable differential system is considered, its Cartan type connection mapping and related Chern type differential invariants are analized.

Досліджено диференціально-геометричні аспекти узагальнених комплексів де Рама – Ходжа, що природним чином пов'язані з інтегровними багатовимірними диференціальними системами типу М. Громова, а також геометричну структуру характеристичних класів Черна. Побудовано спеціальні диференціальні інваріанти типу Черна та розглянуто їх важливість для інтегровності багатовимірних нелінійних диференціальних систем на ріманових многовидах. Розглянуто приклад тривимірної нелінійної інтегровної диференціальної системи типу Деві – Стюартсона і проаналізовано їх сполучне відображення та споріднені диференціальні інваріанти типу Черна.

1. Introduction: Cartan's connection and curvature. Consider a smooth finite-dimensional Riemannian manifold M and two linear bundles over it: the tangent bundle $T(M)$ and a bundle $E(M)$, endowed with some real scalar structure $\langle\cdot,\cdot\rangle_E$ on fibers E. Denote the related vector fields on M as $\mathcal{T}(M)$ and smooth sections of $E(M)$ as $\mathcal{E}(M)$. As usually [1 – 4], one can introduce on $E(M)$ a Cartan's connection Γ by means of a connection mapping

$$d_{\mathrm{A}}\colon \mathcal{E}(M) \to \mathcal{T}^*(M)\otimes\mathcal{E}(M), \tag{1.1}$$

which satisfies the following property:

$$d_{\mathrm{A}}(f\alpha+\beta) := df\otimes\alpha + f d_{\mathrm{A}}\alpha + d_{\mathrm{A}}\beta$$

for any smooth function $f\in\mathcal{D}(M)$ and $\alpha,\,\beta\in\mathcal{E}(M)$. Let $\Lambda(M) := \oplus_{p=0}^{\dim M}\Lambda^p(M)$ denote the usual [1, 2, 4 – 6] Grassmann algebra of differential forms on M. If to define the associated linear bundles $\Lambda^p(M,E) := \Lambda^p(M)\otimes\mathcal{E}(M)$ for $p=\overline{0,m}$, the connection mapping (1.1) can be naturally extended on $\Lambda^p(M,E)$ as

$$d_{\mathrm{A}}\colon \Lambda^p(M,E)\to\Lambda^{p+1}(M,E), \tag{1.2}$$

satisfying the related Leibnitz-rule:

$$d_{\mathrm{A}}(f^{(p)}\wedge\alpha^{(q)}) := df^{(p)}\wedge\alpha^{(q)} + (-1)^p f^{(p)}\wedge d_{\mathrm{A}}\alpha^{(q)}$$

for any $f^{(p)}\in\Lambda^p(M)$ and $\alpha^{(q)}\in\Lambda^q(M,E)$, $q,p=\overline{0,m}$.

The connection operation (1.2) possesses a very interesting and important property: its composition $d_A^2 := d_A d_A$ is linear over functions ring $D(M)$:

$$d_A^2\big(f\alpha^{(p)} + \beta^{(p)}\big) = f d_A^2 \alpha^{(p)} + d_A^2 \beta^{(p)},$$

where $f \in D(M)$ and $\alpha^{(p)}, \beta^{(p)} \in \Lambda^p(M, E)$, $p = \overline{0, m}$, are arbitrary. The resulting linear tensor mapping $\Omega^{(2)} := d_A^2 \colon E(M) \to \Lambda^2(M, E)$ is called the curvature tensor and is of great importance for geometrical analysis of integrable multidimensional differential systems of M. Gromov type [7], generated by means of some Cartan integrable [4, 8–11] ideals $I(\alpha) \subset \Lambda(M, \operatorname{End} E) := \Lambda(M) \otimes \operatorname{End} E(M)$ on Riemannian manifolds M. Moreover, one can construct the smooth integral submanifold imbedding mapping $i_\alpha \colon M_\alpha \to M$ for the ideal $I(\alpha) \subset \Lambda(M, \operatorname{End} E)$, satisfying the following determining condition: the curvature 2-form $\Omega^{(2)} \in \Lambda^2(M, E)$ reduced upon M_α vanishes, that is $i_\alpha^* \Omega^{(2)} = 0$. This implies also that the related reduced co-chain

$$E \to \Lambda^0(M_\alpha, E) \xrightarrow{d_\alpha} \Lambda^1(M_\alpha, E) \xrightarrow{d_\alpha} \ldots \xrightarrow{d_\alpha} \Lambda^{m_\alpha}(M_\alpha, E) \xrightarrow{d_\alpha} 0 \tag{1.3}$$

is a de Rham complex, that is $d_\alpha^2 = 0$, where, by definition, $d_\alpha := i_\alpha^* d_A$ and $m_\alpha :=$ $:= \dim M_\alpha$. Since the submanifold $M_\alpha \subset M$ also possesses the induced from M Riemannian structure $g_\alpha \colon T(M_\alpha) \times T(M_\alpha) \to \mathbb{R}$, we can construct from (1.3) a suitably generalized de Rham–Hodge complex of Hilbert spaces $\mathcal{H}_\Lambda^p(M_\alpha)$, $p = \overline{0, m_\alpha}$, whose properties, as it was shown before in [4, 12–14], make it possible to describe the so called Delsarte–Lions transmutation operators of Volterra type, serving for constructing integrable multidimensional differential systems on Riemannian manifolds and finding their exact of special type solutions.

On the other side, one can consider the following generalized chain of modules over M:

$$E \to \Lambda^0(M, E) \xrightarrow{d_A} \Lambda^1(M, E) \xrightarrow{d_A} \ldots \xrightarrow{d_A} \Lambda^m(M, E) \to 0, \tag{1.4}$$

which is not, evidently, a de Rham–Hodge complex, but determines such very important [1, 3, 15, 16] geometric objects as the Chern characteristic classes and characters. On these and other geometric aspects of the integrability problem of multidimensional differential systems on Riemannian manifolds we will stay in more detail below.

2. The characteristic Chern classes and related differential invariants. The connection mapping (1.2) on M one can equivalently define locally on an open neighborhood $U \subset M$ by means of the following expression:

$$d_A|_U = d + A^{(1)}, \tag{2.1}$$

where $A^{(1)} \in \Lambda(U, \operatorname{End} E)$ is some suitably determined $\operatorname{End} E(U)$-valued differential 1-form on $U \subset M$. In local coordinates of a point $u \in U$ we can write down

$$A^{(1)} := \sum_{i=1}^{m} A_i(u) du^i,$$

where $A_i(u) \in \operatorname{End} E(U)$, $i = \overline{1, m}$. Making use of the representation (2.1), one can obtain easily the following local expression for the curvature 2-form $\Omega^{(2)}|_U \in \Lambda^2(U) \otimes$ $\otimes \operatorname{End} E(U)$:

$$\Omega^{(2)}|_U = dA^{(1)} + A^{(1)} \wedge A^{(1)}. \tag{2.2}$$

The expression (2.2) is very convenient for defining on $U \subset M$ the related with the complex (1.4) local cohomology characteristic Chern classes

$$\mathrm{ch}_j(\mathrm{A})|_U := \mathrm{tr}\left(\Omega^{(2)}|_U\right)^j \in \Lambda^{2j}(U), \tag{2.3}$$

where $j \in \mathbb{Z}_+$. The expression (2.3), owing to the linear vector bundle $E(M)$ properties, can be invariantly extended upon the whole manifold M as correctly defined differential forms on M, thereby determining the characteristic Chern classes

$$\mathrm{ch}_j(\mathrm{A}) = \mathrm{tr}(\Omega^{(2)})^j \in \Lambda^{2j}(M) \tag{2.4}$$

for $j \in \mathbb{Z}_+$, defined already on M. The following lemmas are important for further applications.

Lemma 2.2. *All differential $2j$-forms $\mathrm{ch}_j(\mathrm{A}) \in \Lambda^{2j}(M)$, $j \in \mathbb{Z}_+$, are closed, that is*

$$d\,\mathrm{ch}_j(\mathrm{A}) = 0. \tag{2.5}$$

Proof. A proof is standard by means of the direct substitution of local expressions (2.2) into (2.4) and checking (2.5).

As a consequence of this lemma we really see that inclusions

$$\left[\,\mathrm{ch}_j(\mathrm{A})\right] \in H^{2j}(M, \mathbb{R})$$

hold on M for all $j \in \mathbb{Z}_+$.

Lemma 2.3. *The de Rham cohomology classes $[\mathrm{ch}_j(\mathrm{A})] \in H^{2j}(M, \mathbb{R})$, $j \in \mathbb{Z}_+$, do not depend on the choice of a connection mapping*

$$d_{\mathrm{A}} \colon \Lambda(M, E) \to \Lambda(M, E)$$

and on the choice of a Hermitian metrics on E.

Proof. The homotopy cylinder construction [3, 1, 17] if applied to two different connection mappings gives right away to the independence connection choice, proving the first half of the lemma statement. The same homotopy reasonings prove, respectively, the independence metrics choice, the second part of the lemma statement.

As a consequence of this lemma one can define for every linear Hermitian fiber bundle $E(M)$ over M the set of corresponding characteristic Chern classes

$$\mathrm{ch}_j(E, M) := \left[\,\mathrm{ch}_j(\mathrm{A})\right]$$

for $j \in \mathbb{Z}_+$ by means of which there is determined the Chern character $\mathrm{ch}(E, M)$ of this linear fiber bundle $E(M)$:

$$\mathrm{ch}(E, M) := \oplus_{j \in \mathbb{Z}_+} (j!)^{-1}\,\mathrm{ch}_j(E, M) = \left[\,\mathrm{tr}\exp\Omega^{(2)}\right].$$

The Chern character, as is well known [1, 3, 16], finds a great deal of applications to modern differential topology and mathematical physics problems.

Concerning applications to strongly integrable multidimensional differential systems on Riemannian manifolds, we will consider the Cartan geometric picture, developed before in [4, 8 – 11]. Within this picture a studied nonlinear multidimensional differential system $\hat{\alpha}$ is represented in the form of a Cartan integrable ideal $I(\alpha) \subset \Lambda(M, \mathrm{End}\ E)$

with coefficients from $\operatorname{End} E(M)$, where $E(M)$ is a specially chosen Hermitian linear fiber bundle over some suitably chosen finite-dimensional Riemannian manifold M. The corresponding integral submanifold $M_\alpha \subset M$ of the ideal $I(\alpha)$, in general, is equivalent to the set of independent variables of our multidimensional integrable nonlinear differential system.

Note also here, that we call our multidimensional differential system strongly integrable, if it allows a suitable connection Γ_λ parametrically dependent on $\lambda \in \mathbb{R}$, whose curvature 2-form $\Omega_\lambda^{(2)} \in \Lambda^2(M, \operatorname{End} E)$ is vanishing upon the integral submanifold $M_\alpha \subset M$ of the ideal $I(\alpha) \subset \Lambda(M, \operatorname{End} E)$. The latter condition is, evidently, equivalent to the following inclusion:

$$\Omega_\lambda^{(2)} \in I(\alpha) \tag{2.6}$$

for all allowed values of $\lambda \in \mathbb{R}$. If the connection Γ does not depend nontrivially on parameter $\lambda \in \mathbb{R}$, a multidimensional differential system is called integrable.

On the other side, the condition (2.6) serves [4, 9] for finding the corresponding connection mapping (1.2), if it a priori assumes to exist. The resulting search algorithm details depend [4, 9, 11] strongly on the related so called holonomy group properties of the connection Γ_λ, $\lambda \in \mathbb{R}$, on the naturally associated with $E(M)$ principal fiber bundle $P(M, G)$, where G is a so called structure Lie group of the connection Γ_λ, $\lambda \in \mathbb{R}$. Concerning the mentioned algorithm details one can consult further [4, 9, 11].

The condition (2.6) is very important regarding the result of Lemma 2.3. Really, as the exactness relationships (2.5) hold, we can obtain under imposed conditions $H^{2j}(M, \mathbb{R}) = 0$, $j \in \mathbb{Z}_+$, right away that

$$\operatorname{ch}_j(\mathrm{A}, \lambda) := d\chi_j(\mathrm{A}, \lambda)$$

for some suitably determined global differential $(2j-1)$-form $\chi_j(\mathrm{A}, \lambda) \in \Lambda^{2j-1}(M)$, $j \in \mathbb{Z}_+$, on the manifold M. Moreover, since, evidently, all degrees $(\Omega^{(2)})^j \in I(\alpha)$ too for $j \in \mathbb{Z}_+$, from the condition $i_\alpha^* I(\alpha) = 0$, where $i_\alpha \colon M_\alpha \to M$ is the integral submanifold imbedding mapping, one gets easily that $i_\alpha^* \operatorname{ch}_j(\mathrm{A}) = 0$, or equivalently

$$d\chi_j(A) = 0 \tag{2.7}$$

for all $j \in \mathbb{Z}_+$, giving rise to new differential Chern type invariants on M_α. The obtained result one can formulate as the following theorem.

Theorem 2.1. *If an integrable multidimensional nonlinear differential system $\hat{\alpha}$ is equivalent to the Cartan integrable ideal $I(\alpha) \subset \Lambda(M, \operatorname{End} E)$ on a Riemannian manifold M, satisfying the cohomology conditions $H^{2j}(M, \mathbb{R}) = 0$, $j \in \mathbb{Z}_+$, then it possesses a set of differential Chern type invariants* (2.7) *on the suitable integral submanifold $M_\alpha \subset M$ of the ideal $I(\alpha)$. If nontrivial, these differential invariants describe, in particular, a related moduli space of the linear fiber bundle $E(M)$.*

It is useful to mention here that most of differential invariants (2.7) reduce at higher indices $j \in \mathbb{Z}_+$ to identical zero. Really, all of differential invariants $\chi_j(A)$ for $j \geq [\dim M_\alpha/2] + 1$ are identically zero. In particular, for the case of multidimensional integrable nonlinear differential systems, for which $\dim M_\alpha = 2$ or 3 one gets easily, that only one differential invariant can exist if any. Moreover, if for such differential systems the corresponding structure Lie groups are special linear ones, for which the Lie algebras a traceless, one easily gets that, on the whole, no invariant exists on M_α.

This result is instructive enough for the mathematical theory of geometrically integrable multidimensional nonlinear differential systems, possessing suitable connection mappings (1.2) on $\Lambda(M,E)$ under additional cohomology conditions $H^{2j}(M,\mathbb{R})=0$, $j\in\mathbb{Z}_+$. Namely, for these connection mappings to exist on $\Lambda(M,E)$ in the higher-dimensional case $\dim M_\alpha\geq 4$, the nontrivial differential invariants can prove to exist. The latter gives rise, in particular, to some topological obstacles to be satisfied. Really, a nontrivial differential invariant entails right away nontrivial cohomology constraints $H^s(M,\mathbb{R})\neq 0$ for some $s\in\mathbb{Z}_+$, thereby contradicting with the above zero cohomology conditions at these values $s\in\mathbb{Z}_+$. Otherwise, if a priori $H^{2s}(M,\mathbb{R})\neq 0$ for some $s\in\mathbb{Z}_+$, the related characteristic Chern classes $\mathrm{ch}_s(E,M)$ for these $s\in\mathbb{Z}_+$ are, in general, strongly nontrivial, thereby defining differential $(2s-1)$-forms $\chi_s(A)\in\Lambda^{2s-1}_{\mathrm{loc}}(M)$ only locally, owing to the classical Poincaré lemma. These local differential forms, if reduced upon the integral submanifold $M_\alpha\subset M$, give rise to a set of differential multi-valued quasi-invariants $\chi_s(A)\in\Lambda^{2s-1}_{\mathrm{loc}}(M_\alpha)$, where $d\chi_s(A)=0$ and whose existence can mean, in particular, that our nonlinear differential system is in some sense ill-posed. Remark here that related differential system structures can be studied also by means of the corresponding exact co-chain de Rham – Hodge complexes (1.3). These aspects will be also discussed below.

3. De Rham – Hodge theory and Delsarte – Lions transmutation operators. Consider the cohomology complex (1.3) and define on spaces $\Lambda^p(M_\alpha,E)$, $p\in\mathbb{Z}_+$, where $M_\alpha\subset M$ is the compact integral submanifold of the Riemannian manifold M, the standard Hodge star $\star$-operation

$$\star:\Lambda^p(M_\alpha,E)\to\Lambda^{m_\alpha-p}(M_\alpha,E),$$

where $m_\alpha:=\dim M_\alpha$ and for any $\beta\in\Lambda^p(M_\alpha,E)$ the form $\star\beta\in\Lambda^{m_\alpha-p}(M_\alpha,E)$ is such that the following [1, 4, 5, 18] conditions hold:

i) $\langle\gamma,\star\beta\rangle_{(m_\alpha-p)}:=\langle\langle\gamma,\star\beta\rangle_E\rangle_{m_\alpha-p}=\langle\langle\gamma\wedge,\beta\rangle_E,d\mu_{g_\alpha}\rangle_r$ for any $\gamma\in\Lambda^{m_\alpha-p}(M_\alpha,E)$, where $d\mu_{g_\alpha}$ is the invariant measure on M_α, induced at the dual imbedding mapping $i^*_\alpha\colon\Lambda(M_\alpha)\to\Lambda(M_\alpha)$ from the corresponding invariant measure $d\mu_g$ on the Riemannian manifold M, endowed with the positive definite Riemannian metrics $g\colon T(M)\times T(M)\to\to\mathbb{R}$, the scalar product

$$\left\langle\beta^{(1)}_1\wedge\beta^{(1)}_2\wedge\ldots\wedge\beta^{(1)}_k,\gamma^{(1)}_1\wedge\gamma^{(1)}_2\wedge\ldots\wedge\gamma^{(1)}_k\right\rangle_k:=$$
$$:=\det\left\{\langle\beta^{(1)}_i,\gamma^{(1)}_j\rangle_1\colon i,j=\overline{1,k}\right\},$$

where $\left\langle\beta^{(1)}_i,\gamma^{(1)}_j\right\rangle_1:=\left\langle\hat g^{-1}_\alpha\beta^{(1)}_i,\hat g^{-1}_\alpha\gamma^{(1)}_j\right\rangle_{g_\alpha}$ for any $\beta^{(1)}_i,\gamma^{(1)}_j\in\Lambda^1(M_\alpha)$, $i,j=\overline{1,k}$, and $\hat g_\alpha\colon T(M_\alpha)\to T^*(M_\alpha)$ is the canonical isomorphism, generated by the corresponding metrics $\langle\cdot,\cdot\rangle_{g_\alpha}$ on $T(M_\alpha)$;

ii) $(m_\alpha-p)$-dimensional volume $|\star\beta|$ of the form $\star\beta\in\Lambda^{m_\alpha-p}(M_\alpha,E)$ equals p-dimensional volume $|\beta|$ of a form $\beta\in\Lambda^p(M_\alpha,E)$;

iii) m_α-dimensional measure $\langle\beta\wedge,\star\beta\rangle_E\geq 0$ at a fixed orientation on M_α.

Owing to the above conditions i) – iii) one can endow the spaces $\Lambda^p(M_\alpha,E)$, $p\in\mathbb{Z}_+$, with the natural scalar product

$$(\beta,\gamma):=\int\limits_{M_\alpha}\langle\beta\wedge,*\gamma\rangle_E=\int\limits_{M_\alpha}\langle\beta,\gamma\rangle_{(p)}d\mu_{g_\alpha}\tag{3.1}$$

for any $\beta, \gamma \in \Lambda^p(M_\alpha, E)$. Subject to the scalar product (3.1) we can naturally construct the corresponding Hilbert space

$$\mathcal{H}_\Lambda(M_\alpha) := \overset{m_\alpha}{\underset{k=0}{\oplus}} \mathcal{H}^k_\Lambda(M_\alpha)$$

well suitable for our further consideration. Notice also here that the Hodge star $\star$-operation satisfies the following easily checkable property: for any $\beta, \gamma \in \mathcal{H}^k_\Lambda(M_\alpha)$, $k = \overline{0, m_\alpha}$,

$$(\beta, \gamma)_{(k)} = (\star\beta, \star\gamma)_{(m_\alpha - k)}, \qquad (\beta, \gamma) = (\star\beta, \star\gamma),$$

that is the Hodge operation $\star\colon \mathcal{H}_\Lambda(M_\alpha) \to \mathcal{H}_\Lambda(M_\alpha)$ is isometry and its standard adjoint with respect to the scalar product (3.1) operation satisfies the condition $(\star)' = (\star)^{-1}$.

Denote by d'_α the formally adjoint expression to the Cartan type connection mapping $d_\alpha\colon \mathcal{H}_\Lambda(M_\alpha) \to \mathcal{H}_\Lambda(M_\alpha)$ in the Hilbert space $\mathcal{H}_\Lambda(M_\alpha)$. Here $d_\alpha := i^*_\alpha d_\mathrm{A}$, where $d_\mathrm{A}\colon \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ is a suitable Cartan connection mapping and $i_\alpha\colon M_\alpha \to M$ is the corresponding integral submanifold imbedding mapping, associated with a given multidimensional nonlinear integrable differential system on the Riemannian manifold M. Making use of these operations d'_α and d_α in $\mathcal{H}_\Lambda(M_\alpha)$, one can naturally define [1, 18, 19] a generalized Laplace – Hodge operator $\Delta_\alpha\colon \mathcal{H}_\Lambda(M_\alpha) \to \mathcal{H}_\Lambda(M_\alpha)$ as

$$\Delta_\alpha := d'_\alpha d_\alpha + d_\alpha d'_\alpha. \tag{3.2}$$

Take a form $\beta \in \mathcal{H}_\Lambda(M_\alpha)$ satisfying the equality

$$\Delta_\alpha \beta = 0.$$

Such a form is called *harmonic*. One can also verify that a harmonic form $\beta \in \mathcal{H}_\Lambda(M_\alpha)$ satisfies simultaneously the following two adjoint conditions:

$$d'_\alpha \beta = 0, \quad d_\alpha \beta = 0, \tag{3.3}$$

easily stemming from (3.2) and (3.3).

It is not hard to check that the following differential operation in $\mathcal{H}_\Lambda(M_\alpha)$:

$$d^*_\alpha := \star d'_\alpha (\star)^{-1} \tag{3.4}$$

defines also a usual [10, 15, 17, 20, 21] Cartan type connection mapping in $\mathcal{H}_\Lambda(M_\alpha)$. The corresponding dual to (3.1) co-chain

$$E \to \Lambda^0(M_\alpha, E) \overset{d^*_\alpha}{\to} \Lambda^1(M_\alpha, E) \overset{d^*_\alpha}{\to} \ldots \overset{d^*_\alpha}{\to} \Lambda^m(M_\alpha, E) \overset{d^*_\alpha}{\to} 0 \tag{3.5}$$

is, evidently, α de Rham complex too, as the property $d^*_\alpha d^*_\alpha = 0$ holds owing to the definition (3.4).

Denote further by $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$, $k = \overline{0, m_\alpha}$, the cohomology groups of d_α-closed and by $\mathcal{H}^k_{\Lambda(\alpha^*)}(M_\alpha)$, $k = \overline{0, m_\alpha}$, the cohomology groups of d^*_α-closed differential forms, respectively, and by $\mathcal{H}^k_{\Lambda(\alpha^*\alpha)}(M_\alpha)$, $k = \overline{0, m_\alpha}$, the abelian groups of harmonic differential forms from the sub-spaces $\mathcal{H}^k_\Lambda(M_\alpha, E)$, $k = \overline{0, m_\alpha}$. Before formulating next results, define the standard sub-spaces for harmonic forms $\mathcal{H}^k_{\Lambda(\alpha^*\alpha)}(M_\alpha)$ and cohomology groups $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$, $\mathcal{H}^k_{\Lambda(\alpha^*)}(M_\alpha)$ for $k = \overline{0, m_\alpha}$. Assume also that the Laplace – Hodge operator (3.2) is elliptic in $\mathcal{H}^0_\Lambda(M_\alpha)$. Now by reasonings similar to those in [1, 14, 17 – 19] one can formulate the following a little generalized de Rham – Hodge theorem.

Theorem 3.1. *The groups of harmonic forms* $\mathcal{H}^k_{\Lambda(\alpha^*\alpha)}(M_\alpha)$, $k = \overline{0, m_\alpha}$, *are, respectively, isomorphic to the cohomology groups* $(\mathcal{H}^k(M_\alpha, \mathbb{R}))^{|\Sigma|}$, $k = \overline{0, m_\alpha}$, *where* $\mathcal{H}^k(M_\alpha, \mathbb{R})$ *is the k-th cohomology group of the manifold* M_α *with real coefficients,* $\Sigma \subset \mathbb{R}^p$, $p \in \mathbb{Z}_+$, $|\Sigma| := \operatorname{card} \Sigma$, *is a set of suitable "spectral" parameters labeling the linear space of independent* d^*_α*-closed* 0*-forms from* $\mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)$ *and, moreover, the following direct sum decompositions*

$$\mathcal{H}^k_{\Lambda(\alpha^*\alpha)}(M_\alpha) \oplus \Delta\mathcal{H}^k_\Lambda(M_\alpha) =$$

$$= \mathcal{H}^k_\Lambda(M_\alpha) = \mathcal{H}^k_{\Lambda(\alpha^*\alpha)}(M_\alpha) \oplus d_\alpha \mathcal{H}^{k-1}_\Lambda(M_\alpha) \oplus d^{'}_\alpha \mathcal{H}^{k+1}_\Lambda(M_\alpha)$$

hold for any $k = \overline{0, m_\alpha}$.

Another variant of the statement similar to that above was formulated in [13, 14, 22, 23] and reads as the following generalized de Rham – Hodge theorem.

Theorem 3.2. *The generalized cohomology groups* $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha), k = \overline{0, m_\alpha}$, *are isomorphic, respectively, to the cohomology groups* $(H^k(M_\alpha, \mathbb{R}))^{|\Sigma|}$, $k = \overline{0, m_\alpha}$.

Proof. A proof of this theorem is based on some special sequence [14, 22] of differential Lagrange type identities.

Define the following closed subspace:

$$\mathcal{H}^*_0 := \left\{ \varphi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\alpha^*)}(M_\alpha) \colon d^*_\alpha \varphi^{(0)}(\lambda) = 0, \quad \lambda \in \Sigma \right\} \tag{3.6}$$

for some set $\Sigma \subset \mathbb{R}^p$, where $\mathcal{H}^0_{\Lambda(\alpha^*)}(M_\alpha)$ is, as above, a suitable zero-order cohomology group space from the co-chain given by (3.5). Thereby, the dimension $\dim \mathcal{H}^*_0 = \operatorname{card} \Sigma$ is assumed to be known.

The next lemma [14, 22 – 25] is fundamental for the proof of the above isomorphism Theorem 3.2.

Lemma 3.1. *There exists a set of differential* $(k+1)$*-forms* $Z^{(k+1)}\big[\varphi^{(0)}(\lambda), d_\alpha \psi^{(k)}\big] \in \Lambda^{k+1}(M_\alpha, \mathbb{R})$, $k = \overline{0, m_\alpha - 1}$, *and a set of k-forms* $Z^{(k)}\big[\varphi^{(0)}(\lambda), \psi^{(k)}\big] \in \Lambda^k(M_\alpha, \mathbb{R})$, $k = \overline{0, m_\alpha - 1}$, *parametrized by a set* $\Sigma \ni \lambda$ *and semilinear in* $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,}(M_\alpha)$, *such that*

$$Z^{(k+1)}\Big[\varphi^{(0)}(\lambda), d_\alpha \psi^{(k)}\Big] = dZ^{(k)}\Big[\varphi^{(0)}(\lambda), \psi^{(k)}\Big]$$

for all $k = \overline{0, m_\alpha - 1}$ *and* $\lambda \in \Sigma$.

Proof. A proof is based on the following generalized Lagrange type identity holding for any pair $\big(\varphi^{(0)}(\lambda), \psi^{(k)}\big) \in \mathcal{H}^*_0 \times \mathcal{H}^k_\Lambda(M_\alpha)$:

$$0 = \Big\langle d^*_\alpha \varphi^{(0)}(\lambda), \wedge(\psi^{(k)} \wedge \bar{\gamma}) \Big\rangle_E = \Big\langle d^*_\alpha \varphi^{(0)}(\lambda), \wedge * (*)^{-1}(\psi^{(k)} \wedge \bar{\gamma}) \Big\rangle_E =$$

$$= \Big\langle \star\, d'_\alpha (\star)^{-1} \varphi^{(0)}(\lambda), \wedge * (*)^{-1}\big(\psi^{(k)} \wedge \bar{\gamma}\big) \Big\rangle_E =$$

$$= \Big\langle d'_\alpha (\star)^{-1} \varphi^{(0)}(\lambda), \wedge * (*)^{-2}\big(\psi^{(k)} \wedge \bar{\gamma}\big) \Big\rangle_E =$$

$$= \Big\langle (\star)^{-1} \varphi^{(0)}(\lambda), \wedge(*)^{-1}(*)^2 d_\alpha\big(\psi^{(k)} \wedge \bar{\gamma}\big) \Big\rangle_E + Z^{(k+1)}\Big[\varphi^{(0)}(\lambda), d_\alpha \psi^{(k)}\Big] \wedge \bar{\gamma} =$$

$$= \Big\langle \varphi^{(0)}(\lambda), \wedge\big(d_\alpha \psi^{(k)} \wedge \bar{\gamma}\big) \Big\rangle_E + dZ^{(k)}\Big[\varphi^{(0)}(\lambda), \psi^{(k)}\Big] \wedge \bar{\gamma}, \tag{3.7}$$

where $Z^{(k+1)}\big[\varphi^{(0)}(\lambda), d_\alpha\psi^{(k)}\big] \in \Lambda^{k+1}(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, and $Z^{(k)}\big[\varphi^{(0)}(\lambda), \psi^{(k)}\big] \in \Lambda^k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, are some semilinear differential forms parametrized by the parameter $\lambda\in\Sigma$, and $\bar\gamma\in\Lambda^{m_\alpha-k-1}(M_\alpha,\mathbb{R})$ is an arbitrary constant $(m_\alpha-k-1)$-form. Thereby, the semilinear differential $(k+1)$-forms $Z^{(k+1)}\big[\varphi^{(0)}(\lambda), d_\alpha\psi^{(k)}\big] \in \Lambda^{k+1}(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, and k-forms $Z^{(k)}\big[\varphi^{(0)}(\lambda),\psi^{(k)}\big]\in\Lambda^k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, $\lambda\in\Sigma$, constructed above, exactly constitute those searched for in the lemma.

Based now on Lemma 3.1, one can construct the cohomology group isomorphism claimed in Theorem 3.2 formulated above. Namely, following [14, 22, 23, 26], let us take some singular simplicial [1, 2, 17, 19, 21] complex $K(M_\alpha)$ of the manifold M_α and introduce linear mappings $B_\lambda^{(k)}\colon \mathcal{H}^k_\Lambda(M_\alpha)\to C_k(M_\alpha,\mathbb{R}))$, $k=\overline{0,m_\alpha-1}$, $\lambda\in\Sigma$, where $C_k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, are as before free abelian groups over the field $\mathbb{R}$ generated, respectively, by all k-chains of simplexes $S^{(k)}\in C_k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, from the singular simplicial complex $K(M_\alpha)$, as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)}\in C_k(M_\alpha,\,\mathbb{R}))} S^{(k)} \int\limits_{S^{(k)}} Z^{(k)}\big[\varphi^{(0)}(\lambda),\psi^{(k)}\big] \tag{3.8}$$

with $\psi^{(k)}\in\mathcal{H}^k_\Lambda(M_\alpha)$, $k=\overline{0,m_\alpha-1}$. The following theorem [14, 22, 25] based on mappings (3.8) holds.

Theorem 3.3. *The set of operations* (3.8) *parametrized by* $\lambda\in\Sigma$ *realizes the cohomology groups isomorphism formulated in Theorem* 3.2.

Proof. A proof of this theorem one can get passing over in (3.8) to the corresponding cohomology $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$ and homology $\mathcal{H}_k(M_\alpha,\mathbb{R})$ groups of M_α for every $k=\overline{0,m_\alpha-1}$. If one to take an element $\psi^{(k)}:=\psi^{(k)}(\mu)\in\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$, $k=\overline{0,m_\alpha-1}$, solving the equation $d_\alpha\psi^{(k)}(\mu)=0$ with $\mu\in\Sigma_k$ being some set of the related "spectral" parameters marking elements of the subspace $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$, then one finds easily from (3.8) and the identity (3.7) that

$$dZ^{(k)}\big[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)\big]=0$$

for all pairs $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m_\alpha-1}$. This, in particular, means owing to the Poincaré lemma [1, 17, 20, 21] that there exist differential $(k-1)$-forms $\Omega^{(k-1)}\big[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)\big]\in\Lambda^{k-1}(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, such that

$$Z^{(k)}\big[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)\big]=d\Omega^{(k-1)}[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)]$$

for all pairs $(\varphi^{(0)}(\lambda),\psi^{(k)}(\mu))\in\mathcal{H}_0^*\times\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)$ parametrized by $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m_\alpha-1}$. As a result of passing on the right-hand side of (3.8) to the homology groups $\mathcal{H}_k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, one gets owing to the standard Stokes theorem [1, 17, 20, 21] that the mappings

$$B_\lambda^{(k)}\colon \mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)\rightleftarrows\mathcal{H}_k(M_\alpha,\mathbb{R})$$

are isomorphisms for every $\lambda\in\Sigma$ and $\lambda\in\Sigma$. Making further use of the Poincaré duality [1, 5, 6, 17, 21] between the homology groups $\mathcal{H}_k(M_\alpha,\mathbb{R})$, $k=\overline{0,m_\alpha-1}$, and the cohomology groups $\mathcal{H}^k(M_\alpha)$, $k=\overline{0,m_\alpha-1}$, respectively, one obtains finally the statement claimed in Theorem 3.2, that is $\mathcal{H}^k_{\Lambda(\alpha)}(M_\alpha)\simeq(\mathcal{H}^k(M_\alpha,\mathbb{R}))^{|\Sigma|}$.

The theorem is proved.

Assume now that the Riemannian compactified manifold $M = M_\alpha \times \mathbb{R}^s$, $\dim M = s + \dim M_\alpha \in \mathbb{Z}_+$, and $E := \mathbb{R}^N$, where $M_\alpha \simeq \overset{m_\alpha}{\underset{j=1}{\times}} M_{\alpha,j}$, $M_{\alpha,j} := [0, T_j) \subset \mathbb{R}_+$, $j = \overline{1, m_\alpha}$, and put

$$d_\alpha = \sum_{j=1}^{m_\alpha} dt_j \wedge \mathrm{A}_j(t|\partial), \qquad \mathrm{A}_j(t|\partial) := \frac{\partial}{\partial t_j} + A_j(t),$$

with $A_j(t)$, $j = \overline{1, m_\alpha}$, being matrices parametrically dependent on $t \in M_\alpha^{m_\alpha}$. It is assumed also that operators $\mathrm{A}_j \colon \mathcal{H}^0_\Lambda(M_\alpha) \to \mathcal{H}^0_\Lambda(M_\alpha)$, $j = \overline{1, m_\alpha}$, are commuting to each other.

Take now such a fixed pair $\big(\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dt\big) \in \mathcal{H}^*_0 \times \mathcal{H}^{m_\alpha}_{\Lambda(\alpha)}(M_\alpha)$, parametrized by elements $(\lambda, \mu) \in \Sigma \times \Sigma$, for which owing to both Theorem 3.3 and the Stokes theorem [1, 10, 17, 20] the following equality:

$$B^{(m_\alpha)}_\lambda\big(\psi^{(0)}(\mu)dt\big) = S^{(m_\alpha)}_{(t)} \int\limits_{\partial S^{(m_\alpha)}_{(t)}} \Omega^{(m_\alpha - 1)}\Big[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)d\tau\Big] \tag{3.9}$$

holds, where $S^{(m_\alpha)}_{(t)} \in C_{m_\alpha}(M_\alpha, \mathbb{R})$ is some fixed element parametrized by an arbitrarily chosen point $t \in S^{(s)}_{(t)} \subset M_\alpha$. Consider the next integral expressions

$$\Omega_{(t)}(\lambda, \mu) := \int\limits_{\partial S^{(m_\alpha)}_{(t)}} \Omega^{(m_\alpha - 1)}\Big[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)d\tau\Big],$$

$$\Omega_{(t_0)}(\lambda, \mu) := \int\limits_{\partial S^{(m_\alpha)}_{(t_0)}} \Omega^{(m_\alpha - 1)}\Big[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)d\tau\Big],$$

with a point $t_0 \in S^{(m_\alpha)}_{(t_0)} \subset M_\alpha$, being taken fixed, $\lambda, \mu \in \Sigma$, and interpret them as the corresponding kernels [27, 28] of the integral invertible operators of Hilbert – Schmidt type $\Omega_{(t)}, \Omega_{(t_0)} \colon L^{(\rho)}_2(\Sigma, \mathbb{R}) \to L^{(\rho)}_2(\Sigma, \mathbb{R})$, where ρ is some finite Borel measure on the parameter set Σ. It assumes also above that the boundaries $\partial S^{(m_\alpha)}_{(t)} := \sigma^{(m_\alpha - 1)}_{(t)}$ and $\partial S^{(m_\alpha)}_{(t_0)} := \sigma^{(m_\alpha - 1)}_{(t_0)}$ are taken homological to each other as $t \to t_0 \in M_\alpha$. Define now the expressions

$$\mathbf{\Omega}_\pm \colon \psi^{(0)}(\eta) \to \tilde{\psi}^{(0)}(\eta)$$

for $\psi^{(0)}(\eta) \in \mathcal{H}^0_\Lambda(M_\alpha)$, $\eta \in \Sigma$, and some $\tilde{\psi}^{(0)}(\eta)dt \in \mathcal{H}^0_\Lambda(M_\alpha)$, where, by definition, for any $\eta \in \Sigma$

$$\tilde{\psi}^{(0)}(\eta) := \psi^{(0)}(\eta)\Omega^{-1}_{(t)}\Omega_{(t_0)} =$$

$$= \int\limits_\Sigma d\rho(\mu) \int\limits_\Sigma d\rho(\xi) \psi^{(0)}(\mu)\Omega^{-1}_{(t)}(\mu, \xi)\Omega_{(t_0)}(\xi, \eta), \tag{3.10}$$

being motivated by the expression (3.9). Suppose now that the elements (3.10) are ones being related to some another Delsarte – Lions transformed cohomology group $\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)$, that is the following condition:

$$d_{\tilde{\alpha}}\tilde{\psi}^{(0)}(\eta) = 0 \tag{3.11}$$

for $\tilde{\psi}^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)$, $\eta \in \Sigma$, and some new connection mapping in $\mathcal{H}_\Lambda(M_\alpha)$

$$d_{\tilde{\alpha}} := \sum_{j=1}^{m_\alpha} dt_j \wedge \tilde{\mathrm{A}}_j(t|\partial),$$

where $\tilde{\mathrm{A}}_j(t;\partial) := \partial/\partial t_j + \tilde{A}_j(t)$, $j = \overline{1, m_\alpha}$, are parametrically dependent on $t \in M_\alpha$.

Put now

$$\tilde{\mathrm{A}}_j := \mathbf{\Omega}_\pm \mathrm{A}_j \mathbf{\Omega}_\pm^{-1} \tag{3.12}$$

for each $j = \overline{1, m_\alpha}$, where $\mathbf{\Omega}_\pm : \mathcal{H}^0_\Lambda(M_\alpha) \to \mathcal{H}^0_\Lambda(M_\alpha)$ are the corresponding Delsarte–Lions transmutation operators related with some elements $S_\pm\left(\sigma^{(m_\alpha-1)}_{(t)}, \sigma^{(m_\alpha-1)}_{(t_0)}\right) \in$ $\in C_{m_\alpha}(M_\alpha, \mathbb{R})$ related naturally with homological to each other boundaries $\partial S^{(m_\alpha)}_{(t)} =$ $= \sigma^{(m_\alpha-1)}_{(t)}$ and $\partial S^{(m_\alpha)}_{(t_0)} = \sigma^{(m_\alpha-1)}_{(t_0)}$. Since all of differential expressions $\mathrm{A}_j : \mathcal{H}^0_\Lambda(M_\alpha) \to$ $\to H^0_\Lambda(M_\alpha)$, $j = \overline{1, m_\alpha}$, were taken commuting, the same property also holds for the transformed operators (3.12), that is $[\tilde{\mathrm{A}}_j, \tilde{\mathrm{A}}_k] = 0$, $k, j = \overline{0, m_\alpha}$. The latter is, evidently, equivalent owing to (3.12) to the following general expression:

$$d_{\tilde{\alpha}} = \mathbf{\Omega}_\pm d_\alpha \mathbf{\Omega}_\pm^{-1}. \tag{3.13}$$

For the conditions (3.13) and (3.11) to be satisfied, let us consider the corresponding to (3.9) expressions

$$\tilde{B}^{(m_\alpha)}_\lambda(\tilde{\psi}^{(0)}(\eta)dt) = S^{(m_\alpha)}_{(t)}\tilde{\Omega}_{(t)}(\lambda, \eta),$$

related with the corresponding external differentiation (3.13), where $S^{(m_\alpha)}_{(t)} \in C_{m_\alpha}(M, \mathbb{R})$ and $(\lambda, \eta) \in \Sigma \times \Sigma$. Assume further that there are also defined mappings

$$\mathbf{\Omega}^{\circledast}_\pm : \varphi^{(0)}(\lambda) \to \tilde{\varphi}^{(0)}(\lambda)$$

with $\mathbf{\Omega}^{\circledast}_\pm : \mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha) \to \mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)$, being some operators associated (but not necessary adjoint!) with the corresponding Delsarte–Lions transmutation operators $\mathbf{\Omega}_\pm :$ $\mathcal{H}^0_\Lambda(M_\alpha) \to \mathcal{H}^0_\Lambda(M_\alpha)$ and satisfying the standard relationships $\tilde{\mathrm{A}}^*_j := \mathbf{\Omega}^{\circledast}_\pm \mathrm{A}^*_j \mathbf{\Omega}^{\circledast,-1}_\pm$, $j = \overline{1, m_\alpha}$. The proper Delsarte–Lions type operators $\mathbf{\Omega}_\pm : \mathcal{H}^0_\Lambda(M_\alpha) \to \mathcal{H}^0_\Lambda(M_\alpha)$ are related with two different realizations of the action (3.10) under the necessary conditions

$$d_{\tilde{\alpha}}\tilde{\psi}^{(0)}(\eta) = 0, \qquad d^*_{\tilde{\alpha}}\tilde{\varphi}^{(0)}(\lambda) = 0, \tag{3.14}$$

needed to be satisfied and meaning, evidently, that the embeddings $\tilde{\varphi}^{(0)}(\lambda) \in$ $\in \mathcal{H}^0_{\Lambda(\tilde{\alpha}^*)}(M_\alpha)$, $\lambda \in \Sigma$, and $\tilde{\psi}^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)$, $\eta \in \Sigma$, are satisfied. Now we need to formulate a lemma being important for the conditions (3.14) to hold.

Lemma 3.2. *The following invariance property*

$$Z^{(m_\alpha)} = \Omega_{(t_0)}\Omega^{-1}_{(t)} Z^{(m_\alpha)} \Omega^{-1}_{(t)} \Omega_{(t_0)} \tag{3.15}$$

holds for any t and $t_0 \in M_\alpha$.

As a result of (3.15) and the symmetry invariance between cohomology spaces $\mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)$ and $\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)$ one obtains the following pairs of related mappings:

$$\begin{aligned} \psi^{(0)} &= \tilde{\psi}^{(0)}\tilde{\Omega}_{(t)}^{-1}\tilde{\Omega}_{(t_0)}, \qquad \varphi^{(0)} = \tilde{\varphi}^{(0)}\tilde{\Omega}_{(t)}^{\circledast,-1}\tilde{\Omega}_{(t_0)}^{\circledast}, \\ \tilde{\psi}^{(0)} &= \psi^{(0)}\Omega_{(t)}^{-1}\Omega_{(t_0)}, \qquad \tilde{\varphi}^{(0)} = \varphi^{(0)}\Omega_{(t)}^{\circledast,-1}\Omega_{(t_0)}^{\circledast}, \end{aligned} \tag{3.16}$$

where the integral operator kernels from $L_2^{(\rho)}(\Sigma,\mathbb{R})\otimes L_2^{(\rho)}(\Sigma,\mathbb{R})$ are defined as

$$\tilde{\Omega}_{(t)}(\lambda,\mu) := \int\limits_{\sigma_{(t)}^{(m_\alpha)}} \tilde{\Omega}^{(m_\alpha-2)}\Big[\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)d\tau\Big],$$

$$\tilde{\Omega}_{(t)}^{\circledast}(\lambda,\mu) := \int\limits_{\sigma_{(t)}^{(m_\alpha)}} \bar{\tilde{\Omega}}^{(m_\alpha-2),\intercal}\Big[\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)d\tau\Big]$$

for all $(\lambda,\mu)\in\Sigma\times\Sigma$, giving rise to finding proper Delsarte – Lions transmutation operators ensuring the pure differential nature of the transformed expressions (3.12).

Note here also that owing to (3.15) and (3.16) the following operator property

$$\Omega_{(t_0)}\Omega_{(t)}^{-1}\Omega_{(t_0)} + \tilde{\Omega}_{(t_0)}\Omega_{(t)}^{-1}\Omega_{(t_0)} = 0 \tag{3.17}$$

holds for any t and $t_0\in M_\alpha$ meaning that $\tilde{\Omega}_{(t)} = -\Omega_{(t_0)}$.

One can now define similar to (3.6) the additional closed and dense in $\mathcal{H}_\Lambda^0(M_\alpha)$ three subspaces

$$\begin{aligned} \mathcal{H}_0 &:= \Big\{\psi^{(0)}(\mu)\in\mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)\colon d_\alpha\psi^{(0)}(\mu)=0, \quad \mu\in\Sigma\Big\}, \\ \tilde{\mathcal{H}}_0 &:= \Big\{\tilde{\psi}^{(0)}(\mu)\in\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)\colon d_{\tilde{\alpha}}\tilde{\psi}^{(0)}(\mu)=0, \quad \mu\in\Sigma\Big\}, \\ \tilde{\mathcal{H}}_0^* &:= \Big\{\tilde{\varphi}^{(0)}(\eta)\in\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)\colon d^*_{\tilde{\alpha}}\tilde{\varphi}^{(0)}(\eta)=0, \quad \tilde{\varphi}^{(0)}(\eta)|_{\tilde{\Gamma}}=0,\ \eta\in\Sigma\Big\}, \end{aligned} \tag{3.18}$$

and construct the actions

$$\mathbf{\Omega}_\pm : \psi^{(0)}\to\tilde{\psi}^{(0)} := \psi^{(0)}\Omega_{(t)}^{-1}\Omega_{(t_0)}, \qquad \mathbf{\Omega}^{\circledast}_\pm\colon \varphi^{(0)}\to\tilde{\varphi}^{(0)} := \varphi^{(0)}\Omega_{(t)}^{\circledast,-1}\Omega_{(t_0)}^{\circledast} \tag{3.19}$$

on arbitrary but fixed pairs of elements $\big(\varphi^{(0)}(\lambda),\psi^{(0}(\mu)\big)\in\mathcal{H}_0^*\times\mathcal{H}_0$, parametrized by the set Σ, where by definition, one needs that all obtained pairs $\big(\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)\big)$, $\lambda,\mu\in\Sigma$, belong to $\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)\times\mathcal{H}^0_{\Lambda(\tilde{\alpha})}(M_\alpha)$. Note also that related operator property (3.17) can be compactly written down as follows:

$$\tilde{\Omega}_{(t)} = \tilde{\Omega}_{(t_0)}\Omega_{(t)}^{-1}\Omega_{(t_0)} = -\Omega_{(t_0)}\Omega_{(t)}^{-1}\Omega_{(t_0)}.$$

Construct now from the expressions (3.19) the following operator kernels from the Hilbert space $L_2^{(\rho)}(\Sigma,\mathbb{R})\otimes L_2^{(\rho)}(\Sigma,\mathbb{R})$:

$$\Omega_{(t)}(\lambda,\mu)-\Omega_{(t_0)}(\lambda,\mu) = \int\limits_{\partial S_{(t)}^{(m_\alpha)}} \Omega^{(m_\alpha-1)}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] =$$

$$:= -\int\limits_{\partial S_{(t_0)}^{(m_\alpha)}} \Omega^{(m_\alpha-1)}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] =$$

$$= \int\limits_{S^{(m_\alpha)}_{(\pm)}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)})} d\Omega^{(m_\alpha-1)}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] =$$

$$= \int\limits_{S^{(m_\alpha)}_{(\pm)}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)})} Z^{(m_\alpha)}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big],$$

and, similarly,

$$\Omega^{\circledast}_{(t)}(\lambda,\mu) - \Omega^{\circledast}_{(t_0)}(\lambda,\mu) = \int\limits_{\partial S^{(m_\alpha)}_{(t)}} \bar\Omega^{(m_\alpha-1),\mathrm{T}}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] -$$

$$- \int\limits_{\partial S^{(m_\alpha)}_{(t_0)}} \bar\Omega^{(m_\alpha-1),\mathrm{T}}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] =$$

$$= \int\limits_{S^{(m_\alpha)}_{\pm}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)})} d\bar\Omega^{(m_\alpha-1),\mathrm{T}}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big] =$$

$$= \int\limits_{S^{(m_\alpha)}_{\pm}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)})} \bar Z^{(m_\alpha-1),\mathrm{T}}\Big[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)d\tau\Big], \tag{3.20}$$

where $\lambda,\mu\in\Sigma$, and by definition, m_α-dimensional surfaces $S^{(m_\alpha)}_{+}\Big(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)}\Big)$ and $S^{(m_\alpha)}_{-}\Big(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)}\Big)\subset C_{m_\alpha-1}(M_\alpha)$ are spanned smoothly without self-intersection between two homological cycles $\sigma^{(m_\alpha-1)}_{(t)}=\partial S^{(m_\alpha)}_{(t)}$ and $\sigma^{(m_\alpha-1)}_{(t_0)}:=$ $:=\partial S^{(m_\alpha)}_{(t_0)}\in C_{m_\alpha-1}(M_\alpha,\mathbb{R})$ in such a way that the boundary $\partial(S^{(m_\alpha)}_{+}\Big(\sigma^{(m_\alpha-1)}_{(t)},$ $\sigma^{(m_\alpha-1)}_{(t_0)}\Big)\cup S^{(m_\alpha)}_{-}\Big(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)}\Big)\Big)=\varnothing$. Since the integral operator expressions $\Omega_{(t_0)},\Omega^{\circledast}_{(t_0)}\colon L^{(\rho)}_2(\Sigma,\mathbb{R})\to L^{(\rho)}_2(\Sigma,\mathbb{R})$ are at a fixed point $t_0\in M_\alpha$, evidently, constant and assumed to be invertible, for extending the actions given (3.19) on the whole Hilbert space $\mathcal{H}^0_\Lambda(M_\alpha)\times\mathcal{H}^0_\Lambda(M_\alpha)$ one can apply to them the classical constants variation approach, making use of the expressions (3.20). As a result, we obtain easily the following Delsarte – Lions transmutation integral operator expressions

$$\begin{gathered}
\mathbf{\Omega}_{\pm} = \mathbf{1} - \int\limits_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde\psi(t;\xi)\Omega^{-1}_{(t_0)}(\xi,\eta)\times \\
\times \int\limits_{S^{(m_\alpha)}_{\pm}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t)})} Z^{(m_\alpha)}\Big[\varphi^{(0)}(\eta),\cdot\Big], \\
\mathbf{\Omega}^{\circledast}_{\pm} = \mathbf{1} - \int\limits_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde\varphi(t;\eta)\Omega^{\circledast,-1}_{(t_0)}(\xi,\eta)\times \\
\times \int\limits_{S^{(m_\alpha)}_{\pm}(\sigma^{(m_\alpha-1)}_{(t)},\sigma^{(m_\alpha-1)}_{(t_0)})} \bar Z^{(m_\alpha),\mathrm{T}}\Big[\cdot,\psi^{(0)}(\xi)d\tau\Big]
\end{gathered} \tag{3.21}$$

for fixed pairs $\left(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)\right) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $\left(\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)\right) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, being bounded invertible integral operators of Volterra type on the whole space $\mathcal{H} \times \mathcal{H}^*$. Applying the same arguments as in Section 1, one can show also that respectively transformed sets of operators $\tilde{\mathrm{A}}_j := \mathbf{\Omega}_\pm \mathrm{A}_j \mathbf{\Omega}_\pm^{-1}$, $j = \overline{1, m_\alpha}$, and $\tilde{\mathrm{A}}_k^* := \mathbf{\Omega}_\pm^{\circledast} \mathrm{A}_k^* \mathbf{\Omega}_\pm^{\circledast, -1}$, $k = \overline{1, m_\alpha}$, prove to be purely differential too. Thereby, one can formulate [4, 13, 14] the following final theorem.

Theorem 3.4. *The expressions* (3.21) *are bounded invertible Delsarte – Lions transmutation integral operators of Volterra type onto $\mathcal{H}_\Lambda^0(M_\alpha) \times \mathcal{H}_\Lambda^0(M_\alpha)$, transforming, respectively, given commuting sets of expressions A_j, $j = \overline{1, m_\alpha}$, and their formally adjoint ones A_k^*, $k = \overline{1, m_\alpha}$, into the pure differential sets of expressions $\tilde{\mathrm{A}}_j := \mathbf{\Omega}_\pm \mathrm{A}_j \mathbf{\Omega}_\pm^{-1}$, $j = \overline{1, m_\alpha}$, and $\tilde{\mathrm{A}}_k^* := \mathbf{\Omega}_\pm^{\circledast} \mathrm{A}_k^* \mathbf{\Omega}_\pm^{\circledast, -1}$, $k = \overline{1, m_\alpha}$. Moreover, the suitably constructed closed subspaces $\mathcal{H}_0 \subset \mathcal{H}_{\Lambda(\alpha)}^0(M_\alpha)$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_{\Lambda(\tilde{\alpha})}^0(M_\alpha)$ such that the operators $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\circledast} \in B(\mathcal{H}_\Lambda^0(M_\alpha))$ depend strongly on the topological structure of the generalized cohomology groups $\mathcal{H}_{\Lambda(\alpha)}^0(M_\alpha)$ and $\mathcal{H}_{\Lambda(\tilde{\alpha})}^0(M_\alpha)$, being parametrized by elements $S_\pm^{(m_\alpha)}\left(\sigma_{(t)}^{(m_\alpha - 1)}, \sigma_{(t_0)}^{(m_\alpha - 1)}\right) \in C_{m_\alpha}(M_\alpha, \mathbb{R})$.*

Some applications of the results obtained to multidimensional integrable nonlinear differential systems on Riemannian manifolds we discuss in the next section.

4. An example: three-dimensional Davey – Stewartson type integrable differential system. Consider the generalized de Rham – Hodge theory for a commuting set $\mathcal{A}$ of three differential expressions in a Hilbert space $\mathcal{H}_\Lambda^0(M_\alpha)$, for the special case when the Riemannian compactified manifold $M := M_\alpha^3 \times \bar{\mathbb{C}}^6$ and

$$\mathcal{A} := \left\{ \mathrm{A}_j := \frac{\partial}{\partial t_j} + A_j(t; \lambda | \partial) \colon t_j \in M_{\alpha, j} := [0, T_j) \subset \mathbb{R}_+, \quad j = \overline{1, 3} \right\},$$

where, by definition, the integral submanifold of a suitable multidimensional differential system $\hat{\alpha}$ is $M_\alpha^3 := \mathop{\times}\limits_{j=1}^{3} M_{\alpha, j}$ and

$$\mathrm{A}_1 := \frac{\partial}{\partial t_1} + \begin{pmatrix} -\lambda & -u & f \\ -\bar{u} & \lambda & -g \\ -f^* & -g^* & 0 \end{pmatrix}, \qquad \mathrm{A}_2 = \frac{\partial}{\partial t_2} + \begin{pmatrix} -\lambda & 0 & f \\ 0 & -\lambda & g \\ -f^* & g^* & 0 \end{pmatrix},$$

$$\mathrm{A}_3 = \frac{\partial}{\partial t_3} + i \begin{pmatrix} -(\lambda^2 + f f^*) & f g^* & \lambda f + \dfrac{\partial f}{\partial t_2} \\ -f^* g & -\lambda^2 + g g^* & \lambda g + \dfrac{\partial g}{\partial t_2} \\ -\lambda f^* + \dfrac{\partial f^*}{\partial t_2} & \lambda g^* - \dfrac{\partial g^*}{\partial t_2} & -g g^* + f f^* \end{pmatrix}$$

for some smooth functions $f, f^*, g, g^*, u, \bar{u} \colon M_\alpha \to \mathbb{R}$ and arbitrary "spectral" parameter $\lambda \in \mathbb{R}$. The scalar product in $\mathcal{H}_\Lambda^0(M_\alpha)$ is given as

$$(\varphi, \psi) := \int_{M_\alpha^3} dt \langle \varphi, \psi \rangle_{\mathbb{E}^3}$$

for any pair $(\varphi, \psi) \in \mathcal{H}_\Lambda^0(M_\alpha) \times \mathcal{H}_\Lambda^0(M_\alpha)$. Respectively, the Cartan connection mapping $d_{\mathrm{A}} \colon \Lambda(M, E) \to \Lambda(M, E)$ on M is given as

$$d_{\mathrm{A}} := \sum_{j=1}^{3} dt_j \wedge \mathrm{A}_j + du \wedge \frac{\partial}{\partial u} + d\bar{u} \wedge \mathrm{I}\frac{\partial}{\partial \bar{u}} + df \wedge \mathrm{I}\frac{\partial}{\partial f} +$$

$$+ df^* \wedge \mathrm{I}\frac{\partial}{\partial f^*} + dg \wedge \mathrm{I}\frac{\partial}{\partial g} + dg^* \wedge \mathrm{I}\frac{\partial}{\partial g^*}.$$

It is easy to check that for all $t \in M_\alpha^3 \subset M$ the zero curvature condition $d_\alpha^2 := i_\alpha^* d_{\mathrm{A}}^2 = 0$ holds, where $d_\alpha := \sum_{j=1}^{3} dt_j \wedge \mathrm{A}_j$, and commutators $[\mathrm{A}_j, \mathrm{A}_k] = 0$ for $j, k = \overline{1,3}$. The latter is equivalent to the following three-dimensional Davey – Stewartson type integrable differential system $\hat{\alpha}$ on the Riemannian manifold M:

$$\frac{du}{dt_3} = i\left(\frac{\partial^2 u}{\partial t_1 \partial t_2} + 2u(ff^* + gg^*)\right),$$

$$\frac{d\bar{u}}{dt_3} = -i\left(\frac{\partial^2 \bar{u}}{\partial t_1 \partial t_2} + 2\bar{u}(ff^* + gg^*)\right),$$

augmented with such a set of compatible differential relationships:

$$\frac{\partial g}{\partial t_2} = -\frac{\partial g}{\partial t_1} + \bar{u} f, \qquad \frac{\partial g^*}{\partial t_2} = -\frac{\partial g^*}{\partial t_1} + u f^*,$$

$$\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial t_1} - ug, \qquad \frac{\partial f^*}{\partial t_2} = \frac{\partial f^*}{\partial t_1} - \bar{u} g^*,$$

$$\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial t_1} - ug, \qquad \frac{\partial f^*}{\partial t_2} = \frac{\partial f^*}{\partial t_1} - \bar{u} g^*,$$

$$\frac{\partial u}{\partial t_2} = -2fg^*, \qquad \frac{\partial \bar{u}}{\partial t_2} = -2f^* g,$$

$$\frac{\partial (ff^*)}{\partial t_2} - \frac{\partial (ff^*)}{\partial t_1} = \frac{1}{2}\frac{\partial (u\bar{u})}{\partial t_2} = -\left(\frac{\partial (gg^*)}{\partial t_1} + \frac{\partial (gg^*)}{\partial t_2}\right),$$

$$\frac{df}{dt_3} = i\left(\frac{\partial^2 f}{\partial t_1^2} + (2ff^* - u\bar{u})f - \frac{\partial u}{\partial t_1} g\right),$$

$$\frac{df^*}{dt_3} = -i\left(\frac{\partial^2 f^*}{\partial t_1^2} + (2ff^* - u\bar{u})f^* - \frac{\partial \bar{u}}{\partial t_1} g^*\right),$$

$$\frac{dg}{dt_3} = i\left(\frac{\partial^2 g}{\partial t_1^2} - (2gg^* + u\bar{u})g - \frac{\partial \bar{u}}{\partial t_1} f\right),$$

$$\frac{dg^*}{dt_3} = -i\left(\frac{\partial^2 g^*}{\partial t_1^2} - (2gg^* + u\bar{u})g^* - \frac{\partial u}{\partial t_1} f^*\right).$$

In particular, this means, evidently, that the corresponding generalized co-chains of modules

$$E \to \Lambda^0(M_\alpha, E) \overset{d_\alpha}{\to} \Lambda^1(M_\alpha, E) \overset{d_\alpha}{\to} \ldots \overset{d_\alpha}{\to} \Lambda^{m_\alpha}(M_\alpha, E) \overset{d_\alpha}{\to} 0,$$

$$E \to \Lambda^0(M_\alpha, E) \overset{d_\alpha^*}{\to} \Lambda^1(M_\alpha, E) \overset{d_\alpha^*}{\to} \ldots \overset{d_\alpha^*}{\to} \Lambda^{m_\alpha}(M_\alpha, E) \overset{d_\alpha^*}{\to} 0$$

are de Rham complexes. It is easy to check that all Chern type characteristic invariants $\chi_j(\alpha)$, $j \in \mathbb{Z}_+$, on the integral submanifold $M_\alpha \subset M$ are trivial. Nonetheless, the Chern character $\mathrm{ch}(E, M)$ of the suitable connection mapping $d_{\mathrm{A}}\colon \Lambda(M,E) \to \Lambda(M,E)$ on the whole Riemannian manifold M is nontrivial, giving rise to some sets of relationships, describing moduli space [3, 16, 17] of the Hermitian linear fiber bundle $E(M)$ over the manifold M.

Based on relationships (3.6) and (3.18), proceed to constructing closed subspaces $\mathcal{H}_0^{\circledast}$ and $\mathcal{H}_0$, making possible to construct suitable Delsarte – Lions transmutation operators:

$$\begin{gathered}
\mathcal{H}_0 := \Big\{ \psi^{(0)}(\lambda,\eta) \in \mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)\colon \\
\frac{\partial \psi^{(0)}(\lambda,\eta)}{\partial t_j} = A_j(t)\psi^{(0)}(\lambda,\eta), \quad j = \overline{1,3}, \\
\psi^{(0)}(\lambda,\eta)\Big|_{t=t_0} = \psi_\lambda(\eta) \in E, \\
(\lambda,\eta) \in \Sigma \times \Sigma_\sigma \Big\}, \\
\mathcal{H}_0^{*} := \Big\{ \varphi^{(0)}(\lambda,\eta) \in \mathcal{H}^0_{\Lambda(\alpha)}(M_\alpha)\colon \\
-\frac{\partial \varphi^{(0)}(\lambda,\eta)}{\partial t_j} = A_j^{*}(t)\varphi^{(0)}(\lambda,\eta), \quad j = \overline{1,3}, \\
\varphi^{(0)}(\lambda,\eta)\Big|_{t=t_0} = \varphi_\lambda(\eta) \in E, \\
(\lambda,\eta) \in \Sigma \times \Sigma_\sigma \Big\}
\end{gathered} \tag{4.1}$$

for some “spectral” set $\Sigma_\sigma \in \mathbb{C}^{p-1}$. By means of subspaces (4.1) one can now proceed to construction of Delsarte – Lions transmutation operators $\mathbf{\Omega}_\pm\colon \mathcal{H}^0_\Lambda(M_\alpha) \leftrightarrows \mathcal{H}^0_\Lambda(M_\alpha)$ in the general form like (3.21) with kernels $\Omega_{(t_0)}(\lambda;\xi,\eta) \in L_2^{(\rho)}(\Sigma_\sigma,\mathbb{R}) \otimes L_2^{(\rho)}(\Sigma_\sigma,\mathbb{R})$ for every $\lambda \in \Sigma$, being defined as

$$\Omega_{(t_0)}(\lambda;\xi,\eta) := \int\limits_{\sigma^{(m_\alpha-1)}_{(t_0)}} \Omega^{(m_\alpha-1)}\Big[\varphi^{(0)}(\lambda;\xi), \psi^{(0)}(\lambda;\eta)d\tau\Big],$$

$$\Omega^{\circledast}_{(t_0)}(\lambda;\xi,\eta) := \int\limits_{\sigma^{(m_\alpha-1)}_{(t_0)}} \bar{\Omega}^{(m_\alpha-1),\mathrm{T}}\Big[\varphi^{(0)}(\lambda;\xi), \psi^{(0)}(\lambda;\eta)d\tau\Big]$$

for all $(\lambda;\xi,\eta) \in \Sigma\times\Sigma_\sigma^3$. As a result one gets for the corresponding product $\rho := \rho_\sigma \circ \rho_{\Sigma_\sigma^2}$ such integral expressions:

$$\mathbf{\Omega}_\pm = \mathbf{1} - \int\limits_{\Sigma} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \times$$

$$\times \int\limits_{S_{\pm}^{(m_\alpha)}(\sigma_{(t)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)})} d\tau \qquad \tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(t_0;t_{10})}(\lambda;\xi,\eta)\bar{\varphi}^{(0),\mathrm{T}}(\lambda;\eta)(\cdot),$$

$$\boldsymbol{\Omega}_{\pm}^{\circledast} = \mathbf{1} - \int\limits_{\Sigma} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta)\times \tag{4.2}$$

$$\times \int\limits_{S_{\pm}^{(m_\alpha)}(\sigma_{(t)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)})} d\tau \qquad \tilde{\varphi}_\lambda^{(0)}(\xi)\bar{\Omega}^{\mathrm{T},-1}_{(t_0)}(\lambda;\xi,\eta)\times\bar{\psi}^{(0),\mathrm{T}}(\lambda;\eta)(\cdot),$$

where $S_{+}^{(m_\alpha)}\left(\sigma_{(t_0)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)}\right)\in C_{m_\alpha}(M_\alpha,\mathbb{R})$ is some smooth m_α-dimensional surface spanned between two homological cycles $\sigma_{(t)}^{(m_\alpha-1)}$ and $\sigma_{(t_0)}^{(m_\alpha-1)}\in\mathcal{K}(M_\alpha)$ and $S_{-}^{(m_\alpha)}\left(\sigma_{(t)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)}\right)\in C_{m_\alpha}(M_\alpha,\mathbb{R})$ is its smooth counterpart such that the boundary $\partial\left(S_{+}^{(m_\alpha)}\left(\sigma_{(t)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)}\right)\cup S_{-}^{(m_\alpha)}\left(\sigma_{(t)}^{(m_\alpha-1)},\sigma_{(t_0)}^{(m_\alpha-1)}\right)\right)=\varnothing$. Concerning the related results obtained above one can construct from (4.2) the corresponding factorized Fredholm operators $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{\circledast}$: $\mathcal{H}_\Lambda^0(M_\alpha)\to\mathcal{H}_\Lambda^0(M_\alpha)$, as follows:

$$\boldsymbol{\Omega}:=\boldsymbol{\Omega}_{+}^{-1}\boldsymbol{\Omega}_{-},\ \boldsymbol{\Omega}^{\circledast}:=\boldsymbol{\Omega}_{+}^{\circledast-1}\boldsymbol{\Omega}_{-}^{\circledast}.$$

It is also important to notice here that kernels $\hat{K}_{\pm}(\boldsymbol{\Omega})$ and $\hat{K}_{\pm}(\boldsymbol{\Omega}^{\circledast})\in\mathcal{H}_\Lambda^0(M_\alpha)\otimes\otimes\mathcal{H}_\Lambda^0(M_\alpha)$ satisfy exactly the generalized [14, 27, 28] determining equations in the following tensor form

$$(\tilde{\mathrm{A}}_{j,\mathrm{ext}}\otimes\mathbf{1})\hat{K}_{\pm}(\boldsymbol{\Omega})=(\mathbf{1}\otimes\mathrm{A}^{*}_{j,\mathrm{ext}})\hat{K}_{\pm}(\boldsymbol{\Omega}),$$

$$(\tilde{\mathrm{A}}^{*}_{j,\mathrm{ext}}\otimes\mathbf{1})\hat{K}_{\pm}(\boldsymbol{\Omega}^{\circledast})=(\mathbf{1}\otimes\mathrm{A}_{j,\mathrm{ext}})\hat{K}_{\pm}(\boldsymbol{\Omega}^{\circledast}).$$

Since, evidently, $\operatorname{supp}\hat{K}_{+}(\boldsymbol{\Omega})\cap\operatorname{supp}\hat{K}_{-}(\boldsymbol{\Omega})=\varnothing$ and $\operatorname{supp}\hat{K}_{+}(\boldsymbol{\Omega}^{\circledast})\cap\cap\operatorname{supp}\hat{K}_{-}(\boldsymbol{\Omega}^{\circledast})=\varnothing$, one derives from results [14, 24, 25, 29] the corresponding Gelfand – Levitan – Marchenko equations

$$\hat{K}_{+}(\boldsymbol{\Omega})+\hat{\Phi}(\boldsymbol{\Omega})+\hat{K}_{+}(\boldsymbol{\Omega})\cdot\hat{\Phi}(\boldsymbol{\Omega})=\hat{K}_{-}(\boldsymbol{\Omega}),$$

$$\hat{K}_{+}(\boldsymbol{\Omega}^{\circledast})+\hat{\Phi}(\boldsymbol{\Omega}^{\circledast})+\hat{K}_{+}(\boldsymbol{\Omega}^{\circledast})\cdot\hat{\Phi}(\boldsymbol{\Omega}^{\circledast})=\hat{K}_{-}(\boldsymbol{\Omega}^{\circledast}),$$

where, by definition, $\boldsymbol{\Omega}:=\mathbf{1}+\hat{\Phi}(\boldsymbol{\Omega})$, $\boldsymbol{\Omega}^{\circledast}:=1+\hat{\Phi}(\boldsymbol{\Omega}^{\circledast})$, which can be solved [30, 31] in the space $\mathcal{B}_\infty^{\pm}(\mathcal{H}_\Lambda^0(M_\alpha))$ for kernels $\hat{K}_{\pm}(\boldsymbol{\Omega})$ and $\hat{K}_{\pm}(\boldsymbol{\Omega}^{\circledast})\in\mathcal{H}_\Lambda^0(M_\alpha)\otimes\mathcal{H}_\Lambda^0(M_\alpha)$ depending on $t\in M_\alpha^2$. Thereby, Delsarte – Lions transformed differential expressions $\mathrm{A}_j\colon\mathcal{H}_\Lambda^0(M_\alpha)\to\mathcal{H}_\Lambda^0(M_\alpha)$, $j=\overline{1,3}$, will be, evidently, commuting to each other too, satisfying the following differential relationships:

$$\tilde{\mathrm{A}}_j=\frac{\partial}{\partial t_j}+\boldsymbol{\Omega}_{\pm}A_j\boldsymbol{\Omega}_{\pm}^{-1}-\left(\frac{\partial\boldsymbol{\Omega}_{\pm}}{\partial t_j}\right)\boldsymbol{\Omega}_{\pm}^{-1}:=\frac{\partial}{\partial t_j}+\tilde{A}_j, \tag{4.3}$$

where expressions for $\tilde{A}_j\in\mathcal{L}(\mathcal{H}_\Lambda^0(M_\alpha))$, $j=\overline{1,3}$, prove to be purely matrices. The latter property makes it possible to construct nonlinear partial differential equations on coefficients of differential expressions (4.3) and solve them by means of the standard procedures either of inverse spectral [32 – 36] or the Darboux – Backlund [25, 37] transforms, producing a wide class of exact soliton like solutions. On these and related Chern type differential invariants problems we will stay in detail elsewhere.

5. Conclusion. The study done above presents some of recent results devoted to the development of a generalized de Rham – Hodge theory [1, 4, 14, 17, 19, 21, 22, 25] and related differential-geometric aspects of Chern characteristic classes, concerning special differential complexes with Cartan type connections, which give rise to effective analytical tools of studying multidimensional integrable nonlinear differential systems of M. Gromov type [7] on Riemannian manifolds. Some results on the structure of the Delsarte – Lions transmutation operators are adapted for constructing effective transformations of Cartan type connections for multidimensional integrable Davey – Stewartson type nonlinear differential system on a Riemannian manifold M, vanishing upon three-dimensional integral submanifold $M_\alpha \subset M$. The results obtained can be used for studying a wide class of exact special solutions to this differential systems, having applications [1, 3, 4, 16, 30, 32, 38] at solving some problems of modern differential topology and mathematical physics.

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