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LIE-ALGEBRAIC STRUCTURE OF (2+1)-DIMENSIONAL LAX-TYPE INTEGRABLE NONLINEAR DYNAMICAL SYSTEMS

ЛІ-АЛГЕБРАЇЧНА СТРУКТУРА (2+1)-ВИМІРНИХ ІНТЕГРОВНИХ ЗА ЛАКСОМ НЕЛІНІЙНИХ ДИНАМІЧНИХ СИСТЕМ

The Hamiltonian representation for a hierarchy of Lax-type equations on a dual space to the Lie algebra of integral-differential operators with matrix coefficients, extended by evolutions for eigenfunctions and adjoint eigenfunctions of the corresponding spectral problems, is obtained via some special Blacklund transformation. The connection of this hierarchy with Lax-integrable two-metrizable systems is studied.

Знайдено гамільтонове зображення для ієрархії рівнянь типу Лакса на спряженому просторі алгебри Лі інтегро-диференціальних операторів із матричними коефіцієнтами, розширеної еволюцією власних функцій відповідних спектральних задач, за допомогою деякого спеціального перетворення Беклунда. Досліджено зв'язок цієї ієрархії з інтегровними за Лаксом двометризованими системами.

1. Introduction. Since the paper of M. Adler [1], which had been treated an one-dimensional differential operator algebra, there was understood that a wide class of Lax-integrable Korteweg – de Vries-type nonlinear dynamical systems in partial derivatives [2 – 5] could be described by means of Lie-algebraic techniques. Especially, it was shown that all of them are representable as coadjoint orbits of some Lie groups. The analog of a mentioned above construction for a class of matrix affine groups with central extensions was represented in [4, 6–8], where its relations with the momentum mapping and $\mathcal{R}$-matrix approach had been stated. But the extending problem for the Adler's construction in the case of a multi-dimensional differential operator algebra still stands open. Some preliminary results in this directions was obtained by L. Nizhnik [9] and A. Prykarpatsky [10].

In the article we suggest a new approach to the partial solving of this problem based on the notions of a Backlund transformation [4, 11] and a tensor product of Poisson structures on a dual space of an one-differential operator algebra [12, 13]. By use of the invariant Casimir functionals' property under the Backlund transformations we construct a wide class of Lax-integrable (2 + 1)-dimensional dynamical systems and for the first time represent them as a compatibility condition of three some special linear first order differential equations, called here a triple linear Lax-type representation.

2. The general algebraic scheme. Let $\tilde{\mathcal{G}} := C^{\infty}(\mathbb{S}^1; \mathcal{G})$ be a Lie algebra of loops, taking values in a matrix Lie algebra $\mathcal{G}$. By means of $\tilde{\mathcal{G}}$ one constructs a Lie algebra $\hat{\mathcal{G}}$ of matrix integral-differential operators [1, 10]:

$$\hat{a} := \sum_{j << \infty} a_j \xi^j,$$

where the symbol $\xi := \partial/\partial x$ signs the differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$. The usual Lie commutator on $\hat{\mathcal{G}}$ is defined as:

$$[\hat{a}, \hat{b}] := \hat{a} \circ \hat{b} - \hat{b} \circ \hat{a}$$

for all $\hat{a}$, $\hat{b} \in \hat{\mathcal{G}}$, where "∘" is a product of integral-differential operators, taking the form:

$$\hat{a} \circ \hat{b} := \sum_{\alpha \in \mathbb{Z}_+} \frac{1}{\alpha!} \frac{\partial^\alpha \hat{a}}{\partial \xi^\alpha} \frac{\partial^\alpha \hat{b}}{\partial x^\alpha}.$$

On the Lie algebra $\hat{\mathcal{G}}$ there exists the *ad*-invariant nondegenerated symmetric bilinear form:

$$(\hat{a}, \hat{b}) := \int_0^{2\pi} \mathrm{Tr}\,(\hat{a} \circ \hat{b})\, dx, \tag{1}$$

where Tr-operation for all $\hat{a} \in \hat{\mathcal{G}}$ is given by the expression:

$$\mathrm{Tr}\,\hat{a} := \mathrm{res}_\xi\, \mathrm{Sp}\,\hat{a} = \mathrm{Sp}\, a_{-1},$$

and Sp is the matrix trace. With the scalar product (1) the Lie algebra $\hat{\mathcal{G}}$ is transformed into a metrizable one. As a consequence, its dual linear space of matrix integral-differential operators $\hat{\mathcal{G}}^*$ is identified with the Lie algebra, that is $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$.

The linear subspaces $\hat{\mathcal{G}}_+ \subset \hat{\mathcal{G}}$ and $\hat{\mathcal{G}}_- \subset \hat{\mathcal{G}}$ such as

$$\hat{\mathcal{G}}_+ := \left\{ \hat{a} := \sum_{j=0}^{n(\hat{a}) << \infty} a_j \xi^j : a_j \in \tilde{\mathcal{G}},\ j = \overline{0, n(\hat{a})} \right\},$$
$$\hat{\mathcal{G}}_- := \left\{ \hat{b} := \sum_{j=0}^{\infty} \xi^{-(j+1)} b_j : b_j \in \tilde{\mathcal{G}},\ j \in \mathbb{Z}_+ \right\}, \tag{2}$$

are Lie subalgebras in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$. Because of the splitting of $\hat{\mathcal{G}}$ into the direct sum of its Lie subalgebras one can construct a so called Lie – Poisson structure [1, 4, 5, 7, 8] on $\hat{\mathcal{G}}^*$, using the special linear endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$:

$$\mathcal{R} := \frac{P_+ - P_-}{2}, \quad P_\pm \hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm, \quad P_\pm \hat{\mathcal{G}}_\mp = 0.$$

For any smooth by Frechet functionals $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}}^*)$ the Lie–Poisson bracket on $\hat{\mathcal{G}}^*$ is given by the expression:

$$\{\gamma, \mu\}_{\mathcal{R}}(\hat{l}) = \left(\hat{l}, \left[\nabla\gamma(\hat{l}), \nabla\mu(\hat{l})\right]_{\mathcal{R}}\right), \tag{3}$$

where $\hat{l} \in \hat{\mathcal{G}}^*$ and for all $\hat{a}$, $\hat{b} \in \hat{\mathcal{G}}$ the $\mathcal{R}$-commutator has the form [6–8]:

$$\left[\hat{a}, \hat{b}\right]_{\mathcal{R}} := \left[\mathcal{R}\hat{a}, \hat{b}\right] + \left[\hat{a}, \mathcal{R}\hat{b}\right], \tag{4}$$

subject to which the linear space $\hat{\mathcal{G}}$ becames a Lie algebra too. The gradient $\nabla\gamma(\hat{l}) \in \hat{\mathcal{G}}$ of a functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}^*)$ at a point $\hat{l} \in \hat{\mathcal{G}}^*$ with respect to the scalar product (1) is defined as

$$\delta\gamma(\hat{l}) := \left(\nabla\gamma(\hat{l}), \delta\hat{l}\,\right),$$

where the linear space isomorphism $\hat{\mathcal{G}} \simeq \hat{\mathcal{G}}^*$ is taken into account.

The Lie–Poisson bracket (3), associated with the $\mathcal{R}$-commutator (4), generates Hamiltonian dynamical systems on $\hat{\mathcal{G}}$ with Casimir invariants $\gamma \in I(\mathcal{G}^*)$, satisfying the condition:

$$\left[\nabla\gamma(\hat{l}), \hat{l}\,\right] = 0, \tag{5}$$

as the corresponding Hamiltonian functions. Due to the expressions (3) and (5) the mentioned above Hamiltonian system takes the form:

$$\frac{d\hat{l}}{dt} := \left[\mathcal{R}\nabla\gamma(\hat{l}), \hat{l}\right] = \left[\nabla\gamma_+(\hat{l}), \hat{l}\right], \tag{6}$$

being equivalent to the usual commutator Lax-type representation [2–5]. The relationship (6) is a compatibility condition for the linear integral-differential equations:

$$\hat{l}f = \lambda f, \tag{7}$$

$$\frac{df}{dt} = \nabla\gamma_+(\hat{l})f, \tag{8}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and the vector-function $f \in W(\mathbb{S}^1; \mathbb{H})$ is an element of some matrix representation for the Lie algebra $\hat{\mathcal{G}}$ in some functional Banach space $\mathbb{H}$.

Algebraic properties of the equation (6) together with (8) and the associated dynamical system on the space of adjoint functions $f^* \in W^*(\mathbb{S}^1; \mathbb{H})$:

$$\frac{df^*}{dt} = -\left(\nabla\gamma(\hat{l})\right)_+^* f^*, \tag{9}$$

where $f^* \in W^*$ is a solution to the adjoint spectral problem:

$$\hat{l}^* f^* = \nu f^*,$$

being considered as some coupled evolution equations on the space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ is an object of our further investigations.

3. The tensor product of Poisson structures and its Backlund transformation. To compactify the description below we will use the designation of the gradient vector

$$\nabla\gamma(\tilde{l}, \tilde{f}, \tilde{f}^*) := \left(\frac{\delta\gamma}{\delta\tilde{l}}, \frac{\delta\gamma}{\delta\tilde{f}}, \frac{\delta\gamma}{\delta\tilde{f}^*}\right)^T$$

for any smooth functional $\gamma \in \mathcal{D}\left(\hat{\mathcal{G}}^* \oplus W \oplus W^*\right)$. On the spaces $\hat{\mathcal{G}}^*$ and $W \oplus W^*$ there exist canonical Poisson structures [5, 11, 13]

$$\frac{\delta\gamma}{\delta\tilde{l}} :\xrightarrow{\tilde{\theta}} \left[\left(\frac{\delta\gamma}{\delta\tilde{l}}\right)_+, \tilde{l}\right] - \left[\frac{\delta\gamma}{\delta\tilde{l}}, \tilde{l}\right]_+ \tag{10}$$

at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$\left(\frac{\delta\gamma}{\delta\tilde{f}}, \frac{\delta\gamma}{\delta\tilde{f}^*}\right)^T :\xrightarrow{\tilde{J}} \left(\frac{\delta\gamma}{\delta\tilde{f}^*}, -\frac{\delta\gamma}{\delta\tilde{f}}\right)^T \tag{11}$$

at a point $\left(\tilde{f}, \tilde{f}^*\right) \in W \oplus W^*$ correspondingly. It should be noted that the Poisson structure (10) is transformed into (6) for any Casimir functional $\gamma \in I\left(\hat{\mathcal{G}}^*\right)$. Thus, on the extended space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of the structures (10) and (11).

Let us consider the following Backlund transformation [4, 11]:

$$\left(\hat{l}, f, f^*\right) :\xrightarrow{B} \left(\tilde{l}\left(\hat{l}, f, f^*\right), \tilde{f} = f, \tilde{f}^* = f^*\right), \tag{12}$$

generating on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ a Poisson structure Θ with respect to variables $\left(\hat{l}, f, f^*\right)$ of the coupled evolution equations (6), (8), (9).

The main condition for the mapping (12) to be define is the coincidence of the dynamical system

$$\left(\frac{d\hat{l}}{dt}, \frac{df}{dt}, \frac{df^*}{dt}\right)^T := -\Theta\nabla\gamma\left(\hat{l}, f, f^*\right) \tag{13}$$

with (6), (8), (9) in the case of $\gamma \in I(\hat{\mathcal{G}}^*)$, i. e. when the functional γ is taken to be not dependent of variables $(f, f^*) \in W \oplus W^*$. To satisfy that condition, one should find a variation of some Casimir functional $\gamma \in I(\hat{\mathcal{G}}^*)$ at $\delta\tilde{l} = 0$, taking into account flows (8), (9) and the Backlund transformation (12):

$$\begin{aligned}
\delta\gamma\left(\tilde{l}, \tilde{f}, \tilde{f}^*\right)\Big|_{\delta\tilde{l}=0} &= \left(\left\langle\frac{\delta\gamma}{\delta\tilde{f}}, \delta\tilde{f}\right\rangle\right) + \left(\left\langle\frac{\delta\gamma}{\delta\tilde{f}^*}, \delta\tilde{f}^*\right\rangle\right) = \\
&= \left(\left\langle-\frac{d\tilde{f}^*}{dt}, \delta\tilde{f}\right\rangle\right) + \left(\left\langle\frac{d\tilde{f}}{dt}, \delta\tilde{f}^*\right\rangle\right)\Big|_{\tilde{f}=f,\, \tilde{f}^*=f^*} = \\
&= \left(\left\langle\left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+^* f^*, \delta f\right\rangle\right) + \left(\left\langle\left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f, \delta f^*\right\rangle\right) = \\
&= \left(\left\langle f^*, \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ \delta f\right\rangle\right) + \left(\left\langle\left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f, \delta f^*\right\rangle\right) = \\
&= \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta f\xi^{-1} \otimes f^*\right) + \left(\frac{\delta\gamma}{\delta\hat{l}}, f\xi^{-1} \otimes \delta f^*\right) = \\
&= \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta\left(f\xi^{-1} \otimes f^*\right)\right) := \left(\frac{\delta\gamma}{\delta\hat{l}}, \delta\hat{l}\right),
\end{aligned} \tag{14}$$

where $\gamma \in I(\hat{\mathcal{G}}^*)$. As a result of the expression (14) one obtains the relationships:

$$\delta\hat{l}\,\Big|_{\delta\tilde{l}=0} = \delta\left(f\xi^{-1} \otimes f^*\right),$$

or having assumed the linear dependence of $\hat{l}$ and $\tilde{l} \in \hat{\mathcal{G}}^*$ one gets right away that

$$\hat{l} = \tilde{l} + f\xi^{-1} \otimes f^*. \tag{15}$$

Thus, the Backlund transformation (12) can be now written as

$$\left(\hat{l}, f, f^*\right) :\xrightarrow{B} \left(\tilde{l} = \hat{l} - f\xi^{-1} \otimes f^*, f, f^*\right). \tag{16}$$

The expression (16) generalizes the result, obtained in the papers [12, 13] for the Lie algebra $\hat{\mathcal{G}}$ of integral-differential operators with scalar coefficients. The existence of the Backlund transformation (12) makes it possible to formulate the following theorem.

Theorem 1. *The dynamical system on* $\hat{\mathcal{G}}^* \oplus W \oplus W^*$, *being the Hamiltonian with respect to the canonical Poisson structure* $\tilde{\Theta}: T^*\left(\hat{\mathcal{G}}^* \oplus W \oplus W^*\right) \to$ $\to T\left(\hat{\mathcal{G}}^* \oplus W \oplus W^*\right)$ *and generated by the evolution equations:*

$$\frac{d\tilde{l}}{dt} = \left[\nabla\gamma_+(\tilde{l}), \tilde{l}\right] - \left[\nabla\gamma(\tilde{l}), \tilde{l}\right]_+, \qquad \frac{d\tilde{f}}{dt} = \frac{\delta\gamma}{\delta\tilde{f}^*}, \qquad \frac{d\tilde{f}^*}{dt} = -\frac{\delta\gamma}{\delta\tilde{f}},$$

where $\gamma \in I(\hat{\mathcal{G}}^*)$ *is a Casimir functional at* $\hat{l} \in \hat{\mathcal{G}}^*$, *connected with* $\tilde{l} \in \hat{\mathcal{G}}^*$ *by* (15), *is equivalent to the system* (6), (8) *and* (9) *via the constructed above Backlund transformation* (16).

By means of simple calculations via the formula (see [4])

$$\tilde{\Theta} = B'\Theta B'^{*},$$

where $B' : T(\hat{\mathcal{G}}^* \oplus W \oplus W^*) \to T(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$ is the Frechet derivative of (16), one brings about the following form of the Poisson structure Θ on $\hat{\mathcal{G}}^* \oplus W \oplus W^* \ni$ $\ni (\hat{l}, f, f^*)$:

$$\nabla\gamma(\hat{l}, f, f^*) \xrightarrow{\Theta} \begin{pmatrix} \left[\hat{l}, \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+\right] - \left[\hat{l}, \frac{\delta\gamma}{\delta\hat{l}}\right]_+ - f\xi^{-1} \otimes \frac{\delta\gamma}{\delta f} + \frac{\delta\gamma}{\delta f^*}\xi^{-1} \otimes f^* \\ \frac{\delta\gamma}{\delta f^*} - \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+ f \\ -\frac{\delta\gamma}{\delta f} + \left(\frac{\delta\gamma}{\delta\hat{l}}\right)_+^* f \end{pmatrix}, \qquad (17)$$

that makes it possible to formulate the theorem.

Theorem 2. *The dynamical system* (13), *being Hamiltonian with respect to the Poisson structure* Θ *in the form* (17) *and a function* $\gamma \in I(\hat{\mathcal{G}}^*)$, *gives the inherited Hamiltonian representation for the coupled evolution equations* (6), (8), (9).

By means of the expression (15) one can construct Hamiltonian evolution equations, describing commutative flows on the extended space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ at a fixed element $\tilde{l} \in \hat{\mathcal{G}}^*$. Due to (17) every equation of such a type is equivalent to the system

$$\begin{aligned} \frac{d\hat{l}}{d\tau_n} &= [\hat{l}_+^n, \hat{l}], \\ \frac{df}{d\tau_n} &= \hat{l}_+^n f, \\ \frac{df^*}{d\tau_n} &= -(\hat{l}^*)_+^n f^*, \end{aligned} \qquad (18)$$

generated by involutive with respect to the Poisson bracket (10) Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}^*)$, $n \in \mathbb{N}$, taking the standard form:

$$\gamma_n = \frac{1}{n+1}(\hat{l}^n, \hat{l})$$

at $\hat{l} \in \hat{\mathcal{G}}^*$.

The compatibility conditions of the Hamiltonian systems (18) for different $n \in \mathbb{N}$ can be used for obtaining Lax-integrable equations on usual spaces of smooth 2π-periodic multivariable functions that will be done in the next section.

4. The Lax-integrable Davey–Stewartson equation and its triple linear representation. Choose the element $\tilde{l} \in \hat{\mathcal{G}}^*$ in an exact form such as

$$\tilde{l} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\xi - \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix},$$

where $u, \bar{u} \in S(\mathbb{S}^1; \mathbb{C})$ and $\mathcal{G} = gl(2; \mathbb{C})$. Then

$$\hat{l} = \tilde{l} + \begin{pmatrix} f_1\xi^{-1}f_1^* & f_1\xi^{-1}f_2^* \\ f_2\xi^{-1}f_1^* & f_2\xi^{-1}f_2^* \end{pmatrix}, \qquad (19)$$

where $f = (f_1, f_2)^T$, $f^* = \left(f_1^*, f_2^*\right)^T$ and "–" can mean the complex or related with it conjugation. Below we will study the evolutions (18) of vector-functions $\left(f, f^*\right) \in W\left(\mathbb{S}^1; \mathbb{C}^2\right) \oplus W^*\left(\mathbb{S}^1; \mathbb{C}^2\right)$ with respect to the variables $y = \tau_1$ and $t = \tau_2$ at the point (19). They can be obtained from the second and third equations in (18), having put $n = 1$ and $n = 2$, as well as from the first one. The latter is the compatibility condition of the spectral problem

$$\hat{l}\,\Phi = \lambda\Phi, \tag{20}$$

where $\Phi = (\Phi_1, \Phi_2)^T \in W\left(\mathbb{S}^1; \mathbb{C}^2\right)$, $\lambda \in \mathbb{C}$ is some parameter, with the following linear equations:

$$\frac{d\Phi}{dy} = \hat{l}_+\Phi, \tag{21}$$

$$\frac{d\Phi}{dt} = \hat{l}_+^2\Phi, \tag{22}$$

arising from (18) at $n = 1$ and $n = 2$ correspondingly. The compatibility of equations (20) and (21) leads to the relationships:

$$\frac{\partial u}{\partial y} = -2f_1f_2^*, \qquad \frac{\partial \bar{u}}{\partial y} = -2f_1^*f_2, \tag{23}$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial x} - uf_2, \qquad \frac{\partial f_1^*}{\partial y} = \frac{\partial f_1^*}{\partial x} - \bar{u}f_2^*,$$

$$\frac{\partial f_2}{\partial y} = -\frac{\partial f_2}{\partial x} + \bar{u}f_1, \qquad \frac{\partial f_2^*}{\partial y} = -\frac{\partial f_2^*}{\partial x} + uf_1^*.$$

Analogously, replacing $t \in \mathbb{R}$ by $it \in i\mathbb{R}$, $i^2 = -1$, one gets from (20) and (22):

$$\frac{du}{dt} = i\left(\frac{\partial^2 u}{\partial x\,\partial y} + 2u\left(f_1f_1^* + f_2f_2^*\right)\right), \qquad \frac{d\bar{u}}{dt} = -i\left(\frac{\partial^2 \bar{u}}{\partial x\,\partial y} + 2\bar{u}\left(f_1f_1^* + f_2f_2^*\right)\right), \tag{24}$$

$$\frac{\partial\left(f_1f_1^*\right)}{\partial y} - \frac{\partial\left(f_1f_1^*\right)}{\partial x} = \frac{1}{2}\frac{\partial(u\bar{u})}{\partial y} = -\left(\frac{\partial\left(f_2f_2^*\right)}{\partial x} + \frac{\partial\left(f_2f_2^*\right)}{\partial y}\right), \tag{25}$$

$$\frac{df_1}{dt} = i\left(\frac{\partial^2 f_1}{\partial x^2} + \left(2f_1f_1^* - u\bar{u}\right)f_1 - \frac{\partial u}{\partial x}f_2\right),$$

$$\frac{df_1^*}{dt} = -i\left(\frac{\partial^2 f_1^*}{\partial x^2} + \left(2f_1f_1^* - u\bar{u}\right)f_1^* - \frac{\partial \bar{u}}{\partial x}f_2^*\right),$$

$$\frac{df_2}{dt} = i\left(\frac{\partial^2 f_2}{\partial x^2} - \left(2f_2f_2^* + u\bar{u}\right)f_2 - \frac{\partial \bar{u}}{\partial x}f_1\right),$$

$$\frac{df_2^*}{dt} = -i\left(\frac{\partial^2 f_2^*}{\partial x^2} - \left(2f_2f_2^* + u\bar{u}\right)f_2^* - \frac{\partial u}{\partial x}f_1^*\right).$$

The relationships (24), (25) take the well known form of the Davey–Stewartson equation [5, 14–16] at $\bar{u} \in S\left(\mathbb{S}^1; \mathbb{C}\right)$ being the complex conjugated to $u \in S\left(\mathbb{S}^1; \mathbb{C}\right)$. The compatibility for every pair of equations (20), (21) and (22), which can be rewritten as the first order linear ordinary differential ones in such a way:

$$\frac{d\Phi}{dx} = \begin{pmatrix} \lambda & u & -f_1 \\ \bar{u} & -\lambda & f_2 \\ f_1^* & f_2^* & 0 \end{pmatrix} \Phi, \tag{26}$$

$$\frac{d\Phi}{dy} = \begin{pmatrix} \lambda & 0 & -f_1 \\ 0 & \lambda & -f_2 \\ f_1^* & -f_2^* & 0 \end{pmatrix} \Phi, \tag{27}$$

$$\frac{d\Phi}{dt} = i \begin{pmatrix} \lambda^2 + f_1 f_1^* & \frac{1}{2}\frac{\partial u}{\partial y} & -\lambda f_1 - \frac{\partial f_1}{\partial y} \\ -\frac{1}{2}\frac{\partial \bar{u}}{\partial y} & \lambda^2 - f_2 f_2^* & -\lambda f_2 - \frac{\partial f_2}{\partial y} \\ \lambda f_1^* - \frac{\partial f_1^*}{\partial y} & -\lambda f_2^* + \frac{\partial f_2^*}{\partial y} & f_2 f_2^* - f_1 f_1^* \end{pmatrix} \Phi, \tag{28}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in W(\mathbb{S}^1; \mathbb{C}^3)$, provide its Lax-type integrability. Thus, the following theorem holds.

Theorem 3. *The Davey – Stewartson equation* (24), (25) *possesses the Lax representation as the compatibility condition for* (26) *and* (27) *under the additional constraint* (23), *arising naturally from the equations* (26) *and* (27).

In fact, one has found above a triple linearization for a (2 + 1)-dimensional dynamical system, that is a new important ingredient of the Lie-algebraic approach to Lax-type integrable flows, based on the Backlund-type transformation (16) developed in this work. It is clear that the similar construction of a triple linearization like (26) – (28) can be done for many other both old and new (2+1)-dimensional dynamical systems, on what we plant to stop in detail in another work under preparation.

5. Conclusion. As it is well known, there existed by now only two regular enough algorithmic approaches [4, 5, 9, 10, 17] to constructing integrable multi-dimensional (mainly 2 + 1) dynamical systems on functional spaces. Our approach, devised in this work, is substantially based on the results previously done in [11, 13], explains completely the computational properties of multi-dimensional flows before delivered in works [5, 15, 16]. As the key points of our approach there used the canonical Hamiltonian structures naturally existing on the extended phase space and the related with them Backlund transformation which saves Casimir invariants of a chosen matrix integral-differential Lie algebra. The latter gives rise to some additional Hamiltonian properties of considered extended evolution flows before studied in [5, 13] making use of the standard inverse scattering transform [2, 3] and the formal symmetry reduction for the KP-hierarchy [15, 16] of commuting operator flows.

As one can convince ourselves analyzing the structure of the Backlund-type transformation (16), that it strongly depends on the type of an *ad*-invariant scalar product chosen on an operator Lie algebra $\hat{\mathcal{G}}$ and its Lie algebra decomposition like (2). Since there exist in general other possibilities of choosing such decompositions and *ad*-invariant scalar products on $\hat{\mathcal{G}}$, they give rise naturally to another resulting types of the corresponding Backlund transformations, which can be a subject of another special investigation. Let us here only mention the choice of a scalar product related with the operator Lie algebra $\hat{\mathcal{G}}$ centrally extended by means of the standard Maurer – Cartan two-cocycle [4, 7, 10], bringing about new types of multi-dimensional integrable flows.

The last aspect of the Backlund approach to constructing Lax-type integrable flows and their partial solutions which is worth of mention is related with

Darboux – Backlund-type transformations [18] and their new generalization recently developed in [19–21]. They give rise to very effective procedures of constructing multi-dimensional integrable flows on functional spaces with arbitrary number of independent variables simultaneously delivering a wide class of their exact analytical solutions, depending on many constant parameters, which can appear to be useful for diverse applications in applied sciences.

All mentioned above Backlund-type transformation aspects can be studied as special investigations, giving rise to new directions in the theory of multi-dimensional evolution flows and their integrability.

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