UDC 517.9

E. D. Belokolos (Inst. Magnetism Nat. Acad. Sci. Ukraine, Kyiv)

INTEGRABLE SUPERCONDUCTIVITY AND RICHARDSON EQUATIONS

ІНТЕГРОВНА НАДПРОВІДНІСТЬ І РІВНЯННЯ РІЧАРДСОНА

For the integrable generalized model of superconductivity a solution of the Richardson equations for a spectrum of model is studied. For the case of narrow band the solution is presented in terms of the generalized Laguerre or Jacobi polynomials. In asymptotic limit, when the Richardson equations are transformed to an integral singular equation, the properties of an integration contour are discussed and a spectral density is calculated. Conditions for appearance of gaps in the spectrum are considered.

Для інтегровної узагальненої моделі надпровідності досліджено розв'язання рівнянь Річардсона для спектра моделі. У випадку вузької зони розв'язок подано в термінах узагальнених поліномів Лагерра та Якобі. В асимптотичному випадку, коли рівняння Річардсона трансформуються в інтегральне сингулярне рівняння, з'ясовано властивості контура інтегрування та розраховано спектральну щільність. Розглянуто умови появи щілин у спектрі.

1. Introduction. In 1957 J. Bardeen, L. N. Cooper and J. R. Schrieffer [1] has introduced the BCS Hamiltonian to describe a superconductivity. Later N. N. Bogolubov et. al. [2] proved the integrability of the BCS Hamiltonian in the thermodynamic limit. The integrability of the BCS Hamiltonian for a finite number of particles was proved in 1963 by R. W. Richardson [3]. M. Gaudin developed in the 1976 the appropriate mathematical theory and proposed the integrable generalizations of the BCS Hamiltonian which present an interest for a number of physical problems [4 – 6].

Later the integrable generalizations of the BCS Hamiltonian have been studied L. Amico, J. Dukelsky, G. Sierra and many others (see, e.g., a review article [7]).

In this paper we study generalizations of the integrable BCS Hamiltonian which is possible to describe the old and new superconductivity on equal level. The first reason for this is a fact that both types of superconductivity are based on a concept of the Cooper pair of electrons bound by attractive force, i.e., they have same origin, and therefore in order to explain new superconductivity we have no necessity in new extravagant interactions, scenarios and so on, we need just to generalize the well known BCS Hamiltonian. The second reason is that the essence of the superconductivity phenomenon is the coherence property, if we use physical language, which is equivalent in our opinion to the integrability property of Hamiltonian, if we use mathematical language. We know that the BCS Hamiltonian is integrable.

These reasons lead us to an idea to consider the superconductivity phenomenon by means of a single integrable Hamiltonian which in one domain of parameters describe the old superconductivity and in other domain will describe possibly the new superconductivity.

In this paper we rewiew the integrable pairing Hamiltonians and discuss a solution of the Richardson equations. A structure of the paper is follows. In Section 2 we present the Gaudin construction of quantum integrable Hamiltonians and in Section 3 give a discussion of the Richardson equations. In Section 4 we show that under special conditions the solutions of the Richardson equations are zeros of the Laguerre or Jacobi polynomials. The asymptotic expression of the Richardson equations in a form of the singular integral equation is presented in Section 5. In Section 6 we remind some facts of a theory of singular integral equations and in Section 7 obtain a solution of the Richardson singular integral equation in a special case.

2. The integrable pairing Hamiltonian. ***2.1. Commuting quadratic operator forms.*** The generators S^α, $\alpha = 1, 2, 3$, of the algebra sl_2 are well known to satisfy the following relations:

$$[S^\alpha, S^\beta] = i\varepsilon_{\alpha\beta\gamma} S^\gamma,$$

$$[S^3, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = 2S^3, \qquad S^\pm = S^1 \pm iS^2,$$

$$[S^\alpha, \mathbf{S}^2] = 0, \qquad \mathbf{S}^2 = (S^1)^2 + (S^2)^2 + (S^3)^2.$$

Now let us take N copies of this algebra,

$$S_k^\alpha, \qquad \alpha \in \{1, 2, 3\}, \qquad k \in \{1, 2, \ldots, N\},$$

$$[S_j^3, S_k^\pm] = \pm \delta_{jk} S_k^\pm, \qquad [S_j^+, S_k^-] = 2\delta_{jk} S_k^3,$$

and construct the quadratic operator forms

$$H_j = \sum_{k=1, k \neq j}^{N} w_{jk}^\alpha S_j^\alpha S_k^\alpha, \qquad j \in \{1, 2, \ldots, N\}.$$

These operator forms commute with each other,

$$[H_j, H_k] = 0,$$

if coefficients of these forms satisfy the equations

$$w_{ij}^\alpha w_{jk}^\gamma + w_{ik}^\beta w_{ki}^\alpha + w_{ki}^\gamma w_{ij}^\beta = 0.$$

A solution of this equations looks as follows,

$$w_{ij}^1 = \frac{\theta_{11}'(0)}{\theta_{10}(0)} \frac{\theta_{10}(u_{ij})}{\theta_{11}(u_{ij})}, \qquad w_{ij}^2 = \frac{\theta_{11}'(0)}{\theta_{00}(0)} \frac{\theta_{00}(u_{ij})}{\theta_{11}(u_{ij})},$$

$$w_{ij}^3 = \frac{\theta_{11}'(0)}{\theta_{01}(0)} \frac{\theta_{01}(u_{ij})}{\theta_{11}(u_{ij})}, \qquad u_{ij} = u_i - u_j,$$

where

$$\theta_{ab}(u) = \theta_{ab}(u; \tau) = \sum_{n \in \mathbb{Z}} \exp\left[\pi i \tau \left(n + \frac{a}{2}\right)^2 + 2\pi i \left(n + \frac{a}{2}\right)\left(u + \frac{b}{2}\right)\right]$$

are theta-functions with characteristics and

$$\theta_{ab}'(0) = \left.\frac{d\theta_{ab}(u)}{du}\right|_{u=0}.$$

Further we impose one more additional commutativity condition,

$$[S^3, H_j] = 0, \qquad S^3 = \sum_{i=1}^{N} S_i^3,$$

which is equivalent to the relations

$$w_{ij}^1 = -w_{ji}^1 = w_{ij}^2 = -w_{ji}^2.$$

In this case we can transform the equations for quantities w_{ij}^α to that ones,

$$w_{ij} v_{jk} + w_{ji} v_{ik} = w_{ik} w_{jk},$$

where

$$w_{ij} := w_{ij}^1 = w_{ij}^2, \qquad v_{ij} := w_{jk}^3.$$

These equations have solution

$$v_{jk} = qB\coth q(u_j - u_k),$$

$$w_{jk} = \frac{qB}{\sinh q(u_j - u_k)},$$

where parameters B, u_j are real and the parameter q may be real or imaginary.

If the parameter q is real then the quantities v_{jk}, w_{jk} are hyperbolic functions, if the parameter q is imaginary then the quantities v_{jk}, w_{jk} are trigonometric functions. There are three different types of integrable models depending of values of the parameter q:

the hyperbolic model at $q = 1$,
the trigonometric model at $q = i$,
the rational model at $q = 0$.

We can write down also the following commuting quadratic operator forms,

$$R_j = S_j^3 - H_j, \qquad [R_j, R_k] = 0.$$

It is possible to consider the operators R_j, $j = 1, 2, \ldots, N$, as a complete set of integrals of motions of a Hamiltonian which is an arbitrary function of these operators.

2.2. The integrable pairing Hamiltonian. Let us construct a following integrable Hamiltonian:

$$H = \sum_j 2\varepsilon_j R_j + A\left(\sum_j R_j\right)^2 + \sum_j \beta_j \sum_{\alpha=1}^{3} (S_j^\alpha)^2,$$

where ε_j, β_j, $j = 1, 2, \ldots, N$, and A are some real constants.

If we express the integrals of motion R_j in terms of operators S^α we transform our Hamiltonian to the form

$$H = \sum_j 2\varepsilon_j S_j^3 - \sum_{j,k} g_{jk} S_j^+ S_k^- + \sum_{j,k} U_{jk} S_j^3 S_k^3,$$

with the following interaction functions:

$$g_{jk} = \frac{qB(\varepsilon_j - \varepsilon_k)}{\sinh q(u_j - u_k)}, \quad j \neq k,$$

$$U_{jk} = A - qB(\varepsilon_j - \varepsilon_k)\coth q(u_j - u_k), \quad j \neq k,$$

$$g_{jj} = -\beta_j, \qquad U_{jj} = A + \beta_j,$$

$$\beta_j = \frac{1}{2} qB \sum_{k \neq j} (\varepsilon_k - \varepsilon_j)\coth q(u_k - u_j) + C.$$

Here parameters A, B, C are arbitrary real constants.

We can express the operators S_k^α in terms of the annihilation and creation operators of electrons $c_{i\sigma}$, $c_{i\sigma}^+$,

$$S_j^- = c_{j\downarrow} c_{j\uparrow}, \qquad S_j^+ = (S_j^-)^\dagger = c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger,$$

$$S_j^3 = \frac{1}{2}(c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow} - 1).$$

Using these expressions we can present the Hamiltonian in a form of generalized BCS Hamiltonian

$$H = \sum_{i=1}^{N} \varepsilon_i n_{i\sigma} - \sum_{i,j=1}^{N} g_{ij} c_{i\uparrow}^{+} c_{i\downarrow}^{+} c_{i\downarrow} c_{i\uparrow} + \sum_{i,j=1}^{N} U_{ij} n_{i\sigma} n_{i\sigma'} .$$

2.3. The rational limit. At $q \to 0$ and $\varepsilon_j = u_j$, $j = 1, \ldots, N$, we come to the isotropic case $g_{ij} = g$, i.e., to the BCS Hamiltonian

$$H_{BCS} = \sum_{i,\sigma} \varepsilon_i c_{i\sigma}^{+} c_{i\sigma} - g \sum_{i,j} c_{i\uparrow}^{+} c_{i\downarrow}^{+} c_{j\downarrow} c_{j\uparrow} = \sum_{j=1}^{N} 2\varepsilon_j \left(S_j^3 + \frac{1}{2} \right) - g \sum_{j=1}^{N} \sum_{k=1}^{N} S_j^{+} S_k^{-} .$$

The BCS Hamiltonian has the commuting integrals of motion

$$R_i = S_i^3 - g \sum_{k=1, k \neq i}^{N} \frac{\mathbf{S}_i \cdot \mathbf{S}_k}{\varepsilon_i - \varepsilon_k},$$

$$[H_{BCS}, R_i] = 0, \qquad [R_j, R_i] = 0.$$

The number of pairs M and the Hamiltonian H_{BCS} are expressed in terms of these integrals,

$$M = \sum_{i=1}^{L} \left(R_i + \frac{1}{2} \right),$$

$$H_{BCS} = \sum_{i=1}^{N} 2\varepsilon_i \left(R_i + \frac{1}{2} \right) + g \left(\sum_{i=1}^{N} R_i \right)^2 - g \sum_{i=1}^{N} \mathbf{S}_i^2 .$$

2.4. Eigenvalues and eigenstates. It is easy to prove that the joint eigenfunction of operators H, R_j is of the form [4 – 6]

$$\Psi = \prod_{\alpha=1}^{M} \sum_{j=1}^{N} \frac{c_{j\uparrow}^{\dagger} c_{j\downarrow}^{\dagger}}{\sinh(\omega_\alpha - u_j)} | 0 \rangle,$$

where $| 0 \rangle = \left| \downarrow, \ldots, \downarrow \right\rangle$ is the vacuum and ω_α, $\alpha = 1, 2, \ldots, M$, satisfy to the generalized Richardson equations

$$2 \sum_{\beta=1, \beta \neq \alpha}^{M} q \coth q(\omega_\beta - \omega_\alpha) = \sum_{l=1}^{N} q \coth q(u_l - \omega_\alpha) - \frac{1}{B}.$$

The eigenvalues E of the Hamiltonian H are

$$E = \sum_{j=1}^{N} \varepsilon_j + \sum_{j,k=1}^{N} U_{jk} + 4ABM^2(1+B) - qB \sum_{j=1}^{N} \sum_{\alpha=1}^{M} \varepsilon_j \coth q(\omega_\alpha - u_j).$$

3. The Richardson equations. Three integrable models are described with the Richardson equations which we can present in the following unified form:

$$2 \sum_{\beta=1, \beta \neq \alpha}^{M} q \coth q(\omega_\beta - \omega_\alpha) = \sum_{l=1}^{N} q \coth q(u_l - \omega_\alpha) - \frac{1}{G}.$$

At the limit $q \to 0$ the Richardson equations are

$$2 \sum_{\beta=1, \beta \neq \alpha}^{M} \frac{1}{\omega_\beta - \omega_\alpha} = \sum_{l=1}^{N} \frac{1}{u_l - \omega_\alpha} - \frac{1}{G}.$$

In variables

$$\xi_\alpha = \exp(2q\omega_\alpha), \quad \zeta_l = \exp(2qu_l)$$

the Richardson equations in general case,

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth q(\omega_\beta - \omega_\alpha) = \sum_{l=1}^{N} q\coth q(u_l - \omega_\alpha) - \frac{1}{G},$$

attain the form

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{\xi_\beta - \xi_\alpha} = \sum_{l=1}^{N} \frac{1}{\zeta_l - \xi_\alpha} - \frac{L}{\xi_\alpha},$$

where

$$L = \frac{1}{2qG} - \frac{N}{2} + M - 1.$$

The Richardson equations have solutions ω_α, $\alpha = 1, \ldots, M$, which form a set symmetric with respect to the real axis.

4. Special exact solutions of the Richardson equations. We can solve the Richardson equations in a special case of narrow band when the parameters u_l, $l = 1, 2, \ldots, N$, coincide.

4.1. The rational case.

Theorem 4.1. *If in the Richardson equations for the rational case*

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{\omega_\beta - \omega_\alpha} = \sum_{l=1}^{N} \frac{1}{u_l - \omega_\alpha} - \frac{1}{G}$$

the following conditions are satisfied:

$$u_l = 0, \quad l = 1, 2, \ldots, N,$$

then the Richardson equations have solutions

$$\omega_\alpha = Gx_\alpha, \quad \alpha = 1, 2, \ldots, M,$$

where x_α *are zeros of the generalized Laguerre polynomial*

$$L_M^{-N-1}(x).$$

If $N = 0, 1, \ldots, M-1$, *then among* ω_α *there are* $M - N - 1$ *positive numbers and* $N + 1$ *numbers* $\omega_\alpha = 0$. *If* $N \geq M - 1$, *then all numbers* ω_α *are complex pairwise conjugated numbers but one negative number for the odd* M.

Proof. Let in the Richardson equations for the rational case

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{\omega_\alpha - \omega_\beta} + \sum_{l=1}^{N} \frac{1}{u_l - \omega_\alpha} - \frac{1}{G} = 0,$$

the following conditions are satisfied:

$$u_l = 0, \quad l = 1, 2, \ldots, N,$$

then these equations attain the form

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{\omega_\alpha - \omega_\beta} - \frac{N}{\omega_\alpha} - \frac{1}{G} = 0.$$

As a result of last equalities the polynomial

$$f(x) = \prod_{\beta=1}^{M} (x - \omega_\beta)$$

is solution of the differential equation

$$x\frac{d^2 f}{dx^2} - \left(\frac{x}{G} + N\right)\frac{df}{dx} + \frac{M}{G} f = 0.$$

The generalized Laguerre polynomial

$$y(x) = L_n^a(cx)$$

satisfies the differential equation

$$x\frac{d^2 y}{dx^2} - (cx - a - 1)\frac{dy}{dx} + cny = 0.$$

Comparing these differential equations we get

$$c = \frac{1}{G}, \quad a = -N - 1, \quad n = M,$$

and therefore

$$f(x) = L_M^{-N-1}\left(\frac{x}{G}\right).$$

Let $0 \leq N < M - 1$. Then for $N = 0, 1, \ldots, M - 1$, the generalized Laguerre polynomial has $M - N - 1$ positive zeros and one zero of the order $N + 1$ at $x = 0$.

Let $N \geq M - 1$. Then all zeros of the generalized Laguerre polynomial are complex pairwise conjugated numbers but one negative zero for the odd M.

The theorem is proved.

4.2. The general case.

Theorem 4.2. *If in the Richardson equations for general case*

$$2 \sum_{\beta=1, \beta\neq\alpha}^{M} q \coth q(\omega_\beta - \omega_\alpha) = \sum_{l=1}^{N} q \coth q(u_l - \omega_\alpha) - \frac{1}{G},$$

the following conditions are satisfied:

$$u_l = \frac{\ln 2}{2q}, \quad l = 1, 2, \ldots, N,$$

then the Richardson equations have solutions

$$\omega_\alpha = \frac{\ln(x_\alpha + 1)}{2q}, \quad \alpha = 1, 2, \ldots, M,$$

where x_α *are zeros of the generalized Jacobi polynomial*

$$P_M^{-(N+1), -(L+1)}(x),$$

and

$$L = \frac{1}{2qG} - \frac{N}{2} + M - 1.$$

The location of ω_β *in complex plane is defined by distribution of zeros* x_β *of the generalized Jacobi polynomials.*

Let us assume that $q \in \mathbb{R}$. *Let us define for the generalized Jacobi polynomial*

$P_n^{(a,b)}(x)$ the number $N_1(a,b)$, $N_2(a,b)$, $N_3(a,b)$ of zeros on the intervals $(-1, +1)$, $(-\infty, -1)$, $(1, +\infty)$ appropriately (see explicit expressions for these quantities in [8, 9]*). If $N_1(a,b) + N_3(a,b) = M$, then all numbers ω_β are all real, if $N_1(a,b) + N_3(a,b) < M$, then some numbers ω_β are complex pairwise conjugated.*

Proof. Let for the Richardson equations in general case

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth q(\omega_\alpha-\omega_\beta) + \sum_{l=1}^{N} q\coth q(u_l-\omega_\alpha) - \frac{1}{G} = 0,$$

we introduce new variables

$$x_\alpha = \exp(2q\omega_\alpha) - 1, \quad \alpha = 1, 2, \ldots, M,$$

$$\zeta_l = \exp(2qu_l), \quad l = 1, 2, \ldots, N,$$

and present these equations as follows:

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{x_\alpha - x_\beta} + \sum_{l=1}^{N} \frac{1}{\zeta_l - x_\alpha - 1} - \frac{L}{x_\alpha + 1} = 0,$$

where

$$L = \frac{1}{2qG} - \frac{N}{2} + M - 1.$$

If we assume

$$\zeta_l = 2, \quad l = 1, 2, \ldots, N,$$

then the equations attain the form

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} \frac{1}{x_\alpha - x_\beta} - \frac{N}{x_\alpha - 1} - \frac{L}{x_\alpha + 1} = 0.$$

As a result of last equalities the polynomial

$$f(x) = \prod_{\beta=1}^{M}(x - x_\beta), \qquad x_\beta = \exp(2q\omega_\beta) - 1$$

satisfies the differential equation

$$(1-x^2)\frac{d^2 f}{dx^2} + [(N-L)+(N+L)x]\frac{df}{dx} + M(M-N-L-1)f = 0.$$

The generalized Jacobi polynomial

$$y(x) = P_n^{(a,b)}(x)$$

satisfies the differential equation

$$(1-x^2)\frac{d^2 y}{dx^2} + [(b-a)-(a+b+2)x]\frac{dy}{dx} + n(n+a+b+1)y = 0.$$

Comparing these equations we get

$$a = -(N+1), \qquad b = -(L+1), \quad n = M,$$

and therefore

$$f(x) = P_M^{-(N+1),\,-(L+1)}(x).$$

The theorem is proved.

We can show than an appearance of complex solutions of the Richardson solutions indicates an appearance of a spectral gap [4, 6].

In order to obtain further results we must use some facts of a theory of ordinary differential equations with polynomial solutions [10, 11].

5. Asymptotic form of the Richardson equations. Let us consider the limit

$$N,\ M \to \infty, \quad G \to 0$$

under the condition

$$\lim_{N\to\infty} \frac{M}{N} = m, \qquad \lim_{N\to\infty} GN = g.$$

5.1. Rational case. In this limit the Richardson equations in the rational case are transformed to a singular integral equation

$$P\int_{\Gamma} \frac{r(\xi')|d\xi'|}{\xi'-\xi} = \int_{\Omega} \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon-\xi} - \frac{1}{g}, \quad \xi \in \Gamma,$$

where the density of states $\rho(\varepsilon)$ and the density of pairs $r(\xi)$ satisfy the conditions

$$\int_{\Omega} \rho(\varepsilon)d\varepsilon = \frac{1}{2}, \qquad \int_{\Gamma} r(\xi)|d\xi| = m, \qquad \int_{\Gamma} \xi r(\xi)|d\xi| = e.$$

Here Ω is a support of unperturbed spectrum and Γ is a support of spectrum of pairs.

The integrals of motion look now as follows:

$$\lambda(\varepsilon) = -\frac{1}{2} + g\left(\int_{\Gamma} \frac{r(\xi)|d\xi|}{\varepsilon-\xi} - P\int_{\Omega} \frac{\rho(\varepsilon')d\varepsilon'}{\varepsilon-\varepsilon'}\right),.$$

5.2. General case. In asymptotic limit the general Richardson equations

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth q(e_\beta - e_\alpha) = \sum_{l=1}^{N} q\coth q(\varepsilon_l - e_\alpha) - \frac{1}{G}$$

are transformed to a singular integral equation

$$P\int_{\Gamma} q\coth q(\xi'-\xi)r(\xi')|d\xi'| = \int_{\Omega} q\coth q(\varepsilon-\xi)\rho(\varepsilon)d\varepsilon - \frac{1}{g},$$

where the density of states $\rho(\varepsilon)$ and the density of pairs $r(\xi)$ satisfy the conditions

$$\int_{\Omega} \rho(\varepsilon)d\varepsilon = \frac{1}{2}, \qquad \int_{\Gamma} r(\xi)|d\xi| = m, \qquad \int_{\Gamma} \xi r(\xi)|d\xi| = e.$$

In new variables

$$\varepsilon = \exp(2qu), \qquad \xi = \exp(2q\omega),$$

the above singular integral equation attains the form

$$P\int_{\tilde{\Gamma}} \frac{\tilde{r}(\xi')|d\xi'|}{\xi'-\xi} = \int_{\tilde{\Omega}} \frac{\tilde{\rho}(\varepsilon)d\varepsilon}{\varepsilon-\xi} - \frac{1}{2\tilde{g}},$$

where

$$\tilde{\rho}(\varepsilon) = \rho(u) = \rho\left(\frac{1}{2q}\ln\varepsilon\right),$$

$$\tilde{r}(\xi) = r(\omega) = r\left(\frac{1}{2q}\ln\xi\right),$$

$$\frac{1}{2\tilde{g}} = \frac{1}{2b} - q\left(\frac{1}{2} - m\right).$$

This integral equation coincide with the similar integral equation for the BCS case

$$P\int_{\Gamma} \frac{r(\xi')|d\xi'|}{\xi' - \xi} = \int_{\Omega} \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon - \xi} - \frac{1}{2g}.$$

Thus our problem now is to solve the singular integral equation. To this end let us remind some appropriate facts of such a type equations (see, e.g., [12, 13]).

6. Bounded solution for the Cauchy type integral inversion problem. First of all let us consider the inversion of the Cauchy type integral

$$\frac{1}{\pi i}\int_L \frac{\phi(s)\,ds}{s - t} = f(t), \quad t \in L,$$

where the contour L is assumed to be of a general form consisting of separate arcs without common ends, i.e., the contour

$$L = \bigcup_{k=1}^{p} L_k = \bigcup_{k=1}^{p} (a_k, b_k).$$

We study various possibilities.

6.1. The free term is a function of the Hölder class. The solution of the inversion problem $\phi(t)$, which belongs to the class of functions H, i.e., the solution $\phi(t)$, which is bounded in all ends of the contour L, exists only at special conditions.

Theorem 6.1. *If a contour L consists of separate arcs and the function $f(t) \in \in H$, then the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t - t_0} = f(t_0), \quad t_0 \in L,$$

has a solution $\phi(t) \in H$ iff the function $f(t)$ satisfy the following conditions:

$$\int_L \frac{f(t)t^k}{\sqrt{R(t)}}dt, \quad k = 0, 1, \ldots, p-1, \qquad R(t) = \prod_{k=1}^{p}(t - a_k)(t - b_k).$$

6.2. The free term is sum of a function of the Hölder class and a polynomial. We remind that in generic case there are no solutions which are bounded at all ends of the contour L consisting of separate arcs. We illustrate this statement in case when the free term is just a polynomial.

Theorem 6.2. *If the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t - t_0} = P(t_0), \quad t_0 \in L,$$

with a contour L, consisting of separate arcs, and a polynomial $P(t)$ of degree $\deg P(t) \le p - 1$ *as the free term $f(t)$ has a solution $\phi(t)$ of the class H, then it is necessary*

$$\phi(t) = 0, \qquad P(t) = 0.$$

Now we prove a following generalization of this statement.

Theorem 6.3. *If a contour* L *consists of separate arcs, the function* $f(t) \in H$, *and the polynomial* $P(t)$ *has a degree* $\deg P(t) \le p-1$, *then the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t-t_0} = f(t_0) + P(t_0), \quad t_0 \in L,$$

has a unique bounded solution $\phi(t) \in H$, *iff the polynomial* $P(z)$ *is of the form*

$$P(z) = \frac{1}{\pi i}\int_L \frac{f(t)}{\sqrt{R(t)}}\frac{Q(z)-Q(t)}{z-t}\,dt,$$

$$Q(z) = \sqrt{R(z)} + O(z^{-1}), \qquad |z| \to \infty,$$

$$R(z) = \prod_{i=1}^{p}(z-a_i)(z-b_i).$$

Proof. If a solution of our singular integral equation exists it is to be of the form

$$\phi(t_0) = \frac{\sqrt{R(t_0)}}{\pi i}\int_L \frac{f(t)\,dt}{\sqrt{R(t)}\,(t-t_0)},$$

$$R(t) = \prod_{k=1}^{p}(t-a_k)(t-b_k).$$

Let us put this solution into the equation. In such a way we ensure that it is indeed solution and obtain in addition the expression for the polynomial $P(t)$.

To prove this statement we introduce two functions

$$\Psi(z) = \frac{\sqrt{R(z)}}{2\pi i}\int_L \frac{f(t)\,dt}{\sqrt{R(t)}\,(t-z)},$$

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)\,dt}{t-z}.$$

We have obviously

$$\Psi(t)_+ - \Psi(t)_- = \phi(t),$$

$$\Phi(t)_+ + \Phi(t)_- = f(t) + P(t).$$

Therefore we have

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)\,dt}{t-z} = \frac{1}{2\pi i}\int_{\tilde{L}} \frac{\Psi(t)\,dt}{t-z} =$$

$$= \frac{1}{2\pi i}\int_{\tilde{L}} \frac{\sqrt{R(t)}\,dt}{t-z}\frac{1}{2\pi i}\int_L \frac{f(t_1)\,dt_1}{\sqrt{R(t_1)}\,(t_1-t)} = \frac{1}{2\pi i}\int_L \frac{f(t_1)\,dt_1}{\sqrt{R(t_1)}}\frac{1}{2\pi i}\int_{\tilde{L}} \frac{\sqrt{R(t)}\,dt}{(t-z)(t_1-t)},$$

where $\tilde{L}$ is a clock-wise closed contour encircling the contour L. Using the Cauchy residue theorem we obtain

$$\frac{1}{2\pi i}\int_{\tilde{L}} \frac{\sqrt{R(t)}\,dt}{(t-z)(t_1-t)} = \frac{\sqrt{R(z)}}{t_1-z} + \frac{Q(z)-Q(t_1)}{z-t_1},$$

where

$$\sqrt{R(z)} = Q(z) + Q(z^{-1}).$$

Here we have taken in account that the point t_1 is inside the contour $\tilde{L}$, and the point z is outside the contour $\tilde{L}$. We have taken in account also that the integral vanishes if z or t_1 goes to infinity. Hence we obtain

$$\Phi(z) = \frac{\sqrt{R(z)}}{2\pi i}\int_L \frac{f(t_1)\,dt_1}{\sqrt{R(t_1)}\,(t_1 - z)} + \frac{1}{2}P(z),$$

$$P(z) = \frac{1}{\pi i}\int_L \frac{f(t)}{\sqrt{R(t)}}\frac{Q(z)-Q(t)}{z-t}\,dt.$$

Using the equality

$$\Phi^+(t) + \Phi^-(t) = f(t) + P(t),$$

we arrive to conclusion of the theorem.

6.3. The free term is sum of a function of the Hölder class and a piecewise constant. As we said earlier, in generic case there is no solution of the inversion problem which is bounded at ends of the contour of integration. We demonstrate this statement in case when the free term is just a piecewise constant.

Theorem 6.4. *If the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t - t_0} = C_k, \quad t_0 \in L_k,$$

with a contour L, consisting of separate arcs, and a piecewise constant as the free term $f(t)$, has a solution $\phi(t)$ of the class H, then it is necessary

$$\phi(t) = 0, \qquad C_k = 0, \quad k = 1, \ldots, p.$$

Now we prove a generalization of this statement.

Theorem 6.5. *If a contour L consists of separate arcs, the function $f(t) \in H$, and C_k, $k = 1, \ldots, p$, are some constants, then the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t - t_0} = f(t_0) + C_k, \quad t_0 \in L_k,$$

has a unique bounded solution $\phi(t) \in H$, wich is of the form

$$\phi(t_0) = \frac{\sqrt{R(t_0)}}{\pi i}\int_L \frac{f(t)}{\sqrt{R(t)}}\left\{\frac{1}{t - t_0} + \sum_{k=1}^{p}\omega_k(t)\int_{L_k}\frac{ds}{\sqrt{R(s)}\,(s - t_0)}\right\}dt,$$

iff there exist such polynomials $\omega_k(t)$, $k = 1, \ldots, p$, of the order $\deg \omega_k(t) \le p - 1$ that

$$C_k = \int_L \frac{\omega_k(t) f(t)}{\sqrt{R(t)}}\,dt, \quad k = 1, \ldots, p.$$

These polynomials $\omega_k(t)$ are defined by the function $f(t)$ and the contour L (more exactly, by the ends a_k, b_k and mutual arrangement of arcs L_k).

There is another formulation of the above theorem.

Theorem 6.6. *If a contour L consists of separate arcs, the function $f(t) \in H$ and C_k, $k = 1, \ldots, p$, are some constants, then the singular integral equation*

$$\frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t-t_0} = f(t_0) + C_k, \quad t_0 \in L_k,$$

has a unique bounded solution $\phi(t) \in H$, *wich is of the form*

$$\phi(t_0) = \frac{\sqrt{R(t_0)}}{\pi i}\left\{\int_L \frac{f(t)\,dt}{\sqrt{R(t)}\,(t-t_0)} + \sum_{k=1}^{p} C_k \int_{L_k} \frac{dt}{\sqrt{R(t)}\,(t-t_0)}\right\},$$

iff the C_k *satisfy the linear equations*

$$\sum_{k=1}^{p} a_{jk} C_k + A_j = 0, \quad j = 0, 1, \ldots, p-1,$$

where

$$a_{jk} = \int_{L_k} \frac{t^j}{\sqrt{R(t)}}\,dt, \qquad A_j = \int_L \frac{t^j f(t)}{\sqrt{R(t)}}\,dt.$$

7. The Richardson singular integral equation. Applying the results presented above to solve the Richardson singular integral equation we should find first of all the integration contour L in course of asymptotic analysis. This is difficult problem even for classical polynomials. We shall assume further that the integration contour is defined by this time.

7.1. The inversion problem for the Cauchy type integral with an open arc. Here we determine for an arc $L = ab$ and a function $f(t) \in H$ a unique solution bounded in all nodes for the equation

$$\frac{1}{\pi i}\int_L \frac{\phi(t)}{t-t_0}\,dt = f(t) + C.$$

Solution is of the form

$$\phi(t_0) = \frac{\sqrt{R(t_0)}}{\pi i}\int_L \frac{f(t)\,dt}{\sqrt{R(t)}\,(t-t_0)},$$

$$C = \frac{1}{\pi i}\int_L \frac{f(t)\,dt}{\sqrt{R(t)}}, \qquad R(t) = (t-a)(t-b).$$

7.2. A special case of solution of the Richardson singular integral equation. Determine for arcs $\Gamma = ab$ and $\Omega = cd \in \mathbb{R}$ and a function $\rho(\varepsilon) \in H$ a unique solution bounded in all nodes of the contour L for the equation

$$\int_\Gamma \frac{r(\xi')\,d\xi'}{\xi'-\xi} = \int_\Omega \frac{\rho(\varepsilon)\,d\varepsilon}{\varepsilon-\xi} + C, \quad \xi \in \Gamma.$$

Hence we should study at first properties of the function

$$f(\xi) = \int_\Omega \frac{\rho(\varepsilon)\,d\varepsilon}{\varepsilon-\xi} + C, \quad \xi \in \Gamma.$$

We use further the polynomial

$$R(t) = (t-a)(t-b).$$

In order to obtain the solution we need the following Cauchy residue formulae:

$$\frac{1}{\pi i}\int_{\Gamma}\frac{dt}{\sqrt{R(t)}\,(\varepsilon-t)} = \frac{1}{2\pi i}\int_{\tilde{\Gamma}}\frac{dt}{\sqrt{R(t)}\,(\varepsilon-t)} = \frac{1}{\sqrt{R(\varepsilon)}},$$

$$\frac{1}{\pi i}\int_{\Gamma}\frac{dt}{\sqrt{R(t)}\,(t-\xi)(\varepsilon-t)} = \frac{1}{2\pi i}\int_{\tilde{\Gamma}}\frac{dt}{\sqrt{R(t)}\,(t-\xi)(\varepsilon-t)} = \frac{1}{\sqrt{R(\varepsilon)}\,(\varepsilon-\xi)}.$$

Here $\tilde{\Gamma}$ is a clock-wise closed curve encircling the arc Γ. In the last formula we have taken in account that the point ξ is inside of the contour $\tilde{L}$, and the point ε is outside of the contour $\tilde{L}$.

The solution is

$$r(\xi) = \frac{\phi(\xi)}{\pi i} = \frac{\sqrt{R(\xi)}}{(\pi i)^2}\int_{\Gamma}\frac{f(t)\,dt}{\sqrt{R(t)}\,(t-\xi)} = \frac{\sqrt{R(\xi)}}{(\pi i)^2}\int_{\Gamma}\frac{dt}{\sqrt{R(t)}\,(t-\xi)}\int_{\Omega}\frac{d\varepsilon\,\rho(\varepsilon)}{\varepsilon-t} =$$

$$= \frac{\sqrt{R(\xi)}}{(\pi i)^2}\int_{\Omega}d\varepsilon\,\rho(\varepsilon)\int_{\Gamma}\frac{dt}{\sqrt{R(t)}\,(t-\xi)(\varepsilon-t)} = \frac{\sqrt{R(\xi)}}{\pi i}\int_{\Omega}\frac{\rho(\varepsilon)\,d\varepsilon}{\sqrt{R(\varepsilon)}\,(\varepsilon-\xi)}.$$

The constant is

$$C = \frac{1}{\pi i}\int_{\Gamma}\frac{f(t)\,dt}{\sqrt{R(t)}} = \frac{1}{\pi i}\int_{\Gamma}\frac{dt}{\sqrt{R(t)}}\int_{\Omega}\frac{\rho(\varepsilon)\,d\varepsilon}{\varepsilon-t} = \frac{1}{\pi i}\int_{\Omega}d\varepsilon\,\rho(\varepsilon)\int_{\Gamma}\frac{dt}{\sqrt{R(t)}\,(\varepsilon-t)} = \int_{\Omega}\frac{\rho(\varepsilon)\,d\varepsilon}{\sqrt{R(\varepsilon)}}.$$

8. Conclusion. Thus we have studied a number of the integrable quantum models and discussed different methods to solve the appropriate Richardson equations. For the case of narrow band a solution of the Richardson equations is presented in terms of zeros of the generalized Laguerre or Jacobi polynomials. We have also formulated the conditions for appearance of gaps in the spectrum, i.e., an appearance of complex solution of the Richardson equations. In asymptotic limit, when the Richardson equations are transformed to an integral singular equation, we have studied possible solutions and their relations to a spectral density.

1. *Bardeen J., Cooper L. N., Schrieffer J. R.* Theory of superconductivity // Phys. Rev. – 1957. – **108**. – P. 1175 – 1204.
2. *Боголюбов Н. Н., Зубарев Д. Н., Церковников Ю. А.* К теории фазового перехода // Докл. АН СССР. – 1957. – **117**, № 5. – С. 788 – 791.
3. *Richardson R. W.* A restricted class of exact eigenstates of the pairing-force Hamiltonian // Phys. Lett. – 1963. – **3**, № 6. – P. 277 – 279.
4. *Gaudin M.* Diagonalization d'une class d'hamiltoniens de spin // J. Phys. – 1976. – **37**, № 10. – P. 1087 – 1098.
5. *Gaudin M.* La fonction d'onde de Bethe. – Paris: Masson, 1983. – 352 p.
6. *Gaudin M.* Modèles exactement résolus // CEA Saclay-Service Phys. Théor. Edit. Phys. – 1995.
7. *Dukelsky J., Pittel S., Sierra G.* Exactly solvable Richardson – Gaudin models for many-body quantum systems // nucl-ph/0405011.
8. *Hilbert D.* Über die Discriminante der im Endlichen abbrechenden hypergeometrischen Reihe // J. Math. – 1888. – **103**. – S. 337 – 345.
9. *Szego G.* Orthogonal polynomials. – New York: Amer. Math. Soc., 1959.
10. *Hille E.* Ordinary differential equations in the complex domain. – New York: John Wiley & Sons, 1976.
11. *Ince E. L.* Ordinary differential equations. – New York: Dover Publ., 1956.
12. *Гахов Ф. Д.* Краевые задачи. – М.: Наука, 1977. – 640 с.
13. *Мусхелишвили Н. И.* Сингулярные интегральные уравнения. – М.: Наука, 1968. – 512 с.

Received 09.10.2006