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MORSE FUNCTIONS ON COBORDISMS

ФУНКЦІЇ МОРСА НА КОБОРДИЗМАХ

We study the homotopy invariants of crossed and Hilbert complexes. These invariants are applied to the calculation of exact values of Morse numbers of smooth cobordisms.

Вивчаються гомотопічні інваріанти схрещених і гільбертових комплексів. Ці інваріанти використовуються для підрахунку точних значень чисел Морса гладких кобордизмів.

1. Introduction. Let W^n bea smooth manifold. By definition the i-th Morse number $\mathcal{M}_i(W^n)$ of a manifold W^n is the minimal number of critical points of index i taken over all Morse functions on W^n.

It is known [8, 17] that for closed smooth manifolds of dimension greater than 6 the i-th Morse numbers are invariant of the homotopy type. There is a very complicated unsolved problem: **find exact values of Morse numbers for every** i ([17] for more details).

In [18] using new homotopy invariants of free cochain complexes and Hilbert complexes of non simply-connected closed manifolds W^n, $n \geq 8$, we gave exact values of i-th Morse numbers for $4 \leq i \leq n-4$.

The Morse number $\mathcal{M}(W^n)$ of a manifold W^n is the minimal number of critical points of all indexes taken over all Morse functions on W^n.

In this paper we calculate exact values of Morse numbers for some cobordisms $(W^n, V_0^{n-1}, V_1^{n-1})$.

2. Crossed modules [2]. A G-**crossed module** is a triple (C, ∂, G), where C and G are groups, $\partial\colon C \longrightarrow G$ is a homomorphism and G acts on C from the left (the action will be denoted by gc). Furthermore, the homomorphism ∂ should satisfy the following conditions:

a) $\partial(gc) = g(\partial)g^{-1}$ for all $g \in G, c \in C$,

b) $cdc^{-1} = (\partial) \cdot d$ for all $c, d \in C$.

Thus, if G acts on itself by conjugation action, then a) says that ∂ is a G-homomorphism. The following statements are immediate consequences of the definition:

1) $K = \operatorname{Ker} \partial$ is contained in the center of G,

2) $N = \operatorname{Im} \partial$ is a normal subgroup of G.

Let $Q = G/N$. The action of G on C induces a natural structure of $\mathbb{Z}[Q]$-module on the center of C, and $K = \operatorname{Ker} \partial$ is a submodule of this module. Moreover the action of G on C induces the structure of $\mathbb{Z}[Q]$-module on $C^{ab} = C/[C, C]$. Obvious and important special cases are: 1) the case when C is a $\mathbb{Z}[G]$-module (so $\partial = 0$) and 2) the case when C is a normal subgroup of G (so ∂ is the inclusion).

A **morphism** (α, β) from the crossed module (C, ∂, G) to (C', ∂', G') is a pair of group homomorphisms $\alpha\colon C \longrightarrow C'$ and $\beta\colon G \longrightarrow G'$ such that $\beta \cdot \partial = \partial' \cdot \alpha$ and $\alpha(g \cdot c) = \beta(g) \cdot \alpha(c)$ $(g \in G,\, c \in C)$.

Let **CM** denote the category of crossed modules. If $\beta = \mathrm{Id}$ on $G = G'$, we say that α is a G-**morphism** and denote this category by $\mathbf{CM}_G$.

An important case of crossed module is so-called free crossed module defined by J. H. C. Whitehead [2].

A G-crossed module (C, ∂, G) is called a **free** crossed module with indexed basis $(c_{i \in I}) \subset C$ if it satisfies the following universal property: given a G'-crossed module (C', ∂', G'), an indexed subset $(c'_{i\in I}) \subset C'$ and homomorphism $f\colon G \longrightarrow G'$ such that $f(\partial(c_i)) = \partial'(c'_i)$ for each $i \in I$, then there is a unique homomorphism $g\colon C \longrightarrow C'$ such that $g(c_i) = c'_i$ for each $i \in I$ and the pair (f, g) is a morphism of crossed modules.

The following fundamental theorem is also due to J. H. C. Whitehead [2].

Theorem. *Let X be a path-connected CW-complex, and Y a CW-complex obtained from X attaching two-dimensional cell. Then $\pi_2(Y, X, x)$ is a free crossed $\pi_1(X, x)$-module with basis corresponding to the cells so attached.*

Fix a group G. A G-crossed module C is said to be **projective** if it is projective in the category $\mathbf{CM}_G$, that is to say, for any surjective morphism of G-crossed modules $f\colon A \to B$ and any $g\colon C \to B$ in $\mathbf{CM}_G$, there is an $h\colon C \to A$ in $\mathbf{CM}_G$ such that $f \cdot h = g$.

Let crossed module (C, ∂, G), $N = \operatorname{Im}\partial$, $Q = G/N$ and $C^{ab} = C/[C, C]$. J. G. Ratcliffe proved that (C, ∂, G) is a projective crossed module if and only if $C^{ab} = C/[C, C]$ is a projective module $\mathbb{Z}[Q]$-module and mapping the two-dimensional homology groups

$$\partial_*\colon H_2(C) \to H_2(N)$$

induced by the homomorphism $\partial\colon C \to G$ is trivial.

The following important theorem is due to M. Dyer [2].

Theorem. *Let X be a connected CW-subcomplex of a connected 2-complex Y, where $\pi_1(X, x) = G$ and $x \in X$ is base point. Then the triple $(\pi_2(Y, X, x), \partial, \pi_1(X, x))$ is a projective crossed module.*

(The homomorphism $\partial\colon \pi_2(Y, X, x) \to \pi_1(X, x)$ is taken from the exact homotopy sequence of the pair (Y, X).)

A **projective crossed chain complex** is sequence of groups and homomorphisms

$$e \longleftarrow \pi \xleftarrow{\partial_1} G \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} C_3 \longleftarrow \ldots \xleftarrow{\partial_n} C_n$$

such that:

a) (C_2, ∂_2, G) is a projective G-crossed module,

b) for each $i \geq 3$ the module C_i is a projective $\mathbb{Z}[\pi]$-module, ∂_i is a homomorphism of $\mathbb{Z}[\pi]$-modules, ∂_2 commutes with the action of the group G and $\partial_3(C_3)$ is a $\mathbb{Z}[\pi]$-module,

c) $\partial_i \cdot \partial_{i+1} = 0$.

Obviously, G acts on each C_i, $i \geq 2$. A crossed chain complex is said to be of dimension n if $C_i = 0$ for $i > n$.

With any projective crossed chain complex (C_i, ∂_i, G) associate the chain complex projective $\mathbb{Z}[\pi]$-modules

$$\xleftarrow{\partial_2^{ab}} C_2^{ab} \xleftarrow{\partial_3} C_3 \longleftarrow \ldots \xleftarrow{\partial_n} C_n.$$

3. Stable invariants of finite generated modules and L^2-modules. In what follows, M will be a left finitely generated Λ-module over a certain associative ring Λ with unit. Rings for which the rank of the free module is uniquely defined are called IBN-rings.

Denoting the minimum number of the generators of the module M by $\mu(M)$, we get $\mu(M \bigoplus F_n) < \mu(M) + n$, where F_n is a free module of rank n. There exist examples (stably-free modules) when the strict inequality holds.

Recall that a Λ-module M is called *stably-free* if the direct sum of M and a free Λ-module F_k is free. We assume that if the module M is zero, then $\mu(M) = 0$.

Definition 3.1. *For a finitely generated module M over IBN-ring Λ let us define the following function* (*stable minimal generators of the module M*) [17]

$$\mu_s(M) = \lim_{n \longrightarrow \infty} \big(\mu(M \oplus F_n) - n\big).$$

If a ring Λ is Hopfian then for any Λ-module M $\mu_s(M) = 0$ if and only if $M = 0$. Recall that a ring Λ is called *Hopfian*, if every epimorphism of a free Λ-module F_n on itself is an isomorphism. It is clear, that for any non-zero module M we have $0 < \mu_s(M) \leqslant \mu(M)$.

Denote the ring of integers by $\mathbb{Z}$ and the field of complex numbers by $\mathbb{C}$. Let G be a discrete group. Denote its integer group ring by $\mathbb{Z}[G]$ and the group ring over the field $\mathbb{C}$ by $\mathbb{C}[G]$. It is known that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are IBN-rings. From theorems of Kaplansky and Cockroft it follows that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are Hopfian.

In the ring $\mathbb{C}[G]$ there exists an involution $*\colon \mathbb{C}[G] \to \mathbb{C}[G]$, $\Big(\sum_i \alpha_i g_i\Big)^* = \sum_i \overline{\alpha}_i g_i^{-1}$, where $\overline{\alpha}$ denotes the conjugation in $\mathbb{C}$. This involution satisfies the following conditions:

a) $(r^*)^* = r$;

b) $(\alpha r_1 + \beta r_2)^* = \overline{\alpha} r_1^* + \overline{\beta} r_2^*, (\alpha, \beta \in C)$;

c) $(r_1 r_2)^* = r_2^\star r_1^\star$.

We can define the trace $\operatorname{tr}\colon \mathbb{C}[G] \longrightarrow \mathbb{C}$ by the rule $\operatorname{tr}\left(\sum_i^k \alpha_i g_i\right) = \alpha_1$, where α_1 is the coefficient of $g_1 = e$, which is the identity of the group G. It is obvious that the trace satisfies the following conditions:

a) $\operatorname{tr}(e) = 1$;

b) tr is $\mathbb{C}$-linear mapping;

c) $\operatorname{tr}(r_1 r_2) = \operatorname{tr}(r_2 r_1)$;

d) $\operatorname{tr}(rr^*) \geqq 0$, and if $\operatorname{tr}(rr^*) = 0$, then $r = 0$.

In the ring $\mathbb{C}[G]$ there is an inner product $\Big\langle \sum_i \alpha_i g_i, \sum_i \beta_i g_i \Big\rangle = \sum_i \alpha_i \overline{\beta}_i$. The norm for an element from $\mathbb{C}[G]$ may be defined by $|r| = \operatorname{tr}(rr^*)^{1/2}$. Consider a completion of the ring $\mathbb{C}[G]$ with respect to this norm and denote it by $L^2(G)$. Then $L^2(G)$ is a Hilbert space (the inner product assigns the same formula as for the group ring $\mathbb{C}[G]$). The Hilbert space $L^2(G)$ has an orthonormal basis consisting of all elements of the group G. Now $\mathbb{C}[G]$ acts faithfully and continuously by left multiplication on $L^2(G)$satisfies the following condition

$$\mathbb{C}[G] \times L^2(G) \longrightarrow L^2(G),$$

so we may regard $\mathbb{C}[G] \subseteq \mathbf{B}(L^2(G))$, where $\mathbf{B}(L^2(G))$ denotes the set of bounded linear operators on $L^2(G)$. Let $N[G]$ denote the (reduced) group von Neumann algebra of G: thus by definition $N[G]$ is a week closure of $\mathbb{C}[G]$ in $\mathbf{B}(L^2(G))$. Therefore the map $w \to w(e)$ allows us to identify $N[G]$ with a subspace of $L^2(G)$, where $w \in N[G]$ and

e is unit element of the group G. Thus algebraically we have $\mathbb{C}[G] \subset N[G] \subset L^2(G)$. The involution and the trace map on $N[G]$ may be defined exactly as for the ring $\mathbb{C}[G]$. For the set $M_n(N[G])$ of $n \times n$ matrices over von Neumann algebra $N[G]$, the trace map can be extended by setting $\mathrm{tr}(W) = \sum_{i=1}^{n} w_{ii}$, where $W = (w_{ij})$ is a matrix with entries in $N[G]$.

Let $L^2(G)^n$ denote the Hilbert direct sum n copies of $L^2(G)$, so $L^2(G)^n$ is a Hilbert space. The von Neumann algebra $N[G]$ acts on $L^2(G)^n$ from the left, so $L^2(G)^n$ is a left $N[G]$-module called a *free Hilbert $N[G]$-module of rank n. The left Hilbert $N[G]$-module M* is a closed left $\mathbb{C}[G]$-submodule of $L^2(G)^n$ for some n. By definition an *Hilbert $N[G]$-submodule* of M is a closed left $\mathbb{C}[G]$-submodule of M, an $L^2(G)$-ideal is an Hilbert $N[G]$-submodule of $L^2(G)$, and *homomorphism* $f\colon M \longrightarrow N$ between Hilbert $N[G]$-modules is a continuous left $\mathbb{C}[G]$-map [3].

Let M be a Hilbert $N[G]$-module and let $p\colon L^2(G)^n \to L^2(G)^n$ be an orthogonal projection onto $M \subset L^2(G)^n$. *Von Neumann dimension* of Hilbert $N[G]$-module M is called the number $\dim_{N[G]}(M) = \mathrm{tr}(p) = \sum_{i=1}^{n} \langle p(e_i), e_i \rangle_{L^2(G)^n}$. Here $e_i = (0, \ldots, g, \ldots, 0)$ is standard basis in $L^2(G)^n$. It is known that $\dim_{N[G]}(V)$ is non-negative real number [10].

Definition 3.2. *Let M be a finite generated $\mathbb{Z}[G]$-module, consider Hilbert $N[G]$-module $L^2(G) \bigotimes_{\mathbb{Z}[G]} M$ and define following number*

$$S(M) = \mu_s(M) - \dim_{N[G]} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} M \right).$$

Lemma 3.1. *For any finite generated $\mathbb{Z}[G]$-module M, the number $S(M)$ is non-negative.*

The proof in [18].

4. Stable invariants of homomorphisms. The next results can be found in [18]. Consider a Λ-homomorphism $f\colon F_k \to F_t$, where F_k, F_t are free modules of ranks k and t respectively over ring Λ. The homomorphism f is a *splitting along a submodule* $\overline{F}_p \subseteq F_k$, if there is a presentation of f of the form $f = f_p \bigoplus f_t \colon \overline{F}_p \bigoplus \overline{F}_{k-p} \to \widetilde{F}_p \bigoplus \widetilde{F}_{t-p}$, such that $f|_{\overline{F}_p \bigoplus 0} = f_p \colon \overline{F}_p \to \widetilde{F}_p$, $f|_{0 \bigoplus \overline{F}_{k-p}} = f_t \colon \overline{F}_{k-p} \to \widetilde{F}_{t-p}$, where f_p is an isomorphism.

From now in this situation we will suppose that submodules $\overline{F}_p$, $\overline{F}_{k-p}$, $\widetilde{F}_p$, $\widetilde{F}_{t-p}$ are free.

Definition 4.1. *The number p above is called the rank of a splitting $f = f_p \bigoplus f_t$. The rank $\mathrm{R}(f)$ of a homomorphism f is the maximal value of possible ranks of splittings of f.*

Definition 4.2. *Stabilization of a homomorphism $f\colon F_k \to F_t$ by a free module F_p is a homomorphism $f_{st}(p)\colon F_k \bigoplus F_p \to F_t \bigoplus F_p$, such that $f_{st}(p)|_{F_k \bigoplus 0} = f$, $f_{st}(p)|_{0 \bigoplus F_p} = \mathrm{Id}$.*

A thickening of a homomorphism $f\colon F_k \to F_t$ by free modules F_m and F_n is the homomorphism $f_{th}(m,n)\colon F_k \bigoplus F_m \to F_t \bigoplus F_n$, such that $f_{th}(m,n)|_{F_k \bigoplus 0} = f$, $f_{th}(m,n)|_{0 \bigoplus F_m} = 0$.

Definition 4.3. *The stable rank $\mathrm{Sr}(f)$ of a homomorphism $f\colon F_k \to F_t$ is the limit of values of*

$$\mathrm{Sr}(f) = \lim_{m,n,p \to \infty} \big(\mathrm{R}(f_{th}(m,n)_{st}(p)) - p\big).$$

This limit always exists. There are examples of stably free modules with $\mathrm{Sr}(f) > > \mathrm{R}(f)$. For any homomorphism $f\colon F_k \to F_t$ the following equality holds: $\mathrm{Sr}(f_{st}(v)) = = \mathrm{Sr}(f) + v$.

Consider a composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, such that $g \cdot f = 0$ (condition (∂)). We say, that the homomorphisms f and g are splitting along submodules $\overline{F}_p \subseteq F_m$ and $\overline{F}_q \subseteq F_n$ if there are presentations of f and g of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{F}_p & \xrightarrow{f_1} & \widetilde{F_p} & \longrightarrow & 0 & & \\
 & & \bigoplus & & \bigoplus & & & & \\
 & & F_{m-p} & \xrightarrow{f_2} & F_{n-p-q} & \xrightarrow{g_2} & F_{t-q} & & \\
 & & & & \bigoplus & & \bigoplus & & \\
 & & 0 & \longrightarrow & \overline{F_q} & \xrightarrow{g_1} & \widetilde{F_q} & \longrightarrow & 0
\end{array}$$

such that $f|_{\overline{F}_p \bigoplus 0} = f_1$, $g|_{0 \bigoplus 0 \bigoplus \overline{F}_q} = g_1$. We admit that the module $\overline{F}_p$ or $\overline{F}_q$ to be zero module. In the sequel we will suppose that submodules $\overline{F}_p, \overline{F}_q, F_{m-p}, F_{t-q}, F_{n-p-q}$ are free.

Definition 4.4. *The number $p+q$ will be called the common rank of a splitting of homomorphisms f and g along submodules $\overline{F}_p \subseteq F_m$ and $\overline{F}_q \subseteq F_n$. The common rank $\mathrm{Cr}(f,g)$ of the homomorphisms f and g is a maximal value of common ranks of a splitting of f and g.*

Definition 4.5. *The stabilization of a composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying the condition (∂) by free modules F_p and F_q is the following composition of homomorphisms*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_p & \xrightarrow{id} & F_p & \longrightarrow & 0 & & \\
 & & \bigoplus & & \bigoplus & & & & \\
 & & F_m & \xrightarrow{f} & F_n & \xrightarrow{g} & F_t & & \\
 & & & & \bigoplus & & \bigoplus & & \\
 & & 0 & \longrightarrow & F_q & \xrightarrow{id} & F_q & \longrightarrow & 0.
\end{array}$$

We will denote it by $(f_{st}(p), g_{st}(q))$.

Definition 4.6. *Consider a composition of homomorphisms f and g $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying the condition (∂). The thickening of this composition by free modules F_p and F_q is the following composition of homomorphisms $F_m \bigoplus F_p \xrightarrow{f_{th}(p)} F_n \xrightarrow{g_{th}(q)} F_t \bigoplus F_q$, such that $f_{th}(p)|_{F_m \bigoplus 0} = f$, $f_{th}(p)|_{0 \bigoplus F_p} = 0$, $g_{th}(q) = = g$. It will be denoted by $(f_{th}(p), g_{th}(q))$.*

Definition 4.7. *The stable common rank $\mathrm{Scr}(f,g)$ of the composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying the condition (∂) is the limit of values of common ranks*

$$\mathrm{Scr}(f,g) = \lim_{p,q,v,w \to \infty} \Big(\mathrm{Cr}(f_{th}(p)_{st}(v), g_{th}(q)_{st}(w)) - v - w \Big).$$

This limit always exists. There are examples of stably free modules showing that $\mathrm{Scr}(f,g) \geq \mathrm{Cr}(f,g)$.

Lemma 4.1. *For arbitrary composition of homomorphisms f and g $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying the condition (∂) the following equality holds true*: $\mathrm{Scr}(f_{st}(x), g_{st}(y)) = \mathrm{Scr}(f,g) + x + y$.

Definition 4.8. *The stable common rank from the left* (*from the right*) $\mathrm{Scr}_l(f,g)$ $(\mathrm{Scr}_r(f,g))$ *of the composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying condition (∂) is the following limit of values of common ranks*:

$$\mathrm{Scr}_l(f,g) = \lim_{p,v,w\to\infty} \Big(\mathrm{Cr}(f_{th,l}(p)_{st}(v),\ g_{st}(w)) - v - w\Big)$$

$$\Big(\mathrm{Scr}_r(f,g) = \lim_{q,v,w\to\infty} \Big(\mathrm{Cr}(f_{st}(v), g_{th,r}(q)_{st}(w)) - v - w\Big)\Big).$$

***Remark* 4.1.** For stable common rank from the left (from the right) $\mathrm{Scr}_l(f,g)$ $\big(\mathrm{Scr}_r(f,g)\big)$ of the composition of the homomorphisms satisfying the condition (∂) the analogues of Lemma 4.1 hold true.

Definition 4.9. *The defect $\mathbb{D}(f,g)$ of the composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying condition (∂) is the following number*:

$$\mathbb{D}(f,g) = \mathrm{Sr}(f) + \mathrm{Sr}(g) - \mathrm{Scr}(f,g).$$

***Remark* 4.2.** If in composition of the homomorphisms f and g the module $F_n/f(F_m)$ is stable free, but non free, then all way $\mathbb{D}(f,g) > 0$.

Lemma 4.2. *Consider two compositions of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, and*

$$\begin{array}{ccccccc}
0 & \longrightarrow & F_v & \xrightarrow{id} & F_v & \longrightarrow & 0 \\
 & & \bigoplus & & \bigoplus & & \bigoplus \\
 & & F_m \bigoplus F_p & \xrightarrow{f_{th,l}(p)} & F_n & \xrightarrow{g_{th,r}(q)} & F_t \bigoplus F_q \\
 & & \bigoplus & & \bigoplus & & \bigoplus \\
 & & 0 & \longrightarrow & F_w & \xrightarrow{id} & F_w & \longrightarrow & 0
\end{array}$$

satisfying the condition (∂) (the numbers $p,\ q,\ v,\ w$ are nonnegative). Then the following equality holds true:

$$\mathbb{D}(f,g) = \mathbb{D}\big(f_{th,l}(p)_{st}(v), g_{th,r}(q)_{st}(w)\big).$$

Definition 4.10. *The defect from the left* (*from the right*) $\mathbb{D}_l(f,g)$ $\big(\mathbb{D}_r(f,g)\big)$ *of a composition of homomorphisms of free modules $F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$, satisfying condition (∂) is the following number*:

$$\mathbb{D}_l(f,g) = \mathrm{Sr}_l(f) + \mathrm{Sr}(g) - \mathrm{Scr}_l(f,g)$$

$$\Big(\mathbb{D}_r(f,g) = \mathrm{Sr}(f) + \mathrm{Sr}_r(g) - \mathrm{Scr}_r(f,g)\Big).$$

***Remark* 4.3.** For the defect from the left (from the right) $\mathbb{D}_l(f,g)$ $\big(\mathbb{D}_r(f,g)\big)$ of a composition of homomorphisms f and g satisfying condition (∂) the analogues of Lemma 4.2 and Remark 4.2 hold true.

5. Homotopy invariants of cochain complexes. If $(C,d)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots$ $\ldots \xrightarrow{d^{n-1}} C^n$ is a free cochain complex over a ring Λ, then the numbers $\mathbb{D}_r(d^0)$, $\mathbb{D}_l(d^{n-1})$, $\mathbb{D}_r(d^0,d^1)$, $\mathbb{D}_l(d^{n-2},d^{n-1})$, $\mathbb{D}(d^i,d^{i+1})$ are defined for $1 \le i \le n-3$. In [18] proof that they are invariants of the homotopy type of a cochain complex (C,d).

Definition 5.1. *Let $(C,d)\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{n-1}} C^n$ be a free cochain complex. Then cochain complex $(C(i),d(i))\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{i-1}} C^i$ is called i-th skeleton of cochain complex (C,d).*

Let $(C^*,d^*))\colon C_0 \xrightarrow{d_1} C_1 \to \ldots \xrightarrow{d_n} C_n$, be a sequence of free Hilbert $N[G]$-modules and bounded $\mathbb{C}[G]$-map such that $d_{i+1}\circ d_i = 0$. It is called a Hilbert complex. *The reduced cohomology of Hilbert complex (C^*,d^*)), it is a collection of Hilbert* $N[G]$-modules $\overline{H^i}_{(2)}(C^*,d^*) = \operatorname{Ker} d^i/\,\overline{\operatorname{Im} d^{i-1}}$.

Definition 5.2. *Consider a free cochain complex over $\mathbb{Z}[G]$*

$$(C^*,d^*)\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{n-1}} C^n.$$

Hilbert complex

$$\left(L^2(G)\bigotimes_{\mathbb{Z}[G]} C^*, \operatorname{Id}\bigotimes_{\mathbb{Z}[G]} d^*\right):$$

$$L^2(G)\bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^0} L^2(G)\bigotimes_{\mathbb{Z}[G]} C^1 \to \ldots \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^{n-1}} L^2(G)\bigotimes_{\mathbb{Z}[G]} C^n$$

of free Hilbert $N[G]$-modules is the Hilbert complex generated by $\mathbb{Z}[G]$-complexes.

Consider the i-th skeletons of these complexes

$$\big(C^*(i),d^*(i)\big)\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{i-1}} C^i,$$

$$L^2(G)\bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^0} L^2(G)\bigotimes_{\mathbb{Z}[G]} C^1 \longrightarrow \ldots \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^{i-1}} L^2(G)\bigotimes_{\mathbb{Z}[G]} C^i,$$

Set $\Gamma^i = C^i/d^{i-1}(C^{i-1})$. It is clear that

$$\widehat{\Gamma^i} = L^2(G)\bigotimes_{\mathbb{Z}[G]} C^i/\overline{\operatorname{Id}\bigotimes_{\mathbb{Z}[G]} d^{i-1}\left(L^2(G)\bigotimes_{\mathbb{Z}[G]} C^{i-1}\right)}$$

is the i-th Hilbert $N[G]$-module of reduced cohomology of the i-th skeleton of the Hilbert complex

$$\left(L^2(G)\bigotimes_{\mathbb{Z}[G]} C^*(i),\ \operatorname{Id}\bigotimes_{\mathbb{Z}[G]} d^*(i)\right).$$

Definition 5.3 [16, 17]**.** *For the cochain complex (C^*,d^*) over $\mathbb{Z}[G]$ set*

$$\widehat{S}^i_{(2)}(C^*,d^*) = \mu_s(\Gamma^i) - \dim_{N[G]}\widehat{\Gamma^i}.$$

Lemma 5.1. *The numbers $\widehat{S}^i_{(2)}(C^*, d^*)$ are non-negative for every i.*

If (C^*, d^*) and (D^*, ∂^*) are two homotopy equivalent free cochain complexes over the group ring $\mathbb{Z}[G]$ then

$$\widehat{S}^i_{(2)}(C^*, d^*) = \widehat{S}^i_{(2)}(D^*, \partial^*).$$

Definition 5.4. *The **Morse number** of a cochain complex*

$$(C, d)\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{n-1}} C^n$$

over a ring Λ is the number $\mathcal{M}(C, d) = \sum_{i=0}^{n} \mu(C_i)$.

Definition 5.5. *The **homotopy Morse number** $\mathcal{M}_h(C, d)$ of a cochain complex (C, d) over a ring Λ is the minimum of Morse numbers taken over all cochain complexes homotopy equivalent to (C, d).*

For a cochain complex (C, d) denote by $\mathbb{D}_0(C, d) = \mathbb{D}_r(d^0, d^1)$, $\mathbb{D}_{n-2}(C, d) = \mathbb{D}_l(d^{n-2}, d^{n-1})$, $\mathbb{D}_i(C, d) = \mathbb{D}(d^i, d^{i+1})$.

Theorem 5.1. *Let $(C, d)\colon C^0 \xrightarrow{d^0} C^1 \to \ldots \xrightarrow{d^{n-1}} C^n$, $n \geq 4$, be a free cochain complex over a group ring $\mathbb{Z}[G]$ such that if $\mathbb{D}_i(C, d) \neq 0$, then $\mathbb{D}_{i+1}(C, d) = 0$ for $0 \leq i \leq n-2$. Then the homotopy Morse number of (C, d) equal*:

$$\mathcal{M}_h(C, d) = 2\sum_{i=0}^{n-2} (\mathbb{D}_i(C, d)) + 2\sum_{i=1}^{n-1} \left(\widehat{S}^i_{(2)}(C, d)\right) +$$
$$+ \sum_{i=0}^{n-1} \left(\dim_{N[G]} \left(H^i(L^2(G) \bigotimes_{\mathbb{Z}[G]} C, \mathrm{id} \bigotimes_{\mathbb{Z}[G]} d)\right)\right) + 2\mu(H^n(C, d)) -$$
$$- \dim_{N[G]} \left(H^n(L^2(G) \bigotimes_{\mathbb{Z}[G]} C, id \bigotimes_{\mathbb{Z}[G]} d)\right).$$

Proof. The number $\mathbb{D}_i(C, d)$ arises in this theorem because in definition of the number $S^i_{(2)}(C, d)$ we take the number $\mu_s(\Gamma^i)$ but not the number $\mu(\Gamma^i)$. For example, in view of Remark 4.2 if the module $C^i/d^{i-1}(C^{i-1})$ is stable free but non free, then $\mathbb{D}_i(C, d) > 0$. Therefore, if $\mathbb{D}_i(C, d) > 0$ then in dimension i and $i-1$ we have additional free modules of the rank $\mathbb{D}_i(C, d)$ such that they not give contribution in $\widehat{S}^i_{(2)}$.

From conditions of theorem it follow that in the homotopy type of (C, d) any cochain complex $(D, d)\colon D^0 \xrightarrow{d^0} D^1 \to \ldots \xrightarrow{d^{n-1}} D^n$ such that $\mathbb{D}_{i_0}(C, d) = 0$ satisfies the following condition $\mu(D_{i_0}/d^{i_0-1}D_{i_0-1}) = \mu_s(D_{i_0}/d^{i_0-1}D_{i_0-1})$ for i_0. From [17] it follow that in the homotopy type of (C, d) there exist minimal cochain complex in dimension i_0 such that $\mu(C_{i_0}) = \widehat{S}^{i_0}_{(2)}(C, d) + \widehat{S}^{i_0+1}_{(2)}(C, d) + \dim_{N[G]} \big(H^{i_0}(L^2(G) \bigotimes_{\mathbb{Z}[G]} \bigotimes_{\mathbb{Z}[G]} C, \mathrm{id} \bigotimes_{\mathbb{Z}[G]} d)\big)$.

The value of Morse number of cochain complex may be find direct calculation.

Theorem 5.1 is proved.

6. Applications. It is well known that all chain complexes constructed from cellular decompositions of the non-simply connected CW-complex K have the same homotopy type. Therefore it follows directly from the previous or from [9,17] that the numbers $\widehat{S}^i_{(2)}(K)$ and $\widehat{\mathbb{D}}^0_r(K)$, $\widehat{\mathbb{D}}^{n-1}_l(K)$, $\mathbb{D}^0_r(K)$, $\mathbb{D}^{n-2}_l(K)$, $\mathbb{D}^i(K)$ for $1 \leq i \leq n-2$. are invariants of the homotopy type of the CW-complex K.

For a smooth manifold W^n there is an approach to the construction of cochain complex via Morse functions. The details can be found in [15].

Let $(W^n, V_0^{n-1}, V_1^{n-1})$ be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1} \cup V_1^{n-1}$. Let $\pi = \pi_1(W^n)$ be the fundamental group of the manifold W^n. Denote by $p\colon (\widetilde{W}^n, \widetilde{V}_0^{n-1}, \widetilde{V}_1^{n-1}) \to (W^n, V_0^{n-1}, V_1^{n-1})$ the universal covering space. Here $\widetilde{V}_i^{n-1} = p^{-1}(V_i^{n-1})$. Let us choose on W^n an ordered Morse function $f\colon W^n \to [0,1]$, $f^{-1}(0) = V_0^{n-1}$, $f^{-1}(1) = V_1^{n-1}$ and a gradient-like vector field ξ [17]. Using the mapping p, lift f and ξ to $\widetilde{W}^n$, and denote a lifted function and a vector field by $\widetilde{f}$ and $\widetilde{\xi}$ respectively. Using f, ξ $(\widetilde{f}, \widetilde{\xi})$ construct chain complexes of abelian group $C_*(W^n, f, \xi)$:

$$C_*(W^n, f, \xi)\colon C_0 \xleftarrow{d_1} C_1 \leftarrow \ldots \xleftarrow{d_n} C_n,$$

$$C_*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})\colon \widetilde{C}_0 \xleftarrow{\widetilde{d}_1} \widetilde{C}_1 \leftarrow \ldots \xleftarrow{\widetilde{d}_n} \widetilde{C}_n,$$

where $C_i = H_i(W_i, W_{i-1}, Z)$, $\widetilde{C}_i = H_i(\widetilde{W}_i, \widetilde{W}_{i-1}, Z)$ and $\widetilde{W}_i = \widetilde{f}^{-1}[0, a_i]$, $W_i = f^{-1}[0, a_i]$ are submanifolds containing all critical points of indices less than or equal to i. For the generators of the chain groups C_i $(\widehat{C_i})$ one can take middle disks of critical points of index i constructed by the vector field ξ $(\widehat{\xi})$. The fundamental group $\pi = \pi_1(W^n)$ acts on manifolds $\widetilde{W^n}$. Making use of this actions, we can turn the chain group $\widetilde{C}_i$ into finitely generated modules over ring $Z[\pi]$. Making use of the involution, we turn the right $Z[\pi]$-module $C^{(i)} = \mathrm{Hom}_{Z[\pi]}(C_i, Z[\pi])$ into a left one and construct the following free cochain complex

$$C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})\colon \widetilde{C}^{(0)} \xrightarrow{\widetilde{d}^{(0)}} \widetilde{C}^{(1)} \to \ldots \xrightarrow{\widetilde{d}^{(n-1)}} \widetilde{C}^{(n)}.$$

Taking the tensor product of $C^*\big(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi}\big)$ and $L^2(\pi)$ as $Z[\pi]$-module, we obtain the cochain complex of abelian groups which can be used for the definition the numbers $\widehat{S}^i_{(2)}(W^n, V_0^{n-1})$ and $\mathbb{D}^i(W^n, V_0^{n-1})$.

On cobordism $(W^n, V_0^{n-1}, V_1^{n-1})$ using (f, ξ) construct crossed projective chain complexes $C_*^{cr}(W^n, f, \xi)$:

$$e \leftarrow \pi \leftarrow \pi_1(V_0^{n-1}) \xleftarrow{d^2} \pi_2(W^n, V_0^{n-1}) \xleftarrow{\widetilde{d}^{(3)}} \widetilde{C}^3 \xleftarrow{\widetilde{d}^4} \ldots \xleftarrow{\widetilde{d}^{(n-1)}} \widetilde{C}^{(n)}.$$

and using $(-f, -\xi)$ construct crossed projective chain complexes $C_*^{cr}(W^n, -f, -\xi)$:

$$e \leftarrow \pi \leftarrow \pi_1(V_1^{n-1}) \xleftarrow{d^2} \pi_2(W^n, V_1^{n-1}) \xleftarrow{\widetilde{d}^{(3)}} \widetilde{D}^3 \xleftarrow{\widetilde{d}^4} \ldots \xleftarrow{\widetilde{d}^{(n-1)}} \widetilde{D}^{(n)}.$$

Definition 6.1. *The Morse number $\mathcal{M}(W^n)$ of a manifold W^n is the minimal number of critical points of all indexes taken over all Morse functions on W^n.*

Theorem 6.1. *Let $(W^n, V_0^{n-1}, V_1^{n-1})$, $n \geq 6$, be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1} \cup V_1^{n-1}$ and $\pi = \pi_1(W^n)$ be the fundamental group of the manifold W^n. Suppose that $\pi(V_i^{n-1}) \longrightarrow \pi_1(W^n)$ is isomorphism, $Wh(\pi) = 0$ (Whitehead group of π) and if $\mathbb{D}^i(W^n, V_0^{n-1}) \neq 0$ then $\mathbb{D}^{i+1}(W^n) = 0$ for all $1 < i < n-2$. The following equality holds for the Morse number of W^n :*

$$\mathcal{M}(W^n, V_0^{n-1}) = 2\sum_{i=1}^{n-2}\big(\mathbb{D}^i(W^n, V_0^{n-1})\big) + 2\sum_{i=3}^{n-2}\big(\widehat{S}^i_{(2)}(W^n, V_0^{n-1})\big) +$$

$$+\sum_{i=2}^{n-2}\Big(\dim_{N[\pi]}(H^i_{(2)}(W^n,V_0^{n-1},Z))\Big).$$

Proof. Let f be an arbitrary ordered Morse function, ξ a gradient-like vector field on W^n, and $C_*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)\colon \widetilde{C}_0 \xleftarrow{\widetilde{d}_1} \widetilde{C}_1 \leftarrow \ldots \xleftarrow{\widetilde{d}_n} \widetilde{C}_n$, the chain complex associated with them.

Denote by $C^*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)\colon \widetilde{C}^{(0)} \xrightarrow{\widetilde{d}^{(0)}} \widetilde{C}^{(1)} \to \ldots \xrightarrow{\widetilde{d}^{(n-1)}} \widetilde{C}^{(n)}$ the cochain complex constructed starting from chain complex $C_*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)$. It is clear that if chain complex $C_*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)$ is minimal in dimension i then and cochain complex $C^*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)$ is minimal in dimension i. It is known that the operation of stabilization of the homomorphisms $\widetilde{d}_i$, can be realized by changing Morse function and gradient-like vector field on W^n. But the inverse operation, the elimination of contractible contractible free chain complex of the form $0 \longrightarrow \overline{C}_i \longrightarrow \overline{C}_{i+1} \longrightarrow 0$ from the chain complex $C^*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)$ can not always be realized by change of Morse function and gradient-like vector field on W^n. This is possible if $n \geq 6$ and $Wh(\pi)=0$ [17]. Let $\big(\overline{C},\overline{d}\big)$ be a chain complex homotopy equivalent to chain complex $C_*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\big)$ such that it Morse number equal homotopy Morse number of $C_*(\widetilde{W}^n,\widetilde{f},\widetilde{\xi})$. By Cockroft – Swan theorem [2] there exist contractible free chain complexes (D,∂) and $\big(\overline{D},\overline{\partial}\big)$ such that the chain complexes $\big(C^*(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\bigoplus D, d\bigoplus\partial)\big)$ and $\big(\overline{C}\bigoplus\overline{D},\overline{d}\bigoplus\overline{\partial}\big)$, are chain-isomorphic. The previous notice ensures the existence of a Morse function g and gradient-like vector field that realize the complex $\Big(C^*\big(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\bigoplus D, d\bigoplus\partial\big)\Big)$. Using elimination of contractible contractible free chain complexes of the form $0 \longrightarrow \overline{C}_i \longrightarrow \overline{C}_{i+1} \longrightarrow 0$ and $0 \longrightarrow \overline{C}_{i-1} \longrightarrow \overline{C}_i \longrightarrow 0$ from the chain complex $\big(C^*(\widetilde{W}^n,\widetilde{f},\widetilde{\xi}\bigoplus D, d\bigoplus\partial)\big)$ we can obtain a chain complex $\big(\widehat{C},\widetilde{d}\big)$ that is minimal. The conditions that $n \geq 6$ and $Wh(\pi)=0$ ensures the existence of a Morse function g and gradient-like vector field η that realize the complex $(\widehat{C},\widehat{d})$. The number of critical points of Morse function g can be computed using previous formulas from Theorem 5.1.

Theorem 6.1 is proved.

Theorem 6.2. *Let $(W^n,V_0^{n-1},V_1^{n-1})$, $n \geq 6$, be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1}\cup V_1^{n-1}$ and $\pi=\pi_1(W^n)$ be the fundamental group of W^n. Suppose that $\pi_2(W^n,V_i^{n-1})$ are free crossed $\pi_1(V_i^{n-1})$-modules, $H_2\big(\widetilde{W}^n,\widetilde{V}_i^{n-1},\mathbb{Z}[\pi]\big)$ are free $\mathbb{Z}[\pi]$-modules, $\mu\big(H_2(\widetilde{W}^n,\widetilde{V}_i^{n-1},\mathbb{Z}[\pi])\big)=\mu(\pi_2(W^n,V_i^{n-1}))$, $Wh(\pi)=0$ and if $\mathbb{D}^i(W^n)\neq 0$ then $\mathbb{D}^{i+1}(W^n)=0$ for all $3<i<n-3$. Then the following equality holds for the Morse number of W^n :*

$$\mathcal{M}(W^n,V_0^{n-1}) = 2\sum_{i=2}^{n-2}\big(\mathbb{D}^i(W^n,V_0^{n-1})\big)+$$

$$+2\sum_{i=3}^{n-2}\Big(\widehat{S}^i_{(2)}(W^n,V_0^{n-1})\Big)+\sum_{i=3}^{n-3}\Big(\dim_{N[\pi]}(H^i_{(2)}(W^n,V_0^{n-1},Z))\Big)+$$

$$+\mu\Big(H_2(\widetilde{W}^n,\widetilde{V}_0^{n-1},\mathbb{Z}[\pi]))+\mu(H_2(\widetilde{W}^n,\widetilde{V}_1^{n-1},\mathbb{Z}[\pi])\Big).$$

Proof. The conditions in the theorem guarantee the existence of a ordered Morse function $f\colon W^n \to [0,1]$, $f^{-1}(0)=V_0^{n-1}$, $f^{-1}(1)=V_1^{n-1}$ without critical points of indexes $0, 1, n-1, n$. The Morse number of $(W^n, V_0^{n-1}, V_1^{n-1})$ can be computed using previous formulas from Theorem 5.1.

Theorem 6.2 is proved.

The estimate for Morse numbers study in works [1, 4 – 7, 10 – 19], where were use and other approaches. In next papers we shall give the values of Morse numbers for other class manifolds.

1. *Andrica D.* Functions with a minimal number of critical points. – Babes-Bolyai, 1995. – Preprint 95-8.
2. *Brown R., Hiqqius P. I., Sivera R.* Nonabelian algebraic topology: filtred spaces, crossed complexes, cubical homotopy groupoids // Tracts Math. – 2011. – **15**. – 516 p.
3. *Cohen J.* Von Neumann dimension and homology of covering spaces // Quart. J. Math. – 1979. – **30**. – P. 133 – 142.
4. *Damian A.* On stable Morse number of a closed manifold // Bull. London Math. Soc. – 2002. – **34**. – P. 420 – 430.
5. *Eliashberg Y., Gromov M.* Lagrangian intersections and stable Morse theory // Boll. Unione mat. ital. – 1997. – **(7)11-B**. – P. 289 – 326.
6. *Farber M.* Homological algebra of Novikov – Shubin invariants and Morse inequalities // Geom. and Funct. Anal. – 1996. – **6**. – P. 628 – 665.
7. *Gromov M., Shubin M.* Near-cohomology of Hilbert complexes and topology of non-siply connected manifolds // Asterisque. – 1992. – **210**. – P. 283 – 294.
8. *Hajduk B.* Comparing handle decomposition of homotopy equivalent manifolds // Fund. math. – 1997. – **95**. – P. 35 – 47.
9. *Kirbu R. C., Siebenmann L. C.* Foundational essays on topological manifolds, smoothings and triangulatiions // Ann. Math. Stud. – 1977. – **88**.
10. *Luck W.* L^2-invariants: theory and applications to geometry and K-theory // Ergeb. Math. – 2002. – **44**. – 620 p.
11. *Mathai V., Shubin M.* Twisted L^2-invariants of non-simply connected manifolds and asymptotic L^2 Morse inequalities // Rus. J. Math. Phys. – 1996. – **4**. – P. 499 – 526.
12. *Novikov S. P., Shubin M. A.* Morse inequalities and von Neumamm algebras // Uspekhi Mat. Nauk. – 1986. – **41**. – S. 163 – 164.
13. *Novikov S. P., Shubin M. A.* Morse inequalities and von Neumamm II_1-factors // Sov. Mat. Dokl. – 1987. – **34**. – S. 79 – 82.
14. *Novikov S. P., Shubin M. A.* Morse theory and von Neumann invariants of non-simply connected manifolds // Uspekhi Mat. Nauk. – 1986. – **41**. – S. 222 – 223.
15. *Pajitniov A.* On the asymptotics Morse numbers // Topology. – 1999. – **38**. – P. 529 – 541.
16. *Sharko V.* The L^2-invariants and Morse numbers // Algebraic topology — old and new. – Warsaw: Inst. Math. Pol. Acad. Sci., 2009. – P. 281 – 295.
17. *Sharko V. V.* Functions on manifolds: algebraic and topological aspects // Transl. Math. Monogr. – 1993. –**131**.
18. *Sharko V. V.* New L^2-invariants of chain complexes and applications, $C*$-algebra and elliptic theory // Trends Math. – Basel, Switzerland: Birkhäuser, 2006. – P. 291 – 312.
19. *Witten E.* Supersymmetry and Morse theory // J. Different. Geom. – 1982. – **17**. – P. 661 – 692.

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