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N. S. Chernikov (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

A NOTE ON *FC*-GROUPS*

ЗАУВАЖЕННЯ ЩОДО *FC*-ГРУП

Let G be an arbitrary FC-group, R be its locally soluble radical, and $L/R = L(G/R)$. We prove that for $N \lhd G$ G/N is residually finite if $R \subseteq N \subseteq L$.

Нехай G — довільна FC-група, R — її локально розв'язний радикал і $L/R = L(G/R)$. Доведено, що для $N \lhd G$ G/N фінітно апроксимовна у випадку, коли $R \subseteq N \subseteq L$.

Recall the following definition.

(Below for the empty set $\{G_\lambda : \lambda \in \Lambda = \varnothing\}$ of subgroups of the group G $\prod_{\lambda\in\Lambda} G_\lambda = 1$.)

Definition (see [1]). *The group G (finite or infinite) is called quasisimple, if $G' = G$ and $G/Z(G)$ is simple. The group G is called semisimple, if $G = \prod_{\lambda\in\Lambda} G_\lambda$ for some family $\{G_\lambda : \lambda \in \Lambda\}$ of its quasisimple subgroups such that $[G_\nu, G_\lambda] = 1$, $\nu \neq \lambda$. Further, the subgroup $L(G)$ of the group G is defined as the product of all its normal semisimple subgroups.*

The subgroup $L(G)$ plays a very important role in the theory of finite simple groups (see [2]).

According to Theorem 2 [1] in an arbitrary group G the subgroup $L(G)$ is semisimple. Obviously, for $N \trianglelefteq G$ $L(G)N/N \subseteq L(G/N)$ and, if G is semisimple, then G/N is semisimple too.

Lemma. *An arbitrary quasisimple FC-group G is finite.*

Proof. According to Baer's Theorem [3] for every FC-group X $X/Z(X)$ is locally finite-normal. So, obviously, $G/Z(G)$ has some subnormal finite simple subgroup $H/Z(G)$. In consequence of the statement 2 (or 4) of Lemma 1 [1] $H = G$. Therefore by Schur's Theorem (see, for instance, [4], Theorem 4.12) $G = G'$ is finite.

Theorem. *Let G be an FC-group, R is its locally soluble radical, $L/R = L(G/R)$ and $R \subseteq N \lhd G$, $N \subseteq L$. Then G/N is residually finite. In particular $G/L(G)R$ is residually finite.*

Proof is reduced in an evident way to the case when $G \neq 1$ and $R = 1$. Then $Z(G) = Z(L(G)) = 1$ and G is locally finite-normal.

Prove, at first, that $G/L(G)$ is residually finite. We may assume that $G \neq L(G)$. Since, clearly, G is hyperfinite, by Proposition 8 [1] $C_G(L(G)) = Z(L(G))$. So $C_G(L(G)) = 1$ and $L(G) \neq 1$. Then in consequence of Theorem 2 [1] $L(G)$ is the direct product of some nonabelian simple subgroups Q_λ, $\lambda \in \Lambda$. By Lemma Q_λ are finite. Let $gL(G)$ be an arbitrary element of some prime order p of $G/L(G)$, and $\Gamma = \{\lambda \in \Lambda : [g, Q_\lambda] \neq 1\}$. Obviously, for some finite $\Delta \supseteq \Gamma$ $g \in N_G(\times_{\lambda\in\Delta} Q_\lambda)$ and $g^p \in \times_{\lambda\in\Delta} Q_\lambda$. Put $\langle g\rangle(\times_{\lambda\in\Delta} Q_\lambda) = H$. Clearly that $H \cap C_G(H) \subseteq C_G(L(G)) = 1$, $C_G(H)L(G) = C_G(H)(\times_{\lambda\in\Delta} Q_\lambda)$ and the index $|G/L(G) : C_G(H)L(G)/L(G)|$ is finite. Then $g \notin C_G(H)(\times_{\lambda\in\Delta} Q_\lambda) = C_G(H)L(G)$. Therefore $gL(G) \notin C_G(H)L(G)/L(G)$. Thus $G/L(G)$ is residually finite.

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Prove that G/N is residually finite. Let $G/N \neq 1$ and $aN \in G/N \setminus \{1\}$. Suppose that $a \notin L(G)$. Since $G/L(G)$ is residually finite, there exists a subgroup $K \lhd G$ such that $a \notin K \supseteq L(G)$ and $|G:K|$ is finite. Then $aN \notin K/N$ and $|G/N : K/N|$ is finite. Now let $a \in L(G)$. Clearly, the index $\left|G/N : C_G(\langle a^G \rangle)N/N\right|$ is finite. Suppose that $aN \in C_G(\langle a^G \rangle)N/N$. Then, obviously, $\langle a^G \rangle \subseteq C_G(\langle a^G \rangle)N$. Consequently, $\langle a^G \rangle' \subseteq N$. It's easy to see that $\langle a^G \rangle$ is a direct product of some Q_λ. Consequently, $\langle a^G \rangle = \langle a^G \rangle'$. Thus $a \in N$. Contradiction.

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