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ON THE SPACE OF SEQUENCES OF p-BOUNDED VARIATION AND RELATED MATRIX MAPPINGS

ПРО ПРОСТІР ПОСЛІДОВНОСТЕЙ p-ОБМЕЖЕНОЇ ВАРІАЦІЇ ТА ПОВ'ЯЗАНИХ МАТРИЧНИХ ВІДОБРАЖЕНЬ

The difference sequence spaces $\ell_\infty(\triangle)$, $c(\triangle)$ and $c_0(\triangle)$ were studied by Kızmaz. The main purpose of the present paper is to introduce the space bv_p consisting all the sequences whose differences are in the space ℓ_p, and is to fill up the gap in the existing literature. Moreover, it is proved that the space bv_p is the BK-space including the space ℓ_p and also showed that the spaces bv_p and ℓ_p are linearly isomorphic for $1 \le p \le \infty$. Furthermore, the basis and the α-, β- and γ-duals of the space bv_p have been determined and some inclusion relations have been given. The last section of the paper has been devoted to theorems on the characterizations of the matrix classes $(bv_p : \ell_\infty)$, $(bv_\infty : \ell_p)$ and $(bv_p : \ell_1)$, and the characterizations of some other matrix classes have been obtained by means of a suitable relation.

Різницева послідовність просторів $\ell_\infty(\triangle)$, $c(\triangle)$ та $c_0(\triangle)$ була вивчена Кізмазом. Головною метою даної статті є введення простору bv_p, що складається із послідовностей, різниці яких належать простору ℓ_p, а також заповнення прогалин в існуючій науковій літературі. Крім того, доведено, що простір bv_p є BK-простором, який включає простір ℓ_p, а також показано, що простори bv_p та ℓ_p є лінійно ізоморфними для $1 \le p \le \infty$. Визначено базис та α-, β- і γ-дуальні простори для bv_p та наведено деякі співвідношення включення. В останньому пункті наведено теореми про характеризацію матричних класів $(bv_p : \ell_\infty)$, $(bv_\infty : \ell_p)$ і $(bv_p : \ell_1)$. За допомогою відповідного співвідношення отримано характеризацію деяких інших матричних класів.

1. Preliminaries, background and notation. By w, we shall denote the space of all real valued sequences. Any vector subspace of w is called as a *sequence space*. We shall write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

For the sequence spaces λ and μ, define the set $S(\lambda,\ \mu)$ by

$$S(\lambda,\ \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{1.1}$$

With the notation of (1.1), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda,\ \ell_1),\ \ \lambda^\beta = S(\lambda,\ cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda,\ bs).$$

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \left\| x - (\alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n) \right\| = 0$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. A sequence space λ with a linear topology is called a K-*space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the complex field and the set of natural numbers, respectively.

A K-space λ is called an FK-*space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-*space* (see Choudhary and Nanda [1, p. 272,273]).

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,\ k \in \mathbb{N}$. Then, the matrix A defines a transformation from λ into μ, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, exists and is in μ, where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \tag{1.2}$$

We shall assume throughout that $p^{-1} + q^{-1} = 1$ for $p,\ q \geq 1$ and write for brevity that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} a_{nj}, \qquad a(n,\ k) = \sum_{j=0}^{n} a_{jk}$$

for all $n,\ k \in \mathbb{N}$, and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. We will also use the similar notations with other letters and use the convention that any term with negative subscript is equal to naught.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [2], Ng and Lee [3], Malkowsky [4] and Altay and Başar [5]. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and c_{N_q} in [2], $(\ell_p)_{C_1} = X_p$ in [3], $(\ell_\infty)_{R_t} = r^t_\infty$, $c_{R_t} = r^t_c$ and $(c_0)_{R_t} = r^t_0$ in [4], and $(\ell_p)_{E_r} = e^r_p$ in [5]; where $1 \leq p \leq \infty$ and N_q, C_1, R_q and E_r denote the Nörlund means, Cesàro means of order 1, Riesz means and Euler means of order r, respectively. The main purpose of the present paper, following [2–5], is to introduce the space bv_p of sequences of p-bounded variation and is to derive some related results which fill up the gap in the existing literature. Furthermore, we have constructed one basis and determined the α-, β- and γ-duals of the space bv_p. Besides this, we have essentially characterized the matrix classes $(bv_p : \ell_\infty)$, $(bv_\infty : \ell_p)$ and $(bv_p : \ell_1)$, and also derived the characterizations of some other classes by means of a suitable relation.

2. The space bv_p of sequences of p-bounded variation. The difference spaces $\ell_\infty(\triangle)$, $c(\triangle)$ and $c_0(\triangle)$ were firstly defined and studied by Kızmaz in [6]. The new sequence space λ_A generated by the limitation matrix A from the space λ either includes the space λ or is included by the space λ, in general, i.e., the space λ_A is the expansion or the contraction of the original space λ. Although, in the existing literature, the matrix domain $\lambda_\triangle$ is called as *the difference sequence space* whenever λ is a normed or paranormed sequence space, in the case $\lambda = \ell_p$ we prefer calling this difference sequence space as *the space of all sequences of p-bounded variation* and denote it by bv_p instead of the usual notation $\ell_p(\triangle)$, where $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined by

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n; \\ 0, & 0 \leq k < n-1 \text{ or } k > n. \end{cases}$$

We treat slightly more different than Kızmaz and the other authors following him, and employ the technique obtaining a new sequence space by the matrix domain of a triangle limitation method. Following this way, we introduce the sequence space bv_p as the set of all sequences such that $\triangle$-transforms of them are in the space ℓ_p, that is

$$bv_p = \left\{ x = (x_k) \in w : \sum_k \left| x_k - x_{k-1} \right|^p < \infty \right\}, \ 1 \le p < \infty,$$

and

$$bv_\infty = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k - x_{k-1}| < \infty \right\}.$$

With the notation of (1.2), we may redefine the space bv_p as

$$bv_p = (\ell_p)_\triangle, \ 1 \le p \le \infty. \tag{2.1}$$

It is obvious that the space bv_p is reduced to the spaces bv and $\ell_\infty(\triangle)$ in the cases $p = 1$ and $p = \infty$, respectively.

Define the sequence $y = (y_k)$, which will be frequently used, as the $\triangle$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = (\triangle x)_k = \begin{cases} x_0, & k = 0; \\ x_k - x_{k-1}, & k \ge 1. \end{cases} \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The set bv_p becomes a linear space with the coordinatewise addition and scalar multiplication which is the BK-space with the norm $\| x \|_{bv_p} = \| \triangle x \|_{\ell_p}$, where $1 \le p \le \infty$.*

Proof. The first part of the theorem is a routine verification and so we omit it. Furthermore, since (2.1) holds and ℓ_p, ℓ_∞ are the BK-spaces with respect to their natural norms (see Maddox [7, p. 217, 218]) and $\triangle_{nn} \ne 0$, $\triangle_{nk} = 0$, $k > n$, for all n, $k \in \mathbb{N}$, Theorem 4.3.2 of Wilansky [8, p. 61] gives the fact that the space bv_p is a BK-space, where $1 \le p \le \infty$.

Therefore, one can easily check that the absolute property does not hold on the space bv_p, that is $\|x\|_{bv_p} \ne \||x|\|_{bv_p}$ for at least one sequence in the space bv_p, and this says that bv_p is a sequence space of nonabsolute type; where $|x| = (|x_k|)$ and $1 \le p \le \infty$.

Theorem 2.2. *The space bv_p of sequences of p-bounded variation of nonabsolute type is linearly isomorphic to the space ℓ_p, i.e., $bv_p \cong \ell_p$; where $1 \le p \le \infty$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces bv_p and ℓ_p for $1 \le p \le \infty$. Consider the transformation T defined, with the notation of (2.2), from bv_p to ℓ_p by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in \ell_p$ for $1 \le p \le \infty$ and define the sequence $x = (x_k)$ by $x_k = \sum_{j=0}^k y_j$, $k \in \mathbb{N}$. Then, we respectively get in the cases of $1 \le p < \infty$ and $p = \infty$ that

$$\|x\|_{bv_p} = \left(\sum_k |x_k - x_{k-1}|^p \right)^{1/p} = \left(\sum_k |y_k|^p \right)^{1/p} = \|y\|_{\ell_p} < \infty$$

and

$$\|x\|_{bv_\infty} = \sup_{k\in\mathbb{N}} |\Delta x_k| = \|y\|_{\ell_\infty} < \infty.$$

Thus, we have that $x \in bv_p$. Consequently T is surjective and is norm preserving, where $1 \le p \le \infty$. Hence T is a linear bijection which therefore says us that the spaces bv_p and ℓ_p are linearly isomorphic for $1 \le p \le \infty$, as was desired.

One may expect the similar result for the space bv_p as was observed for the space ℓ_p, and ask the natural question: Isn't the space bv_p a Hilbert space with $p \neq 2$? The answer is positive and is given by the following theorem.

Theorem 2.3. *Except the case $p = 2$, the space bv_p is not an inner product space, hence not a Hilbert space for $1 \le p < \infty$.*

Proof. We wish to prove that the space bv_2 is the only Hilbert space among the bv_p spaces for $1 \le p < \infty$, firstly. Since the space bv_2 is a BK-space with the norm $\| x \|_{bv_2} = \| \Delta x \|_{\ell_2}$ by Theorem 2.1 and its norm can be obtained from an inner product, i.e.,

$$\| x \|_{bv_2} = \langle \Delta x, \ \Delta x \rangle^{1/2}$$

holds, the space bv_2 is a Hilbert space.

Let us now consider the sequences $u = (u_k)$ and $e^{(0)}$ given by

$$u_k = \begin{cases} 1, & k = 0; \\ 2, & k \ge 1, \end{cases} \quad \text{and} \quad e^{(0)} = (1, \ 0, \ 0, \ldots).$$

Then, we see that

$$\|u + e^{(0)}\|_{bv_p}^2 + \|u - e^{(0)}\|_{bv_p}^2 = 8 \neq 4\left(2^{2/p}\right) = 2\left(\| u \|_{bv_p}^2 + \|e^{(0)}\|_{bv_p}^2\right), \quad p \neq 2,$$

i.e., the norm of the space bv_p does not satisfy the parallelogram equality which means that the norm cannot be obtained from an inner product. Hence, the space bv_p with $p \neq 2$ is a Banach space which is not a Hilbert space. This completes the proof.

We wish to derive some inclusion relations concerning with the space bv_p.

Theorem 2.4. *The inclusion $\ell_p \subset bv_p$ strictly holds for $1 \le p \le \infty$.*

Proof. To prove the validity of the inclusion $\ell_p \subset bv_p$ for $1 \le p \le \infty$, it suffices to show the existence of a number $K > 0$ such that $\|x\|_{bv_p} \le K\|x\|_{\ell_p}$ for every $x \in \ell_p$.

Let $x \in \ell_p$ and $1 < p \le \infty$. Then we obtain, with the notation of (2.2),

$$\sum_k |\Delta x_k|^p \le \sum_k 2^{p-1}\left(|x_k|^p + |x_{k-1}|^p\right) \le 2^{p-1}\left(\sum_k |x_k|^p + \sum_k |x_{k-1}|^p\right)$$

and

$$\sup_{k\in\mathbb{N}} |\Delta x_k| \le 2 \sup_{k\in\mathbb{N}} |x_k|$$

which together yield us as was expected that

$$\|x\|_{bv_p} \le 2\|x\|_{\ell_p} \tag{2.3}$$

for $1 < p \leq \infty$. Besides this, since the sequences $e = (1, 1, \ldots)$ and $x = (x_k) = (k)$ are respectively in $bv_p - \ell_p$ and $bv_\infty - \ell_\infty$, the inclusion $\ell_p \subset bv_p$ is strict for $1 < p \leq \infty$. By the similar discussions, it may be easily proved that the inequality (2.3) also holds in the case $p = 1$ and so we omit the detail. This completes the proof.

Theorem 2.5. *Neither of the spaces bv_p and ℓ_∞ includes the other one, where $1 < p < \infty$.*

Proof. Let us consider the sequences $u = (u_k)$ and $x = (x_k)$ defined by $u_k = \sum_{i=0}^{k} 1/(i+1)$ and $x_k = (-1)^k$ for all $k \in \mathbb{N}$, respectively. Then, since $\triangle u = \{1/(k+1)\} \in \ell_p$ which gives that u is in bv_p but not in ℓ_∞. Nevertheless, x is in ℓ_∞ but not in bv_p. Hence, the sequence spaces bv_p and ℓ_∞ overlap but neither contains the other. This completes the proof.

Theorem 2.6. *If $1 \leq p < s$, then $bv_p \subset bv_s$.*

Proof. Suppose that $1 \leq p < s$ and $x \in bv_p$. With the notation of (2.2), Theorem 2.2 implies that $y \in \ell_p$. Then, the well-known inclusion $\ell_p \subset \ell_s$ yields the fact that $y \in \ell_s$. This means that $x \in bv_s$ and hence the inclusion $bv_p \subset bv_s$ holds, as was asserted.

3. The basis for the space bv_p. In the present section, we will give a sequence of the points of the space bv_p which forms a basis for the space bv_p; where $1 \leq p < \infty$.

Theorem 3.1. *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of the elements of the space bv_p for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} 0, & n < k; \\ 1, & n \geq k. \end{cases} \tag{3.1}$$

Then the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space bv_p and any $x \in bv_p$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}, \tag{3.2}$$

where $\lambda_k = (\triangle x)_k$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$.

Proof. It is clear that $\{b^{(k)}\} \subset bv_p$, since

$$\triangle b^{(k)} = e^{(k)} \in \ell_p, \quad k = 0, 1, 2, \ldots, \tag{3.3}$$

for $1 \leq p < \infty$; where $e^{(k)}$ is the sequence whose only nonzero term is a 1 in k-th place for each $k \in \mathbb{N}$.

Let $x \in bv_p$ be given. For every nonnegative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k b^{(k)}. \tag{3.4}$$

Then, by applying the difference operator $\triangle$ to (3.4), we obtain with (3.3) that

$$\triangle x^{[m]} = \sum_{k=0}^{m} \lambda_k \triangle b^{(k)} = \sum_{k=0}^{m} (\triangle x)_k e^{(k)}$$

and

$$\left\{\triangle(x - x^{[m]})\right\}_i = \begin{cases} 0, & 0 \leq i \leq m; \\ (\triangle x)_i, & i > m. \end{cases} \quad i, m \in \mathbb{N}.$$

Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} \left|(\triangle x)_i\right|^p\right]^{1/p} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$\left\|x - x^{[m]}\right\|_{bv_p} = \left[\sum_{i=m}^{\infty} \left|(\triangle x)_i\right|^p\right]^{1/p} \leq \left[\sum_{i=m_0}^{\infty} \left|(\triangle x)_i\right|^p\right]^{1/p} < \frac{\varepsilon}{2} < \varepsilon$$

for all $m \geq m_0$ which proves that $x \in bv_p$ is represented as in (3.2).

Let us show that the uniqueness of the representation for $x \in bv_p$ given by (3.2). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k b^{(k)}$. Since the linear transformation T, from bv_p to ℓ_p, used in Theorem 2.2 is continuous we have at this stage that

$$(\triangle x)_n = \sum_k \mu_k \left\{\triangle b^{(k)}\right\}_n = \sum_k \mu_k e_n^{(k)} = \mu_n, \ \ n \in \mathbb{N},$$

which contradicts the fact that $(\triangle x)_n = \lambda_n$ for all $n \in \mathbb{N}$. Hence, the representation (3.2) of $x \in bv_p$ is unique. Thus the theorem is proved.

4. The α-, β- and γ-duals of the space bv_p. In this section, we state and prove the theorems determining the α-, β- and γ-duals of the space bv_p. Because of the case $p = 1$ may be proved in the similar fashion and found in the literature, we omit the proof of that case and consider only the case $1 < p \leq \infty$ in the proof of Theorems 4.1 and 4.2, respectively.

We shall begin with to quote the lemmas, due to Sieglitz and Tietz [9], which are needed in proving Theorems 4.1–4.3, below.

Lemma 4.1. *$A \in (\ell_p : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_k \left|\sum_{n \in K} a_{nk}\right|^q < \infty, \ \ 1 < p \leq \infty.$$

Lemma 4.2. *$A \in (\ell_p : c)$ if and only if*

$$\lim_{n \to \infty} a_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{4.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty, \ \ 1 < p < \infty. \tag{4.2}$$

Lemma 4.3. *$A \in (\ell_\infty : c)$ if and only if (4.1) holds, and*

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k \left|\lim_{n \to \infty} a_{nk}\right|. \tag{4.3}$$

Lemma 4.4. *$A \in (\ell_p : \ell_\infty)$ if and only if (4.2) holds with $1 < p \leq \infty$.*

Theorem 4.1. *Define the set a_q as follows:*

$$a_q = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_n \right|^q < \infty \right\}.$$

Then, $(bv_p)^\alpha = a_q$, where $1 < p \leq \infty$.

Proof. Let us define the matrix B whose rows are the product of the rows of the matrix $\triangle^{-1}$ and the sequence $a = (a_n)$, that is to say that

$$B_n = (\triangle^{-1})_n a,$$

where B_n and $(\triangle^{-1})_n$ denote the sequences in the n^{th} rows of the matrices B and $\triangle^{-1}$, respectively. Bearing in mind the relation (2.2) we easily obtain that

$$a_n x_n = \sum_{k=0}^{n} a_n y_k = (By)_n, \quad n \in \mathbb{N}. \tag{4.4}$$

We therefore observe by (4.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in bv_p$ if and only if $By \in \ell_1$ whenever $y \in \ell_p$. Then we derive by Lemma 4.1 that

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_n \right|^q < \infty$$

which yields the consequence that $(bv_p)^\alpha = a_q$.

Theorem 4.2. *Define the sets d_q and d by*

$$d_q = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{n} a_j \right|^q < \infty \right\}$$

and

$$d = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k \left| \sum_{j=k}^{n} a_j \right| = \sum_k \left| \sum_{j=k}^{\infty} a_j \right| < \infty \right\}.$$

Then, $(bv_p)^\beta = d_q$ and $(bv_\infty)^\beta = d$; where $1 < p < \infty$.

Proof. Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} y_j \right) = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} a_j \right) y_k = (Cy)_n, \tag{4.5}$$

where $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j, & 0 \leq k \leq n; \\ 0, & k > n, \end{cases} \quad n,\ k \in \mathbb{N}. \tag{4.6}$$

Thus we deduce from Lemma 4.2 with (4.5) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in bv_p$ if and only if $Cy \in c$ whenever $y = (y_k) \in \ell_p$. Thus, $(a_k) \in cs$ and

$(a_k) \in d_q$ by (4.1) and (4.2), respectively. Nevertheless the inclusion $d_q \subset cs$ holds, thus we have that $(a_k) \in d_q$ which gives us $(bv_p)^\beta = d_q$.

It is of course that one may also prove the case $p = \infty$ by using the technique in proving the case $1 < p < \infty$ with Lemma 4.3 instead of Lemma 4.2. So, we leave the detailed proof to the reader.

Theorem 4.3. *$(bv_p)^\gamma = d_q$, where $1 < p \leq \infty$.*

Proof. This may be obtained in the similar way in the proof of Theorem 4.2 with Lemma 4.4 instead of Lemma 4.2. So, we omit the detail.

Before giving our corollary on the monotonicity of the space bv_p, we give a definition and a lemma concerning with the perfectness, normality and monotonicity of a sequence space (see [10, p. 48, 52]), below.

Definition 4.1. *Let λ be a sequence space. Then, λ is called*

(i) *perfect if $\lambda = \lambda^{\alpha\alpha}$;*

(ii) *normal if $y \in \lambda$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x \in \lambda$;*

(iii) *monotone provided λ contains the canonical preimages of all its stepspace.*

Lemma 4.5. *Let λ be a sequence space. Then, we have:*

(i) *λ is perfect $\Rightarrow$ λ is normal $\Rightarrow$ λ is monotone;*

(ii) *λ is normal $\Rightarrow \lambda^\alpha = \lambda^\gamma$;*

(iii) *λ is monotone $\Rightarrow \lambda^\alpha = \lambda^\beta$.*

Combining Lemma 4.5 and Theorems 4.1 – 4.3, we get the following corollary.

***Corollary* 4.1.** *The space bv_p is not monotone and so it is neither normal nor perfect.*

5. Certain matrix mappings related to the sequence space bv_p. In this section, we desire to characterize the matrix mappings from the sequence space bv_p to some of the known sequence spaces and to the itself. We directly prove the theorems characterized the classes $(bv_p : \ell_\infty)$, $(bv_\infty : \ell_p)$ and $(bv_p : \ell_1)$, and derive the other characterizations from them by using the suitable relation between the concerning matrix classes. We shall begin with a lemma due to Wilansky [8, p. 57] which is needed in the proof of Theorem 5.1, below.

Lemma 5.1. *The matrix mappings between the BK-spaces are continuous.*

Theorem 5.1. *Let $1 < p < \infty$. Then, $A \in (bv_p : \ell_\infty)$ if and only if*

$$\sup_{m \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{m} a_{nj} \right|^q < \infty, \quad n \in \mathbb{N}, \tag{5.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \tilde{a}_{nk} \right|^q < \infty. \tag{5.2}$$

Proof. Let $A \in (bv_p : \ell_\infty)$ and $1 < p < \infty$. Then, Ax exists and is in ℓ_∞ for all $x \in bv_p$. This leads us to the fact that $\{a_{nk}\}_{k \in \mathbb{N}} \in d_q$ for all $n \in \mathbb{N}$ which shows the necessity of (5.1).

Suppose that the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (2.2). Then, since $x_k = \sum_{j=0}^{k} \triangle x_j$ holds for every $k \in \mathbb{N}$, we obtain the equality

$$\sum_k a_{nk}x_k = \sum_k a_{nk}\left(\sum_{j=0}^{k} \triangle x_j\right) = \sum_j \sum_{k=j}^{\infty} a_{nk}\triangle x_j = \sum_j \tilde{a}_{nj}y_j\,, \quad n \in \mathbb{N}. \tag{5.3}$$

Since bv_p and ℓ_∞ are the BK-spaces, there exists some real constant $K > 0$ by Lemma 5.1 such that

$$\|Ax\|_{\ell_\infty} \leq K\|x\|_{bv_p}$$

for all $x \in bv_p$. Therefore, by the Hölder's inequality we derive from (5.3) that

$$\frac{\|Ax\|_{\ell_\infty}}{\|y\|_{\ell_p}} = \sup_{n\in\mathbb{N}} \frac{\left|\sum_k \tilde{a}_{nk}y_k\right|}{\|y\|_{\ell_p}} \leq \sup_{n\in\mathbb{N}} \frac{\left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}\left(\sum_k |y_k|^p\right)^{1/p}}{\|y\|_{\ell_p}} =$$

$$= \sup_{n\in\mathbb{N}}\left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q} < \infty$$

for all $x \in bv_p$ and this proves the necessity of (5.2).

Conversely, suppose the conditions (5.1) and (5.2) hold, and take any $x \in bv_p$. Then, the sequence $\{a_{nk}\}_{k\in\mathbb{N}} \in d_q$ for all $n \in \mathbb{N}$ and this implies the existence of A-transform of x. Therefore, taking into account the fact $y = (y_k) \in \ell_p$ by Theorem 2.2, we again obtain by applying the Hölder's inequality to (5.3) that

$$\|Ax\|_{\ell_\infty} = \sup_{n\in\mathbb{N}}\left|\sum_k \tilde{a}_{nk}y_k\right| \leq \sup_{n\in\mathbb{N}}\left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}\left(\sum_k |y_k|^p\right)^{1/p} < \infty$$

which means that $A \in (bv_p : \ell_\infty)$. This step concludes the proof.

We wish to give a lemma concerning the characterization of the class $(\ell_\infty : \ell_p)$ which is needed in proving the next theorem and due to Stiglitz and Tietz [9].

Lemma 5.2. *$A \in (\ell_\infty : \ell_p)$ if and only if*

$$\sup_{F\in\mathcal{F}} \sum_n \left|\sum_{k\in F} a_{nk}\right|^p < \infty, \quad 1 \leq p < \infty. \tag{5.4}$$

Theorem 5.2. *$A \in (bv_\infty : \ell_p)$ if and only if*

$$\lim_{m\to\infty} \sum_k \left|\sum_{j=k}^{m} a_{nj}\right| = \sum_k |\tilde{a}_{nk}|\,, \quad n \in \mathbb{N}, \tag{5.5}$$

$$\sup_{F\in\mathcal{F}} \sum_n \left|\sum_{k\in F} \tilde{a}_{nk}\right|^p < \infty, \quad 1 \leq p < \infty, \tag{5.6}$$

$$\sup_{n\in\mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty, \quad p = \infty. \tag{5.7}$$

Proof. Let $A \in (bv_\infty : \ell_p)$ and $1 \le p \le \infty$. Then, Ax exists and is in ℓ_p for every $x \in bv_\infty$. This leads us to the necessities of the conditions (5.6) and (5.7) with the sequences x_F and $x = (x_k)$ which are defined by

$$x_F = \sum_{k \in F} b^{(k)} \text{ and } x_k = k+1\,, \quad F \in \mathcal{F},\ k \in \mathbb{N},$$

respectively, where $b^{(k)}$ is defined by (3.1). The necessity of the condition (5.5) in the cases both $1 \le p < \infty$ and $p = \infty$ is immediate, since $\{a_{nk}\}_{k \in \mathbb{N}} \in d$ for all $n \in \mathbb{N}$.

Conversely, suppose the conditions (5.5) and (5.6) hold, and take any $x \in bv_\infty$. Then, Ax exists. Since $x \in bv_\infty$ if and only if $y \in \ell_\infty$ by Theorem 2.2, we therefore have from (5.3) by Lemma 5.2 that $Ax \in \ell_p$ whenever $By \in \ell_p$, where $b_{nk} = \tilde{a}_{nk}$ for all $n,\ k \in \mathbb{N}$. This proves the sufficiency of (5.5) and (5.6).

Let us finally suppose that the conditions (5.5) and (5.7) hold and take any $x \in bv_\infty$. Then, Ax exists, again by (5.3) one can easily see that

$$\|Ax\|_{\ell_\infty} = \sup_{n \in \mathbb{N}} \left| \sum_k \tilde{a}_{nk} y_k \right| \le \|y\|_{\ell_\infty} \cdot \sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty$$

which shows the sufficiency of (5.5) and (5.7), and this completes the proof.

Now, we may give the theorem characterizing the class $(bv_p : \ell_1)$ whose proof is similar to the proof of Theorem 5.1, above and so we leave the detail to the reader.

Theorem 5.3. *$A \in (bv_p : \ell_1)$ if and only if (5.1) holds, and*

$$\sup_{F \in \mathcal{F}} \sum_k \left| \sum_{n \in F} \tilde{a}_{nk} \right|^q < \infty\,, \quad 1 < p < \infty. \tag{5.8}$$

Lemma 5.3. *Let $\lambda,\ \mu$ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

Proof. Let B be a triangle matrix, A be an infinite matrix, and $x \in w_A$. Then, we have by Theorem 1.1.4 of Wilansky [8, p. 8] that

$$B(Ax) = (BA)x.$$

This leads us to the desired consequence that $Ax \in \mu_B$ whenever $x \in \lambda$ if and only if $(BA)x \in \mu$ whenever $x \in \lambda$ which is what we wished to prove.

It is trivial that Lemma 5.3 has several consequences some of them give the necessary and sufficient conditions of matrix mappings between the spaces of sequences of p-bounded variation. Indeed, combining the Lemma 5.3 with Theorems 5.1, 5.2 and 5.3 and choosing B as one of the special matrices Δ, E_r, C_1 and R_t one can easily obtain the following results. Let (t_k) be a sequence of non-negative numbers which are not all zero and $T_n = \sum_{k=0}^n t_k$ for all $n \in \mathbb{N}$. Therefore, we have the following corollaries.

***Corollary* 5.1.** (a) *$C \in (bv_p : bv_\infty)$ if and only if (5.1) and (5.2) hold with d_{nk} instead of a_{nk}, where $d_{nk} = c_{nk} - c_{n-1,k}$ for all $n,\ k \in \mathbb{N}$.*

(b) *$C \in (bv_p : e_\infty^r)$ if and only if (5.1) and (5.2) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^n \binom{n}{j}(1-r)^{n-j} r^j c_{jk}$ for all $n,\ k \in \mathbb{N}$.*

(c) $C \in (bv_p : X_\infty)$ *if and only if* (5.1) *and* (5.2) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/(n+1)\ c(n,\ k)$ *for all* $n,\ k \in \mathbb{N}$.

(d) $C \in (bv_p : r^t_\infty)$ *if and only if* (5.1) *and* (5.2) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/T_n \sum_{j=0}^{n} t_j c_{jk}$ *for all* $n,\ k \in \mathbb{N}$.

(e) $C \in (bv_p : bs)$ *if and only if* (5.1) *and* (5.2) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = c(n,\ k)$ *for all* $n,\ k \in \mathbb{N}$.

***Corollary* 5.2.** (a) $C \in (bv_\infty : bv_p)$ *if and only if* (5.5), (5.6) *and* (5.7) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = c_{nk} - c_{n-1,k}$ *for all* $n,\ k \in \mathbb{N}$.

(b) $C \in (bv_\infty : e^r_p)$ *if and only if* (5.5), (5.6) *and* (5.7) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1-r)^{n-j} r^j c_{jk}$ *for all* $n,\ k \in \mathbb{N}$.

(c) $C \in (bv_\infty : X_p)$ *if and only if* (5.5), (5.6) *and* (5.7) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/(n+1)\ c(n,\ k)$ *for all* $n,\ k \in \mathbb{N}$.

(d) $C \in (bv_\infty : r^t_p)$ *if and only if* (5.5), (5.6) *and* (5.7) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/T_n \sum_{j=0}^{n} t_j c_{jk}$ *for all* $n,\ k \in \mathbb{N}$.

***Corollary* 5.3.** (a) $C \in (bv_p : bv)$ *if and only if* (5.1) *and* (5.8) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = c_{nk} - c_{n-1,k}$ *for all* $n,\ k \in \mathbb{N}$.

(b) $C \in (bv_p : e^r_1)$ *if and only if* (5.1) *and* (5.8) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1-r)^{n-j} r^j c_{jk}$ *for all* $n,\ k \in \mathbb{N}$.

(c) $C \in (bv_p : X_1)$ *if and only if* (5.1) *and* (5.8) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/(n+1)\ c(n,\ k)$ *for all* $n,\ k \in \mathbb{N}$.

(d) $C \in (bv_p : r^t_1)$ *if and only if* (5.1) *and* (5.8) *hold with* d_{nk} *instead of* a_{nk}, *where* $d_{nk} = 1/T_n \sum_{j=0}^{n} t_j c_{jk}$ *for all* $n,\ k \in \mathbb{N}$.

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