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THE REDUCTION METHOD IN THE THEORY OF LIE-ALGEBRAICALLY INTEGRABLE OSCILLATORY HAMILTONIAN SYSTEMS

МЕТОД РЕДУКЦІЙ В ТЕОРІЇ ЛІ-АЛГЕБРАЇЧНО ІНТЕГРОВНИХ ГАМІЛЬТОНОВИХ ОСЦИЛЯЦІЙНИХ СИСТЕМ

We study complete integrability of nonlinear oscillatory dynamical systems connected in particular both with the Cartan decomposition of a Lie algebra $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, where $\mathcal{K}$ is the Lie algebra of a fixed subgroup $\mathcal{K} \subset G$ with respect to an involution $\sigma: G \to G$ on the Lie group G, and with a Poisson action of special type on a symplectic matrix manifold.

Вивчаються питання про повну інтегровність нелінійних осциляційних динамічних систем, що пов'язані, зокрема, як з декомпозицією Картана алгебри Лі $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, де $\mathcal{K}$ — алгебра Лі деякої (фіксованої) підгрупи $\mathcal{K} \subset G$ стосовно інволюції $\sigma: G \to G$ в групі Лі G, так і з дією Пуассона спеціального вигляду на симплектичному матричному многовиді.

1. Introduction. Symmetry analysis of nonlinear dynamical systems on a smooth manifold M is well-known [1, 2] to give rise in many cases to exhibiting its many hidden but interesting properties, in particular such as being integrable by quadratures due to the Liouville – Arnold theorem [3, 4]. In case when the manifold M can be represented as the cotangent space $T^*(K)$ to some subgroup K of a Lie group G naturally acting on it, the study of the corresponding flow can be recast via the reduction method [5] into the Hamiltonian framework due to the existence on $T^*(K)$ the canonical Poisson structure.

Furthermore, if the symmetry group G naturally generalizes to the loop group $G_+(\lambda)$ over $\lambda \in D_0 \subset \mathbf{C}$, then the corresponding momentum mapping $l: T^*(K) \to \mathcal{G}_+^*(\lambda)$ provides us with a Lax type representation and related with it a complete set of commuting invariants. Such a scheme appeared to be very useful when proving the Liouville integrability of many finite-dimensional systems such as Kowalevskaya's top [5], Neumann type systems [6, 7] and other.

Below we study complete integrability of nonlinear oscillatory dynamical systems connected in particular both with the Cartan decomposition of a Lie algebra $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, where $\mathcal{K}$ is the Lie algebra of a fixed subgroup $K \subset G$ with respect to an involution $\sigma: G \to G$ on the Lie group G, and with a Poisson action of special type on a symplectic matrix manifold.

2. Integrable systems on $T^*(K)$: the general scheme. Let consider a Lie group G and an involution σ on G. If $K \subset G$ is its fixed subgroup, then the Lie algebra $\mathcal{G}$ of the Lie group G admits the Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ with the induced involution mapping $\sigma = \mathrm{id}$ on $\mathcal{K}$ and $\sigma = -\mathrm{id}$ on $\mathcal{P}$. Denote also $\mathcal{G}^* = \mathcal{K}^* \oplus \mathcal{P}^*$ via the dual decomposition of the adjoint space $\mathcal{G}^*$. The cotangent space $T^*(K) \simeq K \times \mathcal{K}^*$ results by means of left translations on K.

Assume now that the natural group action of G on $T^*(K)$ is extended to that of the loop group $G_+(\lambda)$, $\lambda \in D_0$, where $D_0 \subset \mathbf{C}^1$ is a disc containing zero. Let $\mathcal{G}_+(\lambda)$ be the analytical Lie algebra of the loop group $G_+(\lambda)$ acting on the cotangent bundle $T^*(K) \simeq K \times \mathcal{K}^*$. If the action is Hamiltonian [1], one can define the corresponding momentum [7] mapping $l\colon T^*(K) \to \mathcal{G}_+^*(\lambda)$.

Here the adjoint space $\mathcal{G}_+^*(\lambda)$ is defined with respect to the following invariant and symmetric scalar product on $\mathcal{G}(\lambda, \lambda^{-1})$:

$$\langle \xi(\lambda), \eta(\lambda) \rangle_{-1} = \operatorname{res}_{\lambda \in D_0} 1/\lambda \, \langle \xi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}} \tag{1}$$

for any $\xi(\lambda),\ \eta(\lambda) \in \mathcal{G}(\lambda, \lambda^{-1})$, where $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ denotes the standard Killing form on $\mathcal{G}$.

Any orbit passing through a point $l(u, v; \lambda) \in \mathcal{G}_+^*(\lambda)$ with $(u, v) \in T^*(K)$ being fixed, is defined naturally as

$$\operatorname*{Span}_{x(\lambda) \in \mathcal{G}_+(\lambda)} \left\{ \operatorname{Pr}_{\mathcal{G}_+^*(\lambda)} \left(\operatorname{Ad}^*_{\exp(-x(\lambda))} l(u, v; \lambda) \right) \right\}, \tag{2}$$

where $\operatorname{Pr}_{\mathcal{G}_+^*(\lambda)}\colon \mathcal{G}^*(\lambda, \lambda^{-1}) \to \mathcal{G}_+^*(\lambda)$ denotes the projection upon $\mathcal{G}_+^*(\lambda)$ parallelly to the subspace $\mathcal{G}_-^*(\lambda)$, $\mathcal{G}^*(\lambda, \lambda^{-1}) := \mathcal{G}_+^*(\lambda) \oplus \mathcal{G}_-^*(\lambda)$ with

$$\mathcal{G}_-(\lambda) = \left\{ \sum_{i \in \mathbf{Z}_+} u_i \lambda^{-(i+1)} : u_i \in \mathcal{G},\ \sigma u_i = (-1)^{i+1} u_i \quad \text{for all} \quad i \in \mathbf{Z}_+ \right\}$$

and

$$\mathcal{G}_+(\lambda) = \left\{ \sum_{i \in \mathbf{Z}_+} x_i \lambda^i : x_i \in \mathcal{G},\ \sigma x_i = (-1)^i x_i \quad \text{for all} \quad i \in \mathbf{Z}_+ \right\},$$

so

$$\mathcal{G}_+^*(\lambda) = \left\{ \sum_{j \in \mathbf{Z}_+} y_j \lambda^{-j} : y_j \in \mathcal{G}^*,\ \sigma y_j = (-1)^j y_j \quad \text{for all} \quad j \in \mathbf{Z}_+ \right\}.$$

Denote now by $[\cdot, \cdot]_{\mathcal{R}}$ a new Lie bracket ($\mathcal{R}$-structure [8]) in $\mathcal{G}(\lambda, \lambda^{-1})$ defined as

$$[x_1(\lambda), x_2(\lambda)]_{\mathcal{R}} := [\mathcal{R} x_1(\lambda), x_2(\lambda)] + [x_1(\lambda), \mathcal{R} x_2(\lambda)],$$

where $x_1(\lambda),\ x_2(\lambda) \in \mathcal{G}(\lambda, \lambda^{-1})$ and for any $x(\lambda) \in \mathcal{G}(\lambda, \lambda^{-1})$

$$(\mathcal{R} x)(\lambda) := \operatorname{Pr}_{\mathcal{G}_+(\lambda)} x(\lambda) - \operatorname{Pr}_{\mathcal{G}_-(\lambda)} x(\lambda).$$

Thereby with respect to this new Lie product in $\mathcal{G}(\lambda, \lambda^{-1})$ one can find the corresponding momentum mapping for the modified group action $G_+(\lambda) \times T^*(K) \overset{\mathcal{R}}{\to}$ $\overset{\mathcal{R}}{\to} T^*(K)$. Having, for instance, taken

$$\bar{l}_{a,b}(u, v; \lambda) = a\lambda + v + \lambda^{-1} b \in \mathcal{G}_+^*(\lambda),$$

one can derive that

$$l_{a,b}(u, v; \lambda) := \operatorname{Ad}^{(\mathcal{R})*}_{\exp(-x(\lambda))} \bar{l}_{a,b}(u, v; \lambda) = a\lambda + v(u) + \lambda^{-1} \operatorname{Ad}^*_{u^{-1}} b \tag{3}$$

for any $(u, v) \in K \times \mathcal{K}$ with $u := \exp x_0 \in K$ and $a,\ b \in \mathcal{G}^*$, where we denoted by $\operatorname{Ad}^{(\mathcal{R})*}$ the corresponding modified adjoint action on $\mathcal{G}^*(\lambda, \lambda^{-1})$.

Consider now an element $a\lambda \in \mathcal{G}^*(\lambda, \lambda^{-1})$, where $a \in \mathcal{G}^*$ is constant. Since also

$$\left(a\lambda, \left[\mathcal{G}_+(\lambda), \mathcal{G}_+(\lambda)\right]_{\mathcal{R}}\right)_{-1} = 0 \tag{4}$$

for any $a \in \mathcal{G}^*$, we see that the element $a\lambda \in \mathcal{G}^*(\lambda, \lambda^{-1})$ is an infinitesimal character of the Lie subalgebra $\mathcal{G}_+(\lambda)$.

Based now on the well known Adler – Kostant – Symes (AKS) theorem [9 – 11], one can formulate the following theorem.

Theorem 1. *All functional*

$$\gamma_{s,n}^{(a,b)}(u, \upsilon) := \operatorname{res}_{\lambda \in D_0}\left(\lambda^s l_{a,b}^n(u, \upsilon; \lambda)\right) \quad s,\ n \in \mathbf{Z},$$

where

$$l_{a,b}(u, \upsilon; \lambda) := a\lambda + \upsilon(u) + \lambda^{-1} \mathrm{Ad}_{u^{-1}}^* b \tag{5}$$

are involutive on the contangent space $T^*(K) \simeq K \times \mathcal{K}^*$ *with respect to the standard Poisson bracket on* $T^*(K)$.

Since under the involution $K \ni u \to u^{-1} \in K$ and $T^*(K) \ni \upsilon \to w \in T^*(K)$ combined with the permutation $\mathcal{G}^* \ni a \leftrightarrow b \in \mathcal{G}^*$ the element $l_{a,b}(u, \upsilon; \lambda) \to l_{b,a}(u, w; \lambda)$, making it possible to represent the flow on $T^*(K)$ generated by the invariant $\gamma_{1-n,n}^{(a,b)}(u, \upsilon) \in D(T^*(K))$, $n \in \mathbf{Z}_+$, as the one generated by $\gamma_{n-3,n}^{(a,b)}(u, w)$.

In case when a Lie algebra $\mathcal{G}$ is the Lie algebra of the connected subgroup G of $SO(4, 3)$, the maximal compact subgroup $K \subset G$ with the Lie algebra $\mathcal{K}$ is isomorphic to $so(4, 3)$. Thereby this pair $(\mathcal{G}, \mathcal{K})$ can be used [12, 13] for constructing integrable flows quadratic in momenta on $T^*(K)$, in particular the four-dimensional top and its generalizations.

3. Oscillatory dynamical systems on $T^*(K)$: an example. Consider now the case when a loop group $G_-(\lambda)$ acts on $T^*(K) \simeq K \times \mathcal{K}^*$, where $\lambda \in D_\infty$ and $D_\infty \subset \subset \mathbf{C}$ is an open disc containing the infinite point.

Put $G_-(\lambda)$ the Lie algebra of the group $G_-(\lambda)$ and $\mathcal{G}_-^*(\lambda)$ its adjoint space with respect to the scalar product

$$\langle \xi(\lambda), \eta(\lambda) \rangle_0 := \operatorname{res}_{\lambda \in D_\infty} \langle \xi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}}$$

for any $\xi(\lambda) \in \mathcal{G}_-^*(\lambda)$ and $\eta(\lambda) \in G_-(\lambda)$.

As before, let

$$\mathcal{G}_+(\lambda) = \left\{ \sum_{i \in \mathbf{Z}_+} y_i \lambda^i : y_i \in \mathcal{G},\ \sigma y_i = (-1)^i y_i,\ i \in \mathbf{Z}_+ \right\},$$

$$\mathcal{G}_-(\lambda) = \left\{ \sum_{i \in \mathbf{Z}_+} x_i \lambda^{-(i+1)} : x_i \in \mathcal{G},\ \sigma x_i = (-1)^{i+1} x_i,\ i \in \mathbf{Z}_+ \right\}.$$

The adjoint space

$$\mathcal{G}_-^*(\lambda) = \left\{ \sum_{i \in \mathbf{Z}_+} a_i \lambda^i : a_i \in \mathcal{G}^*,\ \sigma a_i = (-1)^i a_i,\ i \in \mathbf{Z}_+ \right\}$$

contains one-parametric orbits of the Ad^*-action, which can be interpreted as some finite-dimensional integrable Hamiltonian systems on $T^*(K)$.

For this to be a lot more clarified, let us consider an element $a\lambda^2 + b \in \mathcal{G}_-^*(\lambda)$ with $a,\ b \in \mathcal{P}$ and calculate its orbit under the usual action

$$\mathrm{Ad}^*_{\exp(-x(\lambda))}:\ \mathcal{G}_-^*(\lambda) \to \mathcal{G}_-^*(\lambda),$$

where $x(\lambda) \in \mathcal{G}_-(\lambda)$ is some element specified by a point $(u, v) \in T^*(K)$. We find therefore that the orbit of the element $a\lambda^2 + b \in \mathcal{G}_-^*(\lambda)$ has the form:

$$l_{a,b}(u, v; \lambda) = a\lambda^2 + \lambda[x_0, a] + [x_1, a] + \frac{1}{2}[x_0, [x_0, a]] + b, \tag{6}$$

in which one can make identifications $[x_0, a] := q \in \mathcal{K}_a^\perp$ and $[x_1, a] = p \in \mathcal{P}_a^\perp$ with $u := (\exp x_1) \in \mathcal{K}$ and $x_0 \in \mathcal{P}_a^\perp,\ x_1 \in \mathcal{K}_a^\perp$ due to the natural isomorphisms $\mathrm{ad}\, a$: $\mathcal{K}_a^\perp \to \mathcal{P}_a^\perp$ and $\mathrm{ad}\, a$: $\mathcal{P}_a^\perp \to \mathcal{K}_a^\perp$.

Similarly one can represent the forth element in (6) as

$$\alpha(q) := \mathrm{Pr}_{\mathcal{P}_a} \frac{1}{2}[(\mathrm{ad}\, a)^{-1} q, q], \tag{7}$$

where evidently $\alpha : \mathcal{K}_a^\perp \to \mathcal{P}_a$.

Having assumed further that an element $a \in \mathcal{P}$ is such that $[\mathcal{G}_a^\perp, \mathcal{G}_a^\perp] \subset \mathcal{G}_a$ or equivalently $\mathcal{G} = \mathcal{G}_a \oplus \mathcal{G}_a^\perp$ (the symmetric expansion), one easily verifies that $[\mathcal{P}_a^\perp, \mathcal{K}_a^\perp] \subset \mathcal{P}_a$, or

$$\alpha(q) = \frac{1}{2}[(\mathrm{ad}\, a)^{-1} q, q]$$

since $\alpha(q) \in \mathcal{P}_a$ for all $q \in \mathcal{K}_a^\perp$.

In virtue of the isomorphism between $\mathcal{P}_a^\perp$ and $\mathcal{K}_a^\perp$, the orbit (6) evidently is diffeomorphic both to $\mathcal{K}_a^\perp \oplus \mathcal{P}_a^\perp$ and to the cotangent space $T^*(\mathcal{K}_a^\perp)$.

The space $T^*(\mathcal{K}_a^\perp)$ is endowed with the canonical Poissonian structure being equivalent to the standard Lie – Poisson structure upon the orbit (6):

$$\begin{aligned} \{q_i, q_j\} &:= \langle l, [\nabla q_i(l), \nabla q_j(l)]\rangle_0 = 0, \\ \{q_i, p_j\} &:= \langle l, [\nabla q_i(l), \nabla p_j(l)]\rangle_0 = \langle [f_j, e_i], a\rangle_{\mathcal{G}}, \\ \{p_i, p_j\} &:= \langle l, [\nabla p_i(l), \nabla p_j(l)]\rangle_0 = \langle [f_i, f_j], q\rangle_{\mathcal{G}}, \end{aligned} \tag{8}$$

for all $i, j = \overline{1, n}$ and any $(q, p) \in T^*(\mathcal{K}_a^\perp)$, where $\nabla : \mathcal{D}(T^*(\mathcal{K}_a^\perp)) \to \mathcal{K}_a$ denotes the usual gradient mapping on $\mathcal{D}(T^*(\mathcal{K}_a^\perp))$.

When deriving (8) we made use of the following relationships:

$$q := \sum_{i=1}^n q_i e_i, \quad p := \sum_{i=1}^n p_i f_i,$$

where

$$\left\{e_j = [f_j, a] \in \mathcal{K}_a^\perp : j = \overline{1, n}\right\} \quad \text{and} \quad \left\{f_j \in \mathcal{P}_a^\perp : j = \overline{1, n}\right\}$$

are orthogonal bases in $\mathcal{K}_a^\perp$ and $\mathcal{P}_a^\perp$ correspondingly, that is

$$\langle e_i, e_j \rangle_{\mathcal{G}} = \delta_{ij} = \langle f_i, f_j \rangle_{\mathcal{G}} \quad \text{for all} \quad i, j = \overline{1, n}.$$

As was mentioned in [14, 15] the elements $a \in \mathcal{P}$ satisfying the property $\mathcal{G} = \mathcal{G}_a \oplus \mathcal{G}_a^{\perp}$ can be found easily enough if one to consider a dual compact Lie algebra $\mathcal{G} = \mathcal{K} \oplus i\mathcal{P}$. Then the Hermitian symmetric expansion $\mathcal{G} = \mathcal{G}_{ia} \oplus \mathcal{G}_{ia}^{\perp}$ holds and the problem reduces to recounting all involutions $\sigma: \mathcal{G} \to \mathcal{G}$ in $\mathcal{G}$ commuting with the above Hermitian expansion and equal to "$-$id" upon the center of the Lie algebra $\mathcal{G}_{ia}$.

The condition $\mathcal{G} = \mathcal{G}_a \oplus \mathcal{G}_a^{\perp}$ involved above on an element $a \in \mathcal{P}$ implies obviously that $\mathcal{G}_a^{\perp} = \operatorname{ad} a(\mathcal{G}) = \operatorname{ad} a(\mathcal{G}_a^{\perp})$, since by definition $\operatorname{ad} a(\mathcal{G}_a) = 0$. Thus the element $a \in \mathcal{P}$ defines the projection operator $P_a: \mathcal{G} \to \mathcal{G}$ on $\mathcal{G}$ compatible with the involution $\sigma: \mathcal{G} \to \mathcal{G}$, that is $P_a \sigma = \sigma P_a$, where $P_a^2 = P_a$. The latter appears to be useful for practical calculations on which we shall not dwell here.

To end this section, let us write down the corresponding Hamiltonian flows on $T^*(\mathcal{K}_a^{\perp})$ in the component-wise form. The vector $(q, p) \in T^*(\mathcal{K}_a^{\perp})$ is a set of canonical coordinates on the orbit (6) since due to the imbedding $[\mathcal{P}_a^{\perp}, \mathcal{P}_a^{\perp}] \subset \mathcal{K}_a$, the bracket $\{p_i, p_j\} = 0$ for all $i, j = \overline{1, n}$. As a result one obtains the following expression for the orbit point (6):

$$l_{a,b}(q, p; \lambda) = a\lambda^2 + \lambda \sum_{i=1}^{n} q_i e_i + \left(\sum_{i=1}^{n} p_i f_i + \frac{1}{2} \sum_{i,j=1}^{n} q_i q_j [e_i, f_j] \right) + b, \tag{9}$$

where in virtue of (8)

$$\{q_i, q_j\} = 0 = \{p_i, p_j\}, \qquad \{p_i, q_j\} = \langle f_j, f_i \rangle_{\mathcal{G}} \tag{10}$$

for all $i, j = \overline{1, n}$.

Evaluating the functional

$$H(q, p) = \frac{1}{2} \operatorname{res}_{\lambda \in D_\infty} \lambda^{-1} \langle l_{a,b}(q, p; \lambda), l_{a,b}(q, p; \lambda) \rangle_{\mathcal{G}}$$

on the orbit space $T^*(\mathcal{K}_a^{\perp})$ at $b \in P_a$, one gets the Hamiltonian function

$$H(q, p) = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \frac{1}{2} \sum_{i,j=1}^{n} q_i q_j \langle [e_i, f_j], b \rangle_{\mathcal{G}} +$$

$$+ \frac{1}{8} \sum_{i,j=1}^{n} \sum_{s,l=1}^{n} q_i q_s \langle [e_i, f_j], [e_s, f_l] \rangle_{\mathcal{G}} q_j q_l, \tag{11}$$

describing an unharmonic oscillatory dynamical system of particles on the axis $\mathbf{R} \ni q_j$, $j = \overline{1, n}$, interacting with each other by means of a fourth order potential. .

Based on Theorem 1, one can formulate the following result.

Theorem 2. *The unharmonic oscillatory dynamical system with Hamiltonian function* (11) *on the orbit space* $T^*(\mathcal{K}_a^{\perp})$ *with the Poisson brackets* (10) *is a completely Liouville – Arnold integrable Hamiltonian system.*

Choosing different semisimple Lie algebras $\mathcal{G}$ admitting the Hermitian symmetric expansion $\mathcal{G}_a \oplus \mathcal{G}_a^{\perp} = \mathcal{G}$ for some element $a \in \mathcal{P}$, where $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is the Cartan decomposition, one can build all of fourth order potential canonical Hamiltonian systems on $T^*(\mathcal{K}_a^{\perp}) \simeq T^*(\mathbf{R}^n)$ from [1].

4. Unharmonic oscillatory Hamiltonian systems on a matrix manifold and their Lie-algebraic integrability. Consider now a dual matrix manifold $M := M_{n,2} \times M_{n,2}$ of dimension $2n$, $n \in \mathbf{Z}_+$, endowed with the following natural symplectic structure

$$\omega^{(2)} = \mathrm{Sp}(dQ^{\top} \wedge dF), \tag{12}$$

where $(F, Q) \in M$ and "Sp" means the standard trace operation.

Let $A_+(\lambda)$ mean an analytical inside an open ring $D_0 \ni 0$ loop group acting on the manifold M as follows: for any $(F, Q) \in M$ and $g(\lambda) \in A_+(\lambda)$

$$\begin{aligned} F &: \xrightarrow{g(\lambda)} F_{g(\lambda)} := \operatorname{res}_{\lambda \in D_0} \frac{1}{\lambda - \Omega} F g^{-1}(\lambda), \\ Q^{\top} &: \xrightarrow{g(\lambda)} Q^{\top}_{g(\lambda)} := \operatorname{res}_{\lambda \in D_0} g(\lambda) Q^{\top} \frac{1}{\lambda - \Omega}, \end{aligned} \tag{13}$$

where $\Omega \in M_{n,n}$ is some matrix whose spectrum $\sigma(\Omega) \subset D_0$.

Denote $\mathcal{A}_+(\lambda)$ the Lie algebra of the Lie group $A_+(\lambda)$, and put

$$\mathcal{A}_+(\lambda) = \left\{ \sum_{j \in \mathbf{Z}_+} a_j \lambda^j : \ a_j \in \mathrm{sl}(2; \mathbf{R}), \ j \in \mathbf{Z}_+ \right\}. \tag{14}$$

The group action (13) as one can easily verify is Poissonian, leaving the symplectic structure (12) invariant. Thus if a one parametric subgroup $\{\exp(a(\lambda)t) : a(\lambda) \in \mathcal{A}_+(\lambda),\ t \in \mathbf{R}\}$ acts on M, the corresponding Hamiltonian function comes as follows:

$$H_a = -\operatorname{res}_{\lambda \in D_0} \mathrm{Sp}\left(Q^{\top} \frac{1}{\lambda - \Omega} F a(\lambda) \right) := -2 \langle l(F, Q; \lambda), a(\lambda) \rangle_r, \tag{15}$$

where

$$l(F, Q; \lambda) := \frac{1}{2} Q^{\top} \frac{1}{\lambda - \Omega} F \tag{16}$$

is the momentum mapping [1, 3] and $\langle \cdot, \cdot \rangle_r$, $r \in \mathbf{Z}$, is a scalar product on $\mathcal{A}(\lambda, \lambda^{-1})$ defined by the expression:

$$\langle l(\lambda), a(\lambda) \rangle_r := \operatorname{res}_{\lambda \in D_0} \lambda^{-r} \mathrm{Sp}(l(\lambda) a(\lambda)). \tag{17}$$

It is easy to verify that the momentum mapping $l : M \to \mathcal{A}^*_+(\lambda)$ defined by (17) is equivariant [1], that is the diagram

$$\begin{array}{ccc} M & \xrightarrow{l} & \mathcal{A}^*_+(\lambda) \\ g(\lambda) \downarrow & & \downarrow \mathrm{Ad}^*_{g^{-1}(\lambda)} \\ M & \xrightarrow{l} & \mathcal{A}^*_+(\lambda) \end{array} \tag{18}$$

is commutative for all $g(\lambda) \in \mathcal{A}^*_+(\lambda)$, meaning that the loop group $\mathcal{A}_+(\lambda)$ action on M is Hamiltonian.

Define now a Lie algebras homomorphism

$$\alpha : \mathcal{A}_+(\lambda) \to \mathcal{G}_+(\lambda) \subset \lambda^2 \mathcal{A}_+(\lambda) \oplus \sigma_+ \mathbf{R}, \tag{19}$$

where for any $a(\lambda) \in \mathcal{A}_+(\lambda)$

$$\alpha(a)(\lambda) := \lambda^2 a(\lambda) \oplus a_{21}^{(0)} \sigma_+ \tag{20}$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

being a $sl(2, \mathbf{R})$ matrix basis.

It is verified that the mapping (19) is a homomorphism and the image $\alpha \mathcal{A}_+(\lambda) := := \mathcal{G}_+(\lambda)$ constitutes a Lie algebra over $\mathbf{R}$. Thus there exists a loop group $\mathcal{G}_+(\lambda)$ whose Lie algebra coincides with this Lie algebra $\mathcal{G}_+(\lambda)$. Thereby one can define now another loop group $\mathcal{G}_+(\lambda)$-action on M defined by the formulas (13) but with an element $g(\lambda) \in A_+(\lambda)$ replaced by an element $\alpha g(\lambda) \in G_+(\lambda)$, where $\alpha : A_+(\lambda) \rightarrow \rightarrow G_+(\lambda)$ is the corresponding to the mapping (19) loop groups homomorphism.

Therefore, similarly to (16) one finds a momentum mapping $l_\alpha : M \rightarrow \mathcal{G}_+^*(\lambda)$ with respect to the modified loop group action $G_+(\lambda) \times M \overset{\alpha}{\rightarrow} M$ equivalent to that of $A_+(\lambda) \times M \rightarrow M$.

A simple calculation yields

$$l_\alpha(F, Q; \lambda) = l(F, Q; \lambda) + \lambda l_{12}^{(0)} \sigma^+, \tag{21}$$

where, by definition,

$$l := \sum_{j \in \mathbf{Z}_+} l^{(j)} \lambda^{-(j+1)}.$$

When deriving (21) we based on the Hamiltonian function expression

$$H_a^\alpha = -2 \langle l_\alpha(F, Q; \lambda), \alpha(a)(\lambda) \rangle_{-2} \tag{22}$$

generated by a one parametric subgroup

$$\{\exp(\alpha a(\lambda) t) \in G_+(\lambda) : a(\lambda) \in \mathcal{A}_+(\lambda),\ t \in \mathbf{R}\}$$

and made use of the properties

$$\mathrm{Sp}(\sigma_\pm \sigma^\pm) = 1, \quad \mathrm{Sp}(\sigma_- \sigma^+) = 0 = \mathrm{Sp}(\sigma_+ \sigma^-)$$

for the dual bi-orthogonal bases $\{\sigma^\pm, \sigma^0\} \in sl^*(2; \mathbf{R})$.

Notice now that the element $\eta := \lambda^2 \sigma^+ - 2\sigma^- \in \mathcal{G}^*(\lambda, \lambda^{-1})$ is an infinitesimal character of the Lie subalgebra $\mathcal{G}_+(\lambda)$, where by definition $\mathcal{G}(\lambda, \lambda^{-1}) := \mathcal{G}_+(\lambda) \oplus \oplus\, \mathcal{G}_-(\lambda)$ and

$$\langle \eta, [\mathcal{G}_+(\lambda), \mathcal{G}_+(\lambda)] \rangle_{-2} = 0 = \langle \eta, \mathcal{G}_-(\lambda) \rangle_{-2}. \tag{23}$$

Owing to the property (23) and AKS-theorem [9 – 11], the extended momentum mapping

$$S(F, Q; \lambda) := \lambda^2 \sigma^+ - 2\sigma^- + l_\alpha(F, Q; \lambda) \tag{24}$$

generates on the manifold M an involutive with respect to (12) invariants $\gamma_j \in \mathcal{D}(M)$, $j = \overline{-1, n}$, via the expression:

$$\det S(F, Q; \lambda) = -\lambda^2 + \lambda \gamma_{-1} + \gamma_0 + \sum_{j=1}^{n} \frac{\gamma_j}{\lambda - \Omega_j}, \tag{25}$$

where we have put for definiteness $\Omega := \mathrm{diag}\,\{\Omega_j \in \mathbf{R}/\{0\} : j = \overline{1, n}\}$,

$$Q := Fh, \quad h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad F := \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}^{\mathrm{T}} \in M_{n,2}.$$

As a result of simple calculation one finds from (25) that

$$\gamma_j = -\frac{1}{2}p_j^2 + \frac{1}{4}\langle q, \Omega q\rangle q_j^2 - \langle q, q\rangle \Omega_j^2 q_j^2 + \frac{1}{4}\sum_{k \neq j=1}^{n} \frac{(p_j q_k - p_k q_j)^2}{\Omega_j - \Omega_k}, \qquad (26)$$

where $j = \overline{1, n}$, and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbf{R}^n$.

The corresponding symplectic structure (12) turns into the following canonical one: $\omega^{(2)}(F, Q) = 2\omega^{(2)}(q, p)$, where

$$\omega^{(2)}(q, p) := \sum_{j=1}^{n} dp_j \wedge dq_j.$$

Thus all Hamiltonian flows generated by invariants (26) on the space $M \simeq T^*(\mathbf{R}^n)$ are Liouville – Arnold integrable by quadratures since $\{\gamma_j, \gamma_k\} = 0$ for all $j, k = \overline{-1, n}$.

In particular for the Hamiltonian function

$$H := \sum_{j=1}^{n} \Omega_j \gamma_j$$

the corresponding dynamical system on $T^*(\mathbf{R}^n)$ is given as follows:

$$\frac{dq_j}{dx} = p_j, \quad \frac{dp_j}{dx} + \Omega_j^2 q_j - \Omega_j q_j \langle q, q\rangle = q_j\left(\langle q, \Omega q\rangle - \frac{3}{4}\langle q, q\rangle^3\right), \qquad (27)$$

where $j = \overline{1, n}$.

Conclusion. Similar to (27) oscillatory equations constrained to live on the cotangent space $T^*(\mathbf{S}^{n-1})$ to the unit sphere $\mathbf{S}^{n-1} = \{q \in \mathbf{R}^n;\ \langle q, q\rangle = 1\}$ were for the first time derived and studied in detail in [6, 15, 16], having been based exclusively on the algebraic-geometric techniques [17]. Later on these results where rederived in [6, 18] from the Lie-algebraic viewpoint devised in [7].

One can show straightforwardly based on techniques of [17] that the extended momentum mapping (24) satisfies the following dynamical r-matrix identity:

$$\{S(q, p; \lambda), \otimes S(q, p; \mu)\} = [r_{12}(\lambda, \mu), S(q, p; \lambda) \otimes \mathbf{I}] - [r_{21}(\lambda, \mu), \mathbf{I} \otimes S(q, p; \mu)], \qquad (28)$$

where $r_{21}(\lambda, \mu) := r_{12}(\mu, \lambda)$ and

$$r_{12}(\lambda, \mu) = \frac{P}{\lambda - \mu} - (\langle q, q\rangle - \lambda - \mu)\sigma_- \otimes \sigma^+, \qquad (29)$$

$Px \otimes y := y \otimes x$ for any $x, y \in \mathbf{R}^2$ and all $\lambda \neq \mu \in \mathbf{C}$. There is an important problem of deriving this r-matrix (29) from the pure Lie-algebraic viewpoint as it was done in [19] subject to the Calogero type models.

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