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W. I. Skrypnik (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

LONG-RANGE ORDER IN LINEAR FERROMAGNETIC OSCILLATOR SYSTEMS. STRONG PAIR QUADRATIC n – n POTENTIAL*

ДАЛЕКИЙ ПОРЯДОК У ФЕРОМАГНІТНИХ СИСТЕМАХ ЛІНІЙНИХ ОСЦИЛЯТОРІВ. СИЛЬНИЙ ПАРНИЙ КВАДРАТИЧНИЙ б – с ПОТЕНЦІАЛ

Long-range order is proved to exist for lattice linear oscillator systems with a ferromagnetic potential energy containing a term with a strong nearest neighbour (n – n) quadratic pair potential. A contour bound and a generalized Peierls argument are used in the proof.

Доведено, що далекий порядок існує у гратковій системі лінійних осциляторів з феромагнітною потенціальною енергією, яка містить доданок із сильним парним квадратичним потенціалом взаємодії близьких сусідів (б – с). При доведенні використовуються контурна нерівність та узагальнений аргумент Пайєрлса.

1. Introduction and main result. The corner stone of a generalized Peierls argument, which a scheme for proving existence of lro (long-range-order) in different n – n (nearest-neighbours) systems, is a contour bound (see [1]). For ferromagnetic linear oscillator systems with n – n pair quadratic interaction potential Bricmont and Fontaine in [2] proposed a remarkably simple derivation of the contour bound based on an application of the Griffiths – Kelly – Sherman (GKS) inequalities [3, 4]. But they did not prove existence of lro with its help.

In this paper we prove existence of lro in ferromagnetic linear oscillator classical systems with n – n pair quadratic interaction potential applying the generalized Peierls argument and the BF (Bricmont – Fontain) contour bound (see Remark). We assume that the strength g of the n – n interaction is large and that it determines the depth of minima (wells) of the effective external field u^0 which is a bounded from bellow polynomial. Lro is proved to occur at arbitrary temperature for sufficiently large g. Our argument is based on determining an asymptotics in g of the integral in the right-hand side of the BF contour bound.

For the linear oscillator systems in which the oscillator variables $q_x \in \mathbb{R}$ are indexed by the sites x of the hypercubic lattice $\mathbb{Z}^d$, characterized by the potential energy U, the contour bound is given for the set Γ of n – n by

$$\left\langle \prod_{<x,y>\in\Gamma} \chi_x^+ \chi_y^- \right\rangle_\Lambda \le e^{-|\Gamma|E}, \tag{1}$$

$\langle\cdot\rangle_\Lambda$ denotes the Gibbs average for the system confined to compact domain Λ (hypercube) with $|\Lambda|$ sites

$$\langle F_X \rangle_\Lambda = Z_\Lambda^{-1} \int F_X(q_X) e^{-\beta U(q_\Lambda)} dq_\Lambda, \qquad Z_\Lambda = \int e^{-\beta U(q_\Lambda)} dq_\Lambda,$$

$$\chi_x^+ = \chi_{(0,\infty)}(q_x), \quad \chi_x^- = \chi_{(-\infty,0)}(q_x),$$

β is the inverse temperature, the integration is performed over $\mathbb{R}^{|\Lambda|}$, $\chi_{(a,b)}$ is the characteristic function of the open interval (a,b) and F_X is a measurable function.

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For the BF contour bound, the following equalities are valid (the proof is given in the end of this paper):

$$E = g\beta\langle\sigma\sigma'\rangle,$$

$$\langle\sigma\sigma'\rangle = Z^{-1}(2)\int q_1 q_2 e^{-\beta u(q_1,q_2)} dq_1 dq_2, \qquad Z(2) = \int e^{-\beta u(q_1,q_2)} dq_1 dq_2,$$

where g is the strength of the n$-$n pair quadratic interaction, $u(q,q')$ is the potential energy of two oscillators, containing only the sum of the external potentials $u(q)$ and the n$-$n pair quadratic potential,

$$u(q_1,q_2) = u(q_1) + u(q_2) - g q_1 q_2 = u^0(q_1) + u^0(q_2) + \frac{g}{2}(q_1 - q_2)^2.$$

Let $\sigma_x(q_\Lambda) = q_x$; then the ferromagnetic lro occurs in the system if, for a positive a'' independent of Λ, the following inequality holds for all $x, y \in \Lambda$:

$$\langle f(\sigma_x) f(\sigma_y)\rangle_\Lambda > a'', \tag{2}$$

where f is a measurable function. In simplest cases $f(q) = q$ (see [1]) and $f(q) = \dfrac{q}{|q|}$. The derivation of (2) from (1) is easier for the latter option and if one proves (1) for it then there are bounds which allow to prove (2) for the first one (see [1]).

Inequality (2) indicates that there is no decrease of correlations. It does not hold for the systems at high temperatures or weak pair interaction with the following potential energy:

$$U(q_\Lambda) = \sum_{x\in\Lambda} u(q_x) + \sum_{x,y\in\Lambda} u_{x-y}(q_x, q_y), \tag{3}$$

where the external field u, pair (binary) potentials u_x satisfy certain conditions [5] which are stronger then the condition of superstability [6]. In [5] a reader will find a proof that in the high temperature phase there is a decrease of correlations for such the potentials.

We consider the following ferromagnetic potential energy (as in [2]):

$$U(q_\Lambda) = \sum_{x\in\Lambda} u(q_x) - g \sum_{\langle x,y\rangle\in\Lambda} q_x q_y - U'(q_\Lambda),$$

$$U'(q_\Lambda) = \sum_{(A,n_A),A\subseteq\Lambda} J_{(A,n_A)} q_{[A]}^{n_A}, \qquad q_{[A]}^{n_A} = \prod_{x\in A} q_x^{n_x}, \tag{4}$$

where u is an even growing at infinity entire function, the summation in (A, n_A) (finite in n_x), $n_A = (n_x, x\in A)$ is performed over subsets A of Λ and positive integers n_x,

$$J_{A,n_A} \geq 0, \qquad \sum_{x\in A} n_x = 2n(A), \quad n_x, n(A) \in \mathbb{Z}^+.$$

In the last line the intermediate condition implies that the potential energy is invariant under the change of the signs of all the oscillator variables.

In order to guarantee finiteness of the partition function and correlation function we require that $n \in \mathbb{Z}^+$

$$u(q) \geq \eta q^{2n}, \quad \eta > 0, \qquad \sup_A n(A) < n, \qquad \max_x \sum_{x\in A} J_{A,n_A} < \infty, \quad n > 1.$$

Theorem 1. *Let the potential energy of the linear oscillator system be given by* (4) *and* $u(q) = \eta q^{2n}$, $\eta > 0$. *Then* (2) *holds for* $f(q) = \frac{q}{|q|}$ *and sufficiently large* g *if* $d \geq 2$, $1 < n \in \mathbb{Z}^+$.

The proof of this theorem is based on application of the generalized Peierls argument and the fact that the average in the expression for E tends to infinity together with g. The generalized Peierls argument can be formulated as the following theorem.

Theorem 2. *Let* (1) *hold with sufficiently large* E *then there exist positive numbers* a, a' *independent of* Λ *such that*

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leq a' e^{-aE}. \tag{5}$$

Moreover, if the Gibbs average is invariant under the change of signs of all variables then (2) *is valid for* $f(q) = \frac{q}{|q|}$.

The proof of Theorem 1 follows from Theorem 2, the equality

$$4\langle \sigma\sigma' \rangle = \langle (\sigma + \sigma')^2 \rangle - \langle (\sigma - \sigma')^2 \rangle$$

and the following lemma.

Lemma. *There exist functions* C_+, C_- *in* β *independent of* g *such that*

$$\langle (\sigma + \sigma')^2 \rangle \geq C_+ g^{-1}(e_0 - 1)^2, \qquad \langle (\sigma - \sigma')^2 \rangle \leq (g\beta)^{-1} C_-, \tag{6}$$

where $e_0 = (g^n (2\eta n)^{-1})^{\frac{1}{2(n-1)}}$.

This lemma and Theorem 2 are proved in the second and third sections, respectively.

2. Proof of Lemma. Let's rescale the variables by $g^{-\frac{1}{2}}$ and subtract the constant term $2u_g^0(2e_0)$ from the potential $u(g^{-\frac{1}{2}}q_1, g^{-\frac{1}{2}}q_2)$. Now in the averages in (6) we have to substitute the potential u_g instead of u

$$u_g(q_1, q_2) = u_g^0(q_1) + u_g^0(q_2) + \frac{1}{2}(q_1 - q_2)^2 - 2u_g^0(e_0),$$
$$u_g^0(q) = \frac{1}{2}(2\eta g^{-n} q^{2n} - q^2) = u^0\left(g^{-\frac{1}{2}} q\right).$$

It is not difficult to check that e_0 is the minimum of u_g^0.

After the rescaling we obtain

$$\langle \sigma\sigma' \rangle = g^{-1} Z_g^{-1}(2) \int q_1 q_2 e^{-\beta u_g(q_1,q_2)} dq_1 dq_2, \qquad Z_g(2) = \int e^{-\beta u_g(q_1,q_2)} dq_1 dq_2.$$

We easily derive the following inequality:

$$\langle (\sigma - \sigma')^2 \rangle \leq 2(g\beta)^{-1} \kappa \frac{Z_g^0(2)}{Z_g(2)}, \quad \kappa = \sup_{q \geq 0} q e^{-q}, \tag{7}$$

where

$$Z_g^0(2) = \left(\int e^{-\beta(u_g^0(q) - u_g^0(e_0))} dq \right)^2.$$

Now we have to show that the integral in the expression for $Z_0(2)$ is uniformly bounded in g. In order to do this we have to rescale the variables in the integral by e_0.

Recollecting the expression for e_0 one checks that

$$u_g^0(q) = \frac{1}{2n}(e_0^{-2n+2} q^{2n} - nq^2), \qquad u_g^0(e_0) = -\frac{n-1}{2n} e_0^2. \tag{8}$$

As a result

$$u_g^0(e_0 q) - u_g^0(e_0) = \frac{1}{2n} e_0^2 (q^{2n} - nq^2 + n - 1) = e_0^2 \tilde{u}(q).$$

It's clear that for $\varepsilon << 1$ the integral is equal to

$$e_0 \int e^{-\beta e_0^2 \tilde{u}(q)} dq = 2e_0 \left\{ \int\limits_0^{1-\varepsilon} + \int\limits_{1-\varepsilon}^{1+\varepsilon} + \int\limits_{1+\varepsilon}^{q_n} + \int\limits_{q_n}^{\infty} \right\} e^{-\beta e_0^2 \tilde{u}(q)} dq, \tag{9}$$

where for $q \geq q_n$ the follow inequality is valid $q^{2n} \geq (n+1)q^2$. For $|q| \leq \varepsilon << 1$ from the expansion

$$\tilde{u}(q+1) = (n-1)q^2 + \tilde{p}(q), \qquad \tilde{p}(q) = n^{-1} \sum_{j=3}^{2n} \frac{s!(2n-s)!}{n!} q^s,$$

we derive the inequality $\tilde{u}(q+1) \geq (n - 1 - \tilde{\varepsilon})q^2$, $\tilde{\varepsilon} << 1$.

As a result the second integral in (9) is estimated in the following way:

$$2e_0 \int\limits_{-\varepsilon}^{\varepsilon} e^{-\beta e_0^2 \tilde{u}(q+1)} dq \leq 2e_0 \int\limits_{-\varepsilon}^{\varepsilon} e^{-\beta e_0^2 (n-1-\tilde{\varepsilon}) q^2} dq \leq 2e_0 \int e^{-\beta e_0^2 (n-1-\tilde{\varepsilon}) q^2} dq =$$
$$= 2\sqrt{\pi}(\beta(n-1-\tilde{\varepsilon}))^{-\frac{1}{2}}.$$

The function $\tilde{u}$ is positive and grows in the complement of the interval $[1-\varepsilon, 1+\varepsilon]$. This implies that the sum of first and third integrals is less then

$$2q_n e_0 e^{-\beta e_0^2 \tilde{u}(1-\varepsilon)} < \infty.$$

The last integral in (9) is less then

$$2e_0 e^{-\beta \frac{n-1}{2n} e_0^2} \int e^{-\beta e_0^2 q^2} dq \leq 2e^{-\beta \frac{n-1}{2n} e_0^2} \sqrt{\pi} \beta^{-\frac{1}{2}} < \infty,$$

since the integral in the left-hand side of (3) is uniformely bounded in g. Thus we proved that there exists a constant C_-^0 such that

$$Z_g^0(2) \leq C_-^0. \tag{10}$$

From the representation

$$u_g^0(q + e_0) - u_g^0(e_0) = (n-1)q^2 + p_g(q),$$
$$p_g(q) = n^{-1} \sum_{j=3}^{2n} \frac{s!(2n-s)!}{n!} q^s e_0^{2-s},$$

it follows that the polynomial p_g tends to zero when g tends to infinity.

As a result for $q_1, q_2 \in [e_0 - 1, e_0 + 1]$, $e_0 > 1$, the following bound holds:

$$u_g(q_1, q_2) \leq u_g^0(e_0 + 1) - u_g^0(e_0) + 2 \leq n + 1 + n^{-1} 2^n.$$

Here we took into account that e_0 is the zero of the nonnegative function $u_g^0 - u_g^0(e_0)$ and that it increases if $|e_0 - q|$ increases.

From the previous inequality we derive that

$$Z_g(2) = \int e^{-\beta u_g(q_1,q_2)} dq_1 dq_2 \geq$$
$$\geq \int_{(\times[e_0-1,e_0+1])^2} e^{-\beta u_g(q_1,q_2)} dq_1 dq_2 \geq 4e^{-2\beta[n+1+n^{-1}2^n]}.$$

This inequality together with (7) and (10) yield the second inequality in (6) with
$$C_- = 4\kappa C_-^0 e^{2\beta[n+1+n^{-1}2^n]}.$$

This proves the second inequality in (6). To prove the first inequality in (6) we have to apply the inequality

$$Z_g(2)\langle(\sigma+\sigma')^2\rangle \geq \int_{(\times[e_0-1,e_0+1]^2)} e^{-\beta u_g(q_1,q_2)} (q_1+q_2)^2 dq_1 dq_2 \geq$$
$$\geq 2^4(e_0-1)^2 e^{-2\beta[u_g^0(e_0-1)-u_g^0(e_0)+2]} \geq$$
$$\geq 2^4(e_0-1)^2 e^{-2\beta[n+1+2^n]}, \quad e_0 > 1.$$

The last inequality, (10) and the inequality
$$Z_g(2) \leq Z_g^0(2)$$

give
$$\langle(\sigma+\sigma')^2\rangle \geq 2^4 g^{-1}(e_0-1)^2 e^{-2\beta[n+1+2^n]}(Z_g^0(2))^{-1} \geq$$
$$\geq g^{-1}(e_0-1)^2 2^4 e^{-2\beta[n+1+2^n]}(C_-^0)^{-1} = g^{-1}(e_0-1)^2 C_+.$$

Hence the first inequality in (6) holds with
$$C_+ = 2^4 e^{-2\beta[n+1+2^n]}(C_-^0)^{-1}.$$

Lemma is proved.

3. Proof of Theorem 2. The proposed proof is a slightly modified version of the proof from [1] and we give it for convenience of a reader. The set of all configurations q_Λ can be described by the set of configurations $s_\Lambda, s_x = 1, -1$, or simply $s_x = +, -$, as in the Ising model. The set of all spin configurations can be classified by different contours $\gamma(s_\Lambda)$, i.e. connected union of faces of unit cubes, centered at lattice sites, which is a boundary of a related connected union of the cubes. The main idea is to consider contours $\gamma_{x,y}$, enclosing x, separating it from y and with adjacent cubes, containing spins of different signs from the opposite sides. So inside $\gamma_{x,y}$ there are spins of both signs. The contours may be nonclosed ending on the boundary $\partial\Lambda$. There may be several such the contours in a configuration. In this case the smallest contour is chosen. We have to estimate the l. h. s. of (1) in terms of such the contours. With this aim we express it as a sum over s_Λ and then transform this sum into the sum over the contours $\gamma_{x,y} \in \Lambda$, summing over all configurations, characterized by the contours. So, at first we have to insert the equality

$$1 = \prod_{l\in\Lambda}(\chi_l^+ + \chi_l^-)$$

under the sign of the Gibbs average. As a result

$$\langle\chi_x^+\chi_y^-\rangle_\Lambda = \sum_{s_\Lambda}\langle\chi_x^+\chi_y^- \prod_{l\in\Lambda}\chi_l^{s_l}\rangle_\Lambda =$$

Proof of the contour bound (1).

$$\chi_x^+\chi_y^- = e^{-\frac{g}{2}\beta\sigma_x\sigma_y} e^{\frac{g}{2}\beta\sigma_x\sigma_y}\chi_x^+\chi_y^- \le e^{-\frac{g}{2}\beta\sigma_x\sigma_y}\chi_x^+\chi_y^- \le e^{-\frac{g}{2}\beta\sigma_x\sigma_y}.$$

As a result

$$\left\langle \prod_{\langle x,y\rangle\in\Gamma} \chi_x^+\chi_y^- \right\rangle_\Lambda \le \left\langle e^{-\beta\frac{g}{2}\sum_{\langle x,y\rangle\in\Gamma}\sigma_x\sigma_y}\right\rangle_\Lambda = \left\langle e^{\beta\frac{g}{2}\sum_{\langle x,y\rangle\in\Gamma}\sigma_x\sigma_y}\right\rangle_{\Lambda[\Gamma]}^{-1} \le$$

$$\le e^{-\beta\frac{g}{2}\sum_{\langle x,y\rangle\in\Gamma}\langle\sigma_x\sigma_y\rangle_{\Lambda[\Gamma]}} = e^{-E_\Gamma},$$

where $\langle\cdot,\cdot\rangle_{\Lambda[\Gamma]}$ is the average corresponding to the potential energy

$$U_\Gamma(q_\Lambda) = U(q_\Lambda) + \frac{g}{2}\sum_{x,y\in\Gamma} q_x q_y.$$

In last line, we applied the Jensen inequality. From the second GKS inequality

$$\left\langle q_{[A]}^{n_A} q_{[B]}^{n_B}\right\rangle_{\Lambda[\Gamma]} - \left\langle q_{[A]}^{n_A}\right\rangle_{\Lambda[\Gamma]}\left\langle q_{[B]}^{n_B}\right\rangle_{\Lambda[\Gamma]} \ge 0 \tag{11}$$

it follows that the average $\langle q_x q_y\rangle_{\Lambda[\Gamma]}$ is a monotone increasing function in $J_{(A,n_A)}$ since its derivative in $J_{(A,n_A)}$ coincide with the right-hand side of this inequality multiplied by β for $B=(x,y)$. In the second n$-$n term in (4) the coefficient $J_{(\langle x,y\rangle,1,1)}$ under the sign of the sum is equal to the unity. Hence,

$$\langle\sigma_x\sigma_y\rangle_{\Lambda[\Gamma]} \ge \langle\sigma_x\sigma_y\rangle^0_{\Lambda[\Gamma]},$$

where the average in the right-hand side is generated by the potential energy

$$U^0_\Gamma(q_\Lambda) = \sum_{x\in\Lambda} u(q_x) - \frac{g}{2}\sum_{<x,y>\in\Gamma} q_x q_y.$$

Applying the GKS inequality again we have

$$\langle\sigma_x\sigma_y\rangle^0_{\Lambda[\Gamma]} \ge \langle\sigma\sigma'\rangle$$

and

$$\langle\sigma_x\sigma_y\rangle_{\Lambda[\Gamma]} \ge \langle\sigma\sigma'\rangle, \qquad E_\Gamma \ge |\Gamma|E.$$

Contour bound is proved.

Inequality (11) is proved reducing its left-hand side, at first, to the following form:

$$(2Z^2_{\Lambda[\Gamma]})^{-1}\int\left(q_{[A]}^{n_A} - q'^{n_A}_{[A]}\right)\left(q_{[B]}^{n_B} - q'^{n_B}_{[B]}\right)e^{-\beta[U_\Gamma(q_\Lambda)-U_\Gamma(q'_\Lambda)]}dq_\Lambda dq'_\Lambda, \tag{12}$$

where $Z_{\Lambda[\Gamma]}$ is the partition function corresponding to the potential energy U_Γ. Then it is necessary to expand the exponent in (12) with interaction terms into a series. As a result (12) will be the sum of the following terms with positive coefficients:

$$\int\prod_{j=1}^{s}\left(q_{[A_j]}^{n_{A_j}} + (-1)^{n_{A_j}} q'^{n_{A_j}}_{[A_j]}\right)e^{-\beta\sum_{x\in\Lambda}[u^0(q_x)+u^0(q'_x)]}dq_\Lambda dq'_\Lambda \ge 0, \quad A_j\subset\Lambda.$$

An elegant proof of this inequality can be found in [4] (Theorem VIII,14A).

Remark. Another application of the Peierls argument for proving lro can be found in [8] and references there.

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