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ELEMENTARY REPRESENTATIONS OF THE GROUP $B_0^{\mathbb{Z}}$ OF INFINITE IN BOTH DIRECTIONS UPPER-TRIANGULAR MATRICES. I

ЕЛЕМЕНТАРНІ ЗОБРАЖЕННЯ ГРУПИ $B_0^{\mathbb{Z}}$ НЕСКІНЧЕННИХ В ОБИДВА БОКИ ВЕРХНЬОТРИКУТНИХ МАТРИЦЬ. І

We define the so-called "elementary representations" $T_p^{R,\mu}$, $p \in \mathbb{Z}$, of the group $B_0^{\mathbb{Z}}$ of finite, infinite in both directions upper-triangular matrices using quasi-invariant measures on some homogeneous spaces and give a criterion of irreducibility and equivalence of the constructed representations. We give also a criterion of irreducibility of tensor product of a finite and infinite number of elementary representations.

Визначено так звані елементарні зображення $T_p^{R,\mu}$, $p \in \mathbb{Z}$, групи $B_0^{\mathbb{Z}}$ фінітних нескінченних в обидва боки верхньотрикутних матриць з використанням квазіінваріантних мір на деяких однорідних просторах і наведено критерій незвідності та еквівалентності побудованих зображень. Дано також критерій незвідності тензорного добутку скінченного та нескінченного числа елементарних зображень.

1. G-action, quasiinvariant measures, and representations. The following construction of the unitary representations of a topological group G is well known. Let us have some measurable space X with a probability measure μ on which the group G acts, i. e., we have a group homomorphism $\alpha: G \to \mathrm{Aut}(X)$ such that

1) $\alpha_e(x) = x \quad \forall x \in X$, where $e \in G$ is the identity element;

2) $\alpha_{t_1}(\alpha_{t_2}(x)) = \alpha_{t_1 t_2}(x) \quad \forall t_1, t_2 \in G, \ x \in X$.

Let μ^{α_t}, $t \in G$, be images of the measure μ with respect to the action α, i. e., $\mu^{\alpha_t}(\Delta) = \mu(\alpha_{t^{-1}}(\Delta))$. If $\mu^{\alpha_t} \sim \mu \quad \forall t \in G$, one can define the unitary representation $\pi^{\alpha,\mu}: G \to U(L^2(X, d\mu))$ of the group G by

$$(\pi_t^{\alpha,\mu} f)(x) = \left(\frac{d\mu^{\alpha_t}(x)}{d\mu(x)}\right)^{1/2} f(\alpha_{t^{-1}}(x)), \quad f \in L^2(X, d\mu). \tag{1}$$

2. An analog of the regular representations of infinite-dimensional groups. The regular representation of a locally compact group G is well known (see, for example, [1]). It uses existence of a G-invariant measure on the group G, the Haar measure, and is defined by formula (1) with $X = G$ and α being the right or the left action of the group G on itself.

For a group G that is not locally compact, it is impossible to define a regular representation, since there is no G-invariant measure on the group G [2], nor is there a G-quasiinvariant measure either [3].

An analog of the regular representations of some infinite-dimensional noncommutative groups, current groups, were constructed and studied firstly in [4 – 7].

An analog of the regular representation for any infinite-dimensional group G, using G-quasiinvariant measures μ on some *completions* $\tilde{G}$ *of the group* G is defined firstly in [8 – 10]. It uses the formula (1), where $X = \tilde{G}$ and α is the right or the left action of the group G on $\tilde{G}$. More precisely, let $H_\mu = L^2(\tilde{G}, d\mu)$. We define an *analog of the right* $T^{R,\mu}$ *and the left* $T^{L,\mu}$ *regular representations of the group* G in the space H_μ,

$$T^{R,\mu}, T^{L,\mu} : G \to U(H_\mu),$$

in a natural way,

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \tag{2}$$

$$\left(T_s^{L,\mu} f\right)(x) = \left(\frac{d\mu(s^{-1}x)}{d\mu(x)}\right)^{1/2} f\left(s^{-1}x\right). \tag{3}$$

Obviously $\left[T_t^{R,\mu}, T_s^{L,\mu}\right] = 0 \quad \forall t, s \in G$, hence the right regular representation $T^{R,\mu}$ is reducible if $\mu^{L_s} \sim \mu$ for some $s \in G \backslash e$ or the measure μ is not G-right ergodic. Let μ be a G-right quasiinvariant measure on $\tilde{G}$, i. e., $\mu^{R_t} \sim \mu \ \forall t \in G$.

Conjecture 1. *The right regular representation* $T^{R,\mu} : G \to U(H_\mu)$ *is irreducible if and only if*

1) $\mu^{L_s} \perp \mu \quad \forall s \in G \backslash e$;

2) *the measure* μ *is* G*-right ergodic.*

Remark. This conjecture was formulated by R. S. Ismagilov in 1985 for the group $B_0^{\mathbb{N}}$ of finite, infinite in one direction real upper-triangular matrices with unities on the principal diagonal and any Gaussian centered product measure μ_b.

In this case the conjecture was proved in [8, 9]. For the same group $B_0^{\mathbb{N}}$ and for any product measure $\mu = \otimes_{k<n} \mu_{kn}$, this was proved in [11] with some technical assumption. In [12] the conjecture was proved for the group $B_0^{\mathbb{Z}}$ of finite, infinite in both directions upper-triangular matrices for some Gaussian centered product measures. In [10] a criterion was proved for groups of the interval and circle diffeomorphisms and the Wiener measure.

3. An analog of the regular representations of the group $B_0^{\mathbb{Z}}$. Let $B_0^{\mathbb{Z}}$ be the group of finite, infinite in both directions upper-triangular matrices with unities on the principal diagonal, $B^{\mathbb{Z}}$ be the group of all such matrices (not necessarily finite),

$$B_0^{\mathbb{Z}} = \left\{ I + x = I + \sum_{k<n} x_{kn} E_{kn} \,|\, x \text{ is finite} \right\},$$

$$B^{\mathbb{Z}} = \left\{ I + x = I + \sum_{k<n} x_{kn} E_{kn} \,|\, x \text{ is arbitrary} \right\},$$

where E_{kn}, $k, n \in \mathbb{Z}$, are matrix units of infinite order. Let us denote by R and L the right and the left action of the group $B^{\mathbb{Z}}$ on itself: $R_s(t) = ts^{-1}$, $L_s(t) = st$, $s, t \in B^{\mathbb{Z}}$. Let μ be some probability measure on the group $B^{\mathbb{Z}}$. If $\mu^{R_t} \sim \mu$ and $\mu^{L_t} \sim \mu \ \forall t \in B_0^{\mathbb{Z}}$ we can define, by formulas (2) and (3), an analog of the right $T^{R,\mu}$ and the left $T^{L,\mu}$ regular representations of the group $B_0^{\mathbb{Z}}$ in the space $H_\mu = L^2\left(B^{\mathbb{Z}}, d\mu\right)$, $T^{R,\mu}, T^{L,\mu} : B_0^{\mathbb{Z}} \to U(H_\mu)$.

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt),$$

$$\left(T_t^{L,\mu} f\right)(x) = \left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f\left(t^{-1}x\right).$$

For the generators $A_{kn}^{R,\mu}\left(A_{kn}^{L,\mu}\right)$ of the one-parameter groups $I + tE_{kn}$, $t \in \mathbb{R}^1$, $k < n$, corresponding to the right $T^{R,\mu}$ (respective, left $T^{L,\mu}$) regular representation, we have the following formulas:

$$A_{kn}^{R,\mu} = \frac{d}{dt} T_{I+tE_{kn}}^{R,\mu}\Big|_{t=0} = \sum_{r=-\infty}^{k-1} x_{rk} D_{rn}(\mu) + D_{kn}(\mu), \tag{4}$$

$$A_{kn}^{L,\mu} = \frac{d}{dt} T_{I+tE_{kn}}^{L,\mu}\Big|_{t=0} = -\left(D_{kn}(\mu) + \sum_{m=n+1}^{\infty} x_{nm} D_{km}(\mu)\right), \tag{5}$$

where

$$D_{kn}(\mu) = \frac{\partial}{\partial x_{kn}} + \frac{d}{dt}\left(\frac{d\mu(x(I + tE_{kn}))}{d\mu(x)}\right)^{1/2}\Big|_{t=0}.$$

For an arbitrary product measure $\mu = \otimes_{k<n} \mu_{kn}$, we have

$$D_{kn}(\mu) = \frac{\partial}{\partial x_{kn}} + \frac{\partial}{\partial x_{kn}}\left(\ln \mu_{kn}^{1/2}(x_{kn})\right),$$

where $d\mu_{kn}(x) = \mu_{kn}(x)dx$, $x \in \mathbb{R}^1$. Denote

$$M_{kn}(p) = \int_{\mathbb{R}^1} x^p \mu_{kn}(x)dx, \qquad \tilde{M}_{kn}(p) = \left(\left(i^{-1}D_{kn}(\mu)\right)^p 1, 1\right)_{L^2(\mathbb{R}^1, d\mu_{kn})}, \qquad p \in \mathbb{N}.$$

Let us define the Gaussian measure μ_b on the group $B^{\mathbb{Z}}$ in the following way:

$$d\mu_b(x) = \otimes_{k<n} \left(b_{kn}/\pi\right)^{1/2} \exp\left(-b_{kn}x_{kn}^2\right) dx_{kn} = \otimes_{k<n} d\mu_{b_{kn}}(x_{kn}),$$

where $b = (b_{kn})_{k<n}$ is some set of positive numbers. In this case we have (see, for example, [13], formulas (6) and (7))

$$D_{kn}(\mu_b) = \frac{\partial}{\partial x_{kn}} - b_{kn}x_{kn},$$

$$M_{kn}(2) = \frac{1}{2b_{kn}}, \qquad M_{kn}(4) = \frac{3}{\left(2b_{kn}\right)^2}, \qquad M_{kn}(2m) = \frac{(2m-1)!!}{\left(2b_{kn}\right)^m}, \tag{6}$$

$$\tilde{M}_{kn}(2) = \frac{b_{kn}}{2}, \qquad \tilde{M}_{kn}(4) = 3\left(\frac{b_{kn}}{2}\right)^2, \qquad \tilde{M}_{kn}(2m) = (2m-1)!!\left(\frac{b_{kn}}{2}\right)^m. \tag{7}$$

For an arbitrary Gaussian product measure $\mu_b = \otimes_{k<n} \mu_{b_{kn}}$ it is easy to verify the equivalence $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$. Three following lemmas are proved in [12].

Lemma 1.

$$\mu_b^{R_t} \sim \mu_b \quad \forall t \in B_0^{\mathbb{Z}} \Leftrightarrow$$

$$\Leftrightarrow S_{kn}^R(\mu_b) = \sum_{r=-\infty}^{k-1} M_{rk}(2)\tilde{M}_{rn}(2) = \frac{1}{4}\sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}} < \infty \quad \forall k < n.$$

Lemma 2.

$$\mu_b^{L_t} \sim \mu_b \quad \forall t \in B_0^{\mathbb{Z}} \Leftrightarrow$$

$$S_{kn}^L(\mu_b) = \sum_{m=n+1}^{\infty} \tilde{M}_{km}(2)M_{nm}(2) = \frac{1}{4}\sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}} < \infty \quad \forall k < n.$$

Lemma 3. *For* $k, n \in \mathbb{Z}$, $k < n$, *we have* $\mu_b^{L_{I+tE_{kn}}} \perp \mu_b \quad \forall t \in \mathbb{R}^1 \setminus 0 \Leftrightarrow$ $\Leftrightarrow S_{kn}^L(\mu_b) = \infty$.

4. Elementary representations of the group $B_0^{\mathbb{Z}}$. Let us consider the subgroups X_p, $p \in \mathbb{Z}$, and $X^{\{p\}}$ in the group $B^{\mathbb{Z}}$, where $\{p\}$ is a finite or infinite subset of $\mathbb{Z}$. For infinite in both directions $\{p\}$ we have $\{p\} = (p_k)_{k \in \mathbb{Z}}$, $p_k < p_{k+1}$ $\forall k \in \mathbb{Z}$,

$$X_p = \left\{ I + x \in B^{\mathbb{Z}} \mid I + x = I + \sum_{n=p+1}^{\infty} x_{pn}E_{pn} \right\},$$

$$X^{\{p\}} =$$

$$= \prod_{p_k \in \{p\}} X_{p_k} = \left\{ I + x \in B^{\mathbb{Z}} \mid I + x = I + \sum_{p_k \in \{p\}} \sum_{n=p_k+1}^{\infty} x_{p_k n}E_{p_k n} \right\}.$$

Obviously, the right action of the group $B_0^{\mathbb{Z}}$ is well defined on the groups X_p and $X^{\{p\}}$.

For $B_0^{\mathbb{Z}}$-right quasiinvariant measure μ on X_p (respectively $X^{\{p\}}$), we define a representation $T_p^{R,\mu}$ (respectively $T^{R,\mu,\{p\}}$) by the formulas

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \quad f \in H_p(\mu) := L^2\left(X_p, d\mu\right),$$

$$\left(T_t^{R,\mu,\{p\}} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \quad f \in H^{\{p\}}(\mu) := L^2\left(X^{\{p\}}, d\mu\right).$$

For a particular case $\{p\} = (1, 2, \ldots, q)$ we denote

$$X^q = X^{(1,2,\ldots,q)}, \quad T^{R,\mu,q} = T^{R,\mu,(1,2,\ldots,q)}, \quad H^q(\mu) = L^2\left(X^{(1,2,\ldots,q)}, d\mu\right).$$

Definition 1. *We will call the representations* $T_p^{R,\mu}$, $p \in \mathbb{Z}$ *by the elementary* (see also [14]).

5. Irreducibility and equivalence of elementary representations. For the Gaussian measure $\mu = \mu_b$ and its projections $\mu_{b,p} = \otimes_{n=p+1}^{\infty}\mu_{b_{pn}}$ we have the following theorem.

Theorem 1. 1. *The representation* $T_p^{R,\mu}$ *is irreducible if and only if the measure* μ *on the space* X_p *is* $B_0^{\mathbb{Z}}$*-right-ergodic.*

2. *Two irreducible representations* $T_{p_1}^{R,\mu_1}$ *and* $T_{p_2}^{R,\mu_2}$ *are equivalent if and only if* $p_1 = p_2$ *and* $\mu_1 \sim \mu_2$.

Since $T_p^{R,\mu}$ (respectively $T^{R,\mu,\{p\}}$) is the restriction of the representation $T^{R,\mu}$ to the subspace $H_p(\mu) = L^2(X_p, d\mu_p)$ (respectively $H^{\{p\}}(\mu) = L^2(X^{\{p\}}, d\mu^{\{p\}})$) of the space $H_\mu = L^2(B^{\mathbb{Z}}, d\mu)$, we have

$$A_{p,kn}^{R,\mu} = \begin{cases} 0, & \text{if } k < p; \\ D_{pn}(\mu), & \text{if } p = k < n; \\ x_{pk} D_{pn}(\mu), & \text{if } p < k < n, \end{cases} \tag{8}$$

$$A_{kn}^{R,\mu,q} := A_{kn}^{R,\mu,(1,2,\dots,q)} = \sum_{p=1}^{q} A_{p,kn}^{R,\mu} =$$

$$= \begin{cases} 0, & \text{if } k < 1; \\ \sum_{r=1}^{k-1} x_{rk} D_{rn}(\mu) + D_{kn}(\mu), & \text{if } 1 \le k \le q,\ k < n; \\ \sum_{r=1}^{q} x_{rk} D_{rn}(\mu), & \text{if } q < k < n, \end{cases} \tag{9}$$

$$A_{kn}^{R,\mu,\{p\}} := \sum_{p_m \in \{p\},\, p_m \le k} A_{p_m,kn}^{R,\mu} =$$

$$= \begin{cases} 0, & \text{if } k < p_{\min}; \\ \sum_{p_m \in \{p\},\, p_m < k} x_{p_m k} D_{p_m n}(\mu) + D_{kn}(\mu), & \text{if } k \in \{p\},\ k < n; \\ \sum_{p_m \in \{p\},\, p_m < k} x_{p_m k} D_{p_m n}(\mu), & \text{if } k \notin \{p\},\ p_{\min} < k < n, \end{cases} \tag{10}$$

where $p_{\min} = \min\{p_m \mid p_m \in \{p\}\} \in \mathbb{R}^1 \cup \{-\infty\}$.

Proof. See proof of the Theorem 5 in [14]. 1. Let a bounded operator A on the Hilbert space $H_p(\mu)$ commute with representation $T_p^{R,\mu} : \left[A, T_{p,t}^{R,\mu}\right] = 0 \quad \forall t \in B_0^{\mathbb{Z}}$. We prove that A is trivial, $A = \lambda I$, $\lambda \in \mathbb{C}^1$. To prove this, we consider the commutative set of generators $\left\{i^{-1} A_{p,pn}^{R,\mu}\right\}_{n=p+1}^{\infty}$. By formulas (8) we have $i^{-1} A_{p,pn}^{R,\mu}$ $= = i^{-1} D_{pn}(\mu)$. Since the family of operators $i^{-1}\mathbb{D}_p(\mu) = \left\{i^{-1} D_{pn}(\mu)\right\}_{n=p+1}^{\infty}$ has a common simple spectrum in the space $H_p(\mu) = L^2(X_p, d\mu)$, any bounded operator A on the space $H_p(\mu)$ commuting with this family is some essentially bounded function of this family,

$$A = a\left(i^{-1}\mathbb{D}_p(\mu)\right) = a\left(i^{-1} D_{pp+1}(\mu), i^{-1} D_{pp+2}(\mu), \dots, i^{-1} D_{pn}(\mu), \dots\right).$$

To complete the proof we use some Fourier – Wiener transform defined in [13]. Let us denote by F_{kn}^b the one-dimensional Fourier transform, corresponding to the measure $d\mu_{b_{kn}}(x_{kn}) = (b_{kn}/\pi)^{1/2} \exp\left(-b_{kn} x_{kn}^2\right) d x_{kn}$,

$$F_{kn}^b : L^2\left(\mathbb{R}^1, d\mu_{b_{kn}}\right) \to L^2\left(\mathbb{R}^1, d\mu_{b_{kn}^{-1}}\right),$$

given by the formula

$$\left(F_{kn}^b f\right)(y_{kn}) = \exp\left(\frac{y_{kn}^2}{2b_{kn}}\right)\sqrt{\frac{b_{kn}}{2\pi}}\int_{\mathbb{R}^1} f(x_{kn})\exp(iy_{kn}x_{kn})\exp\left(-\frac{b_{kn}x_{kn}^2}{2}\right)dx_{kn}.$$

Obviously, $F_{kn}^b \mathbb{1} = \mathbb{1}$, where $\mathbb{1}(x) \equiv 1$.

Let us define, for any $p \in \mathbb{Z}$, the Fourier – Wiener transform $F_p^b = \otimes_{n=p+1}^{\infty} F_{pn}^b$. The operator F_p^b is an isometry between two spaces, $F_p^b\colon H_p(\mu_b) \to H_p(\mu_{b^{-1}})$, where $H_p(\mu_b) = L^2(X_p, d\mu_{b,p})$, $H_p(\mu_{b^{-1}}) = L^2\left(X_p, d\mu_{b^{-1},p}\right)$. We have (see [13])

$$F_p^b\left(i^{-1}D_{pn}(\mu_b)\right)\left(F_p^b\right)^{-1} = y_{pn}, \quad p < n, \tag{11}$$

$$F_p^b\left(x_{pn}i^{-1}D_{pm}(\mu_b)\right)\left(F_p^b\right)^{-1} = i^{-1}D_{pn}(\mu_{b^{-1}})y_{pm}, \quad p < n < m,$$

$$F_p^b A\left(F_p^b\right)^{-1} = F_p^b a\left(i^{-1}D_{pp+1}(\mu),\ldots,i^{-1}D_{pn}(\mu),\ldots\right)\left(F_p^b\right)^{-1} =$$

$$= a\left(y_{pp+1},\ldots,y_{pn},\ldots\right).$$

The one-parameter group $\tilde{T}_{I+tE_{nm}}^{R,\mu_b} = F_p^b T_{I+tE_{nm}}^{R,\mu_b}\left(F_p^b\right)^{-1}$ corresponds to the generator $i^{-1}D_{pn}(\mu_{b^{-1}})y_{pm}$ in the space $H_p(\mu_{b^{-1}})$, so it acts by the formula

$$\left(\tilde{T}_{I+tE_{nm}}^{R,\mu_b} f\right)(\ldots, y_{pn},\ldots, y_{pm},\ldots) =$$

$$= \left(\frac{d\mu_{b^{-1},p}(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots)}{d\mu_{b^{-1},p}(\ldots, y_{pn},\ldots, y_{pm},\ldots)}\right)^{1/2} f(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots).$$

So the commutation $\left[\tilde{A}, \tilde{T}_{I+tE_{nm}}^{R,\mu_b}\right] = 0 \ \ \forall t \in \mathbb{R}^1$, where $\tilde{A} = F_p^b A\left(F_p^b\right)^{-1}$, gives us

$$a\left(y_{pp+1},\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\right) = a\left(y_{pp+1},\ldots, y_{pn},\ldots, y_{pm},\ldots\right) \quad \forall t \in \mathbb{R}^1.$$

Indeed, it is sufficient to compare two equations,

$$\left(\tilde{A}\tilde{T}_{I+tE_{nm}}^{R,\mu_b} f\right)(\ldots, y_{pn},\ldots, y_{pm},\ldots) = a(\ldots, y_{pn},\ldots, y_{pm},\ldots)\times$$

$$\times\left(\frac{d\mu_{b^{-1},p}(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots)}{d\mu_{b^{-1},p}(\ldots, y_{pn},\ldots, y_{pm},\ldots)}\right)^{1/2} f(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots),$$

$$\left(\tilde{T}_{I+tE_{nm}}^{R,\mu_b}\tilde{A} f\right)(\ldots, y_{pn},\ldots, y_{pm},\ldots) = \left(\frac{d\mu_{b^{-1},p}(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots)}{d\mu_{b^{-1},p}(\ldots, y_{pn},\ldots, y_{pm},\ldots)}\right)^{1/2}\times$$

$$\times\, a(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots) f(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots).$$

By ergodicity of the measure $\mu_{b^{-1},p}$, the function

$$a = a\left(y_{pp+1},\ldots, y_{pn},\ldots\right)$$

is constant and the operator A is trivial, $A = \lambda I$.

2. Sufficiency is obvious. Let $T_p^{R,\mu} \sim T_{p'}^{R,\mu'}$, we prove that $p = p'$ and $\mu \sim \mu'$. Let us assume that $p \neq p'$, for example, $p > p'$ and consider the restrictions $T|_G$ of the representations $T = T_p^{R,\mu}$ and $T_{p'}^{R,\mu'}$ to the subgroup $G = X_{P,0} =$

$= \{ I + x \in B_0^{\mathbb{Z}} \mid I + x \in X_p \}$. The spectral measure $\mathbb{E}_p^{\mu}$ of the restriction $T_p^{R,\mu}\big|_{X_{p,0}}$ is the spectral measure of the commutative family of self-adjoint operators $i^{-1}\mathbb{D}_p(\mu) = \{ i^{-1} D_{pn}(\mu) \}_{p=n+1}^{\infty}$ and the spectral measure $\mathbb{E}_p^{\mu'}$ of $T_{p'}^{R,\mu'}\big|_{X_{p,0}}$ is trivial (see (8)), so $p = p'$. In this case, the spectral measures $\mathbb{E}_p^{\mu}$ and $\mathbb{E}_p^{\mu'}$ are equivalent, so $\mu \sim \mu'$.

Indeed let use the Fourier – Wiener transform F_p^b. We denote by $\mathbb{E}_p^{\mu_{b^{-1}}}(y)$ the spectral measure of the family of operators of multiplications by independent variables $(y_{pn})_{n=p+1}^{\infty}$ in the Hilbert space $H_p(\mu_{b^{-1}})$. Since the spectral measures $\mathbb{E}_p^{\mu}$ and $\mathbb{E}_p^{\mu'}$ are equivalent so using (11) we see that spectral measures $\mathbb{E}_p^{\mu_{b^{-1}}}(y)$ and $\mathbb{E}_p^{\mu_{(b')^{-1}}}(y)$ are equivalent. Moreover, we have

$$\left(\mathbb{E}_p^{\mu_{b^{-1}}}(y)(\Delta)\mathbb{1}, \mathbb{1}\right)_{H_p(\mu_{b^{-1}})} = \mu_{b^{-1},p}(\Delta).$$

Finally,

$$\mathbb{E}_p^{\mu} \sim \mathbb{E}_p^{\mu'} \Leftrightarrow \mathbb{E}_p^{\mu_{b^{-1}}}(y) \sim \mathbb{E}_p^{\mu_{(b')^{-1}}}(y) \Leftrightarrow \mu_{b^{-1},p} \sim \mu_{(b')^{-1},p} \Leftrightarrow$$

$$\Leftrightarrow \prod_{n=p+1}^{\infty} \frac{4(b_{pn})^{-1}(b'_{pn})^{-1}}{\left((b_{pn})^{-1} + (b'_{pn})^{-1}\right)^2} > 0 \Leftrightarrow \prod_{n=p+1}^{\infty} \frac{4 b_{pn} b'_{pn}}{\left(b_{pn} + b'_{pn}\right)^2} > 0 \Leftrightarrow \mu_{b,p} \sim \mu_{b',p}.$$

6. Tensor product of a finite number of the elementary representations and irreducibility. Let $\{p\} = (p_1, \ldots, p_m)$ be a finite subset of $\mathbb{Z}$.

Theorem 2. 1. *The representation* $T^{R,\mu,\{p\}}$ *is the tensor product of the representations* $T_{p_k}^{R,\mu_{p_k}}$, $1 \le k \le m$,

$$T^{R,\mu,\{p\}} = \otimes_{k=1}^{m} T_{p_k}^{R,\mu_{p_k}}. \tag{12}$$

2. *The representation* $T^{R,\mu,\{p\}}$ *is irreducible if and only if*

i) $S_{p_k p_n}^{L}(\mu) = \infty$, $1 \le k < n \le m$,

ii) *the measure* μ *on the space* $X^{\{p\}}$ *is* $B_0^{\mathbb{Z}}$*-right-ergodic.*

Proof. We prove the theorem for $\{p\} = (1, 2, \ldots, q)$. For other finite $\{p\}$, the proof is the same. We will show that by using the generators $A_{kn}^{R,\mu,q} := A_{kn}^{R,\mu,(1,2,\ldots,q)}$, $k < n$, it is possible to approximate the operators of multiplication by independent variables x_{kn}, $1 \le k < n \le q$, and the set of operators $D_{kn}(\mu)$, $k < n$, $k \le q$. Indeed, according to (9) we have

$$A_{1n}^{R,\mu,q} = D_{1n}(\mu), \quad 1 < n, \qquad A_{2n}^{R,\mu,q} = x_{12} D_{1n}(\mu) + D_{2n}(\mu), \quad 2 < n,$$

$$A_{3n}^{R,\mu,q} = x_{13} D_{1n}(\mu) + x_{23} D_{2n}(\mu) + D_{3n}(\mu), \quad 3 < n,$$

$$A_{kn}^{R,\mu,q} = \sum_{r=1}^{k-1} x_{rk} D_{rn}(\mu) + D_{kn}(\mu), \quad k \le q, \quad k < n.$$

$$A_{kn}^{R,\mu,q} = \sum_{r=1}^{q} x_{rk} D_{rn}(\mu), \text{ if } q < k < n.$$

The proof of approximation is the same as in [9]. It is based on the Lemma 6 in [14].

Let us denote by $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ the von-Neumann algebra, generated by the representation $T^{R,\mu,q}$: $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}}) = \left(T_t^{R,\mu,q} \middle| t \in B_0^{\mathbb{Z}}\right)''$. Let also $\langle f_n | n = 1,2,\ldots \rangle$ be the closure of the linear space, generated by the set of vectors $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H.

Definition 2. *Recall* [15] *that a not necessarily bounded self-adjoint operator A on a Hilbert space H is affiliated to the von-Neumann algebra M of operators on this Hilbert space H (denoted $A\,\eta\,M$) if $\exp(itA) \in M \quad \forall t \in \mathbb{R}^1$.*

Lemma 4 [14]. $\{x_{kn}\}_{1 \le k < n \le q}\ \eta\ \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ *if* $S_{kn}^L(\mu) = \infty$, $k < n \le q$. *In this case we also have* $D_{kn}(\mu)\ \eta\ \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$, $k < n$, $k \le q$.

Finally we have $\{x_{kn}\}_{k<n\le q}\ \eta\ \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$, $\{D_{kn}(\mu)\}_{k<n,\,k\le q}\ \eta\ \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$, so the commutant $\left(\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})\right)'$ of the von-Neumann algebra $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ coincides with essentially bounded functions from the family of operators $i^{-1}\mathbb{D}^q(\mu) = \{i^{-1}D_{kn}(\mu)\}_{k\le q<n}$.

Let now a bounded operator $A \in L(H^q(\mu))$ commute with $T_t^{R,\mu,q}$, $t \in B_0^{\mathbb{Z}}$. Then this operator A is an operator of multiplication in the space $H^q(\mu)$ by some essentially bounded function, $A = a\left(\{i^{-1}D_{kn}(\mu)\}_{k<n,\,k\le q}\right)$.

As in the proof of the Theorem 1 we use here an appropriate Fourier – Wiener transform to prove irreducibility. Let us denote $F^{b,q} = \otimes_{p=1}^{q} F_p^b$. This operator is an isometry between $H^q(\mu_b)$ and $H^q(\mu_{b^{-1}})$. Obviously, $\tilde{A}F^{b,q}A(F^{b,q})^{-1} = a\left(\{y_{kn}\}_{k\le q<n}\right)$ and the operator $\tilde{T}_{I+tE_{kn}}^{R,\mu,q} = F^{b,q}\tilde{T}_{I+tE_{kn}}^{R,\mu,q}(F^{b,q})^{-1}$ acts by the following formula

$$\left(\tilde{T}_{I+tE_{kn}}^{R,\mu,q} f\right)\begin{pmatrix} y_{1q+1} & \ldots & y_{1k} & \ldots & y_{1n} & \ldots \\ & \ldots & & \ldots & & \ldots \\ y_{qq+1} & \ldots & y_{qk} & \ldots & y_{qn} & \ldots \end{pmatrix} =$$

$$= \left(\frac{d\mu_{b^{-1}}^q\left(\tilde{R}_{I+tE_{kn}}(y)\right)}{d\mu_{b^{-1}}^q(y)}\right)^{1/2} f\left(\tilde{R}_{I+tE_{kn}}(y)\right) :=$$

$$:= \left(\frac{d\mu_{b^{-1}}^q\left(\tilde{R}_{I+tE_{kn}}(y)\right)}{d\mu_{b^{-1}}^q(y)}\right)^{1/2} f\begin{pmatrix} y_{1q+1} & \ldots & y_{1k}+ty_{1n} & \ldots & y_{1n} & \ldots \\ & \ldots & & \ldots & & \ldots \\ y_{qq+1} & \ldots & y_{qk}+ty_{qn} & \ldots & y_{qn} & \ldots \end{pmatrix},$$

so the commutation $\left[\tilde{A}, \tilde{T}_{I+tE_{nm}}^{R,\mu,q}\right] = 0 \quad \forall t \in \mathbb{R}^1$ gives us as in the proof of the Theorem 1, the equality

$$a\begin{pmatrix} y_{1q+1} & \ldots & y_{1k} & \ldots & y_{1n} & \ldots \\ & \ldots & & \ldots & & \ldots \\ y_{qq+1} & \ldots & y_{qk} & \ldots & y_{qn} & \ldots \end{pmatrix} =$$

$$= a\begin{pmatrix} y_{1q+1} & \dots & y_{1k}+ty_{1n} & \dots & y_{1n} & \dots \\ & \dots & & \dots & & \dots \\ y_{qq+1} & \dots & y_{qk}+ty_{qn} & \dots & y_{qn} & \dots \end{pmatrix} \quad \forall t \in \mathbb{R}^1, \quad \forall q < k < n.$$

By ergodicity of the measure $\mu^q_{b^{-1}}$ this means that the function $a(\{ y_{kn} \}_{k \le q < n})$ is *constant*, $a(y) = \text{const}$.

7. Regular representations as infinite tensor product of the elementary representations.

Theorem 3. 1. *The representation* $T^{R,\mu}$ *is the infinite tensor product of the representations* T_p^{R,μ_p}, $p \in \mathbb{Z}$,

$$T^{R,\mu} = \otimes_{p \in \mathbb{Z}} T_p^{R,\mu_p}. \tag{13}$$

2. *The representation* $T^{R,\mu}$ *is irreducible if:*

i) $S^L_{kn}(\mu) = \infty \ \ \forall\, k < n$;

ii) *the measure* μ *on the group* $B^{\mathbb{Z}}$ *is* $B_0^{\mathbb{Z}}$*-right-ergodic;*

iii) $\sup\limits_{n,\, n>k} \dfrac{S^R_{kn}(\mu)}{b_{kn}} = C_k < \infty \ \ \forall\, k \in \mathbb{Z}$.

Proof. The irreducibility is proved in [12]. The representation (13) follows from (4) and (10).

8. Tensor product of an infinite number of elementary representations and irreducibility. Let $\{p\}$ be an infinite subset of $\mathbb{Z}$ with only finite number of negative integers.

Theorem 4. 1.*The representation* $\otimes_{p_k \in \{p\}} T_{p_k}^{R,\mu_{p_k}}$ *is irreducible if and only if:*

i) $S^L_{p_k p_n}(\mu) = \infty \ \ \forall\, p_k < p_n,\ p_k, p_n \in \{p\}$;

ii) *the measure* $\otimes_{p_k \in \{p\}} \mu_{p_k}$ *is* $B_0^{\mathbb{Z}}$*-right-ergodic.*

2. *In this case,* $\otimes_{p_k \in \{p\}} T_{p_k}^{R,\mu_{p_k}} = T^{R,\mu,\{p\}}$, *where* $\mu = \otimes_{p_k \in \{p\}} \mu_{p_k}$.

3. $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$ *if and only if* $\{p\} = \{p'\}$ *and* $\mu \sim \mu'$.

4. *The tensor product of two irreducible representations* $T^{R,\mu,\{p\}} \otimes T^{R,\mu',\{p'\}}$ *is irreducible if and only if* $\{p\} \cap \{p'\} = \{\varnothing\}$ *and* $S^L_{p_k p'_n}(\mu \otimes \mu') = \infty$ $\forall\, p_k \in \{p\},\ p'_n \in \{p'\}$.

Proof. The irreducibility and equivalence for $\{p\} = \{p'\} = (p_n)_{n=1}^{\infty}$, $p_n = n$ follows from the Theorem 1.1 and Theorem 3.1 in [9]. For another infinite $\{p\}$ with only a finite number of negative integers, the proof of parts 1 and 2 is the same.

Let us prove the part 3 for a general $\{p\}$. Sufficiency is obvious. Necessity is based on the Theorem 1 part 2 and Theorem 3.1 in [9]. Let $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$, where $\{p\} = (p_1, p_2, \dots)$, $\{p'\} = (p'_1, p'_2, \dots)$. We prove that $\{p\} = \{p'\}$ and $\mu \sim \mu'$. Let us assume that $p_1 \ne p'_1$, for example, $p_1 > p'_1$ and consider the spectral measures $\mathbb{E}^{\mu}_{p_1}$ and $\mathbb{E}^{\mu'}_{p_1}$ of the restrictions of the representations $T^{R,\mu,\{p\}}$ and $T^{R,\mu',\{p'\}}$ on the subgroup $X_{p_1,0}$. The spectral measure $\mathbb{E}^{\mu}_{p_1}$ is the spectral measure of the commutative family of self-adjoint operators $i^{-1}\mathbb{D}_{p_1}(\mu)=\{i^{-1}D_{p_1 n}(\mu)\}_{n=p_1+1}^{\infty}$ and is not trivial but the spectral measure $\mathbb{E}_{p_1}^{\mu'}$ is trivial (see (9), (10)). This contradicts $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$, so $p_1 = p_1'$. In this case the spectral measures $\mathbb{E}_{p_1}^{\mu}$ and $\mathbb{E}_{p_1}^{\mu'}$ are equivalent, so $\mu_{p_1} \sim \mu'_{p_1}$ and $T_{p_1}^{R,\mu_{p_1}} \sim T_{p_1}^{R,\mu'_{p_1}}$. Since, by formula (13), we have

$$T^{R,\mu,\{p\}} = T_{p_1}^{R,\mu_{p_1}} \otimes T^{R,\mu^{\{p_2\}},\{p_2\}}, \quad T^{R,\mu',\{p'\}} = T_{p_1}^{R,\mu'_{p_1}} \otimes T^{R,\mu'^{\{p_2'\}},\{p_2'\}},$$

and the equivalence $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$ holds, we conclude that $T^{R,\mu^{\{p_2\}},\{p_2\}} \sim T^{R,\mu'^{\{p_2'\}},\{p_2'\}}$, where $\{p_2\}=(p_2,p_3,\ldots)$, $\{p_2'\}=(p_2',p_3',\ldots)$, and

$$T^{R,\mu,\{p_2\}} = \otimes_{p_k\in\{p_2\}} T_{p_k}^{R,\mu_{p_k}}, \quad T^{R,\mu',\{p_2'\}} = \otimes_{p_k\in\{p_2'\}} T_{p_k}^{R,\mu'_{p_k}}.$$

Analogously we conclude that $p_2 = p_2'$ and $\mu_{p_2} \sim \mu'_{p_2}$. Finally, $\{p\}=\{p'\}$ and $\mu_{p_k} \sim \mu'_{p_k}$ $\forall p_k \in \{p\}=\{p'\}$. For finite $\{p\}$, $\{p'\}$ the proof is finished since in this case we have $\mu=\otimes_{p_k\in\{p\}}\mu_{p_k} \sim \mu'=\otimes_{p_k\in\{p'\}}\mu'_{p_k}$. In the general case (for infinite $\{p\}$, $\{p'\}$), the equivalence $\mu_{p_k}\sim\mu'_{p_k}$ $\forall p_k\in\{p\}=\{p'\}$ does not imply $\mu=\otimes_{p_k\in\{p\}}\mu_{p_k} \sim \mu' = \otimes_{p_k\in\{p'\}}\mu'_{p_k}$. For the particular case $\{p\}=(p_k)_{k=1}^{\infty}$, $p_k=k$, $k\in\mathbb{N}$, the equivalence of the measures $\mu\sim\mu'$ follows from the Theorem 3.1 in [9]. For general $\{p\}$ the proof is the same.

4. Sufficiency follows from parts 1 and 2, since in this case we have

$$T^{R,\mu,\{p\}} \otimes T^{R,\mu',\{p'\}} = T^{R,\mu\otimes\mu',\{p\}\cup\{p'\}},$$

where $\{p\}\cup\{p'\}=\{p_k,p_n' \mid p_k\in\{p\},\ p_n'\in\{p'\}\}$. Let now $\{p\}\cap\{p'\}=\{p''\}$ be finite, $\{p''\}:=(p_1,\ldots,p_k)$. For infinite $\{p''\}$ the proof is the same. In this case we have $\{p\}=\{q\}\cup\{p''\}$ and $\{p'\}=\{q'\}\cup\{p''\}$, so $\{p\}\cup\{p'\} = \{q\}\cup\{q'\}\cup\{p''\}$ and we have

$$T^{R,\mu,\{p\}} \otimes T^{R,\mu',\{p'\}} = T^{R,\mu^{\{q\}}\otimes\mu^{\{p''\}}\otimes\mu'^{\{q'\}},\{q\}\cup\{p''\}\cup\{q'\}} \otimes T^{R,\mu'^{\{p''\}},\{p''\}}.$$

So the proof that the last tensor product is reducible is similar to the proof that the following tensor product

$$T^{R,\mu,q} \otimes T^{R,\mu',q+k}$$

is reducible.

Consider the essentially bounded function $a: X^q \ni x \mapsto a(x) \in \mathbb{C}^1$ and let A_0 be the operator of multiplication in the space

$$H^q(\mu)\otimes H^{q+k}(\mu') = L^2(X^q,d\mu)\otimes L^2(X^{q+k},d\mu') = L^2(X^q\otimes X^{q+k},d\mu\otimes\mu')$$

by the function $a_0: X^q\times X^{q+k} \ni (x,y,z) \mapsto a_0(x,y,z) = a(yx^{-1}) \in \mathbb{C}^1$. We show that the representation $T^{R,\mu,q}\otimes T^{R,\mu',q+k}$ commutes with the operator A_0. Indeed, for any function $f(x,y,z)\in L^2(X^q\otimes X^{q+k},d\mu\otimes\mu')$, using the property that for any $(y,z)\in X^q\times X^k = X^{q+k}$ in $B^{\mathbb{Z}}$, $(y,z)=zy$ holds, we have

$$\left(T_t^{R,\mu,q}\otimes T_t^{R,\mu',q+k}A_0 f\right)(x,zy) = \left(T_t^{R,\mu,q}\otimes T_t^{R,\mu',q+k}a_0 f\right)(x,zy) =$$

$$= \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}\left(\frac{d\mu'(z\,yt)}{d'\mu(z\,y)}\right)^{1/2} a\big((yt)(xt)^{-1}\big) f(xt, z\,yt) =$$

$$= a(yx^{-1})\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}\left(\frac{d\mu'(z\,yt)}{d\mu'(z\,y)}\right)^{1/2} f(xt, z\,yt) =$$

$$= \left(A_0\left(T_t^{R,\mu,q} \otimes T_t^{R,\mu',q+k}\right) f\right)(x, zy).$$

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