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SPATIALLY HOMOGENEOUS BOLTZMANN HIERARCHY AS AVERAGED SPATIALLY INHOMOGENEOUS STOCHASTIC BOLTZMANN HIERARCHY*

ПРОСТОРОВО-ОДНОРІДНА ІЄРАРХІЯ БОЛЬЦМАНА ЯК УСЕРЕДНЕНА ПРОСТОРОВО-НЕОДНОРІДНА СТОХАСТИЧНА ІЄРАРХІЯ БОЛЬЦМАНА

We introduce the stochastic dynamics in phase space that corresponds the Boltzmann equation and hierarchy and is the Boltzmann–Grad limit of Hamiltonian dynamics of systems of hard spheres. By method of averaging over space of positions, we derive from it the stochastic dynamics in momentum space that corresponds to the space-homogeneous Boltzmann equation and hierarchy. Analogous dynamics in mean-field approximation had been postulated by Kac for explanation of the phenomenon of propagation of chaos and derivation of the Boltzmann equation.

Введено стохастичну динаміку у фазовому просторі, яка відповідає рівнянню та ієрархії Больцмана і є границею Больцмана–Греда гамільтонової динаміки системи пружних куль. Методом усереднення за просторовими змінними з неї виведено стохастичну динаміку в імпульсному просторі, яка відповідає просторово-однорідному рівнянню та ієрархії Больцмана. Аналогічна динаміка в наближенні середнього поля у свій час постульована Кацом для пояснення явища еволюції хаосу та виведення рівняння Больцмана.

Introduction. Equations of classical statistical mechanics are derived from equations of classical mechanics. For example, the BBGKY hierarchy is derived from the Hamilton equations via the Liouville equation for distribution function on the phase space. The Liouville equation are obtained for distribution functions, at given time t, that are result of action of the operator of evolution (the operator of shift along the trajectory) on initial distribution functions.

The BBGKY hierarchy, that is a basis of nonequilibrium classical statistical mechanics, are obtained by the following method:

1) the Hamilton equation → 2) the operator of evolution (the operator of shift along the trajectory) on initial distribution function → 3) the Liouville equation for distribution functions → 4) the BBGKY hierarchy for (reduced) correlation functions [1–5].

The Boltzmann equation are proved to be of great importance in classical statistical mechanics. M. Kac [6, 7], as far as we know, was the first who made an attempt to modify the above described method of derivation of the BBGKY hierarchy and the Boltzmann equation in the spatially homogeneous case when correlation functions depend only on time and momenta and do not depend on positions of particles. He proposed the following method of derivation of the spatially homogeneous Boltzmann equation:

1) certain stochastic Markov process in momentum space → 2) The Kolmogorov–Chapman equation for distribution function → 3) the hierarchy for (reduced) correlation functions in mean field approximation.

It has been shown that in the thermodynamic limit the phenomenon of propagation of chaos takes place and all many-particle correlation functions are products of one-particle correlation function that satisfies the Boltzmann equation in the spatially homogeneous case [6–11].

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This result was a great achievement of nonequilibrium statistical mechanics, but it also creates a series of questions. The first question is connected with the stochastic dynamics in momentum space, because a physical meaning has only dynamics in phase space where states of real particles are determined by their momenta and positions. As well known, the spatially inhomogeneous Boltzmann equation is associated with certain stochastic dynamics in phase space [3–5] and it is natural to suppose that it reduces to the Kac's dynamics in momentum space in the spatially homogeneous case.

The second question is connected with the mean field approximation that is not completely justified from the physical point of view because there it was made an assumption that does not follow from postulates of physics. The last remark become extremely actual with connection with the last achievements concerning the Boltzmann–Grad limit for systems of hard spheres.

Namely, it has been shown that solutions of the BBGKY hierarchy, in the Boltzmann–Grad limit and for initial data that satisfy condition of chaos, (i.e. initial many-particles correlation functions are products of one-particle correlation function), also satisfy condition of chaos in the following sense [2]. Many-particle correlation functions at arbitrary time (from the interval where solutions exist), and outside certain hyperplanes in phase space of lower dimension, are equal to products of one-particle correlation functions and the last is solution of the Boltzmann equation. Any mean-field approximations have not been made.

It has been established that the Hamilton dynamics of system of hard spheres in the Boltzmann–Grad limit degenerates into certain stochastic dynamics of point-particles [3, 4]. The stochastic dynamics consists in the following: point-particles move freely until positions of some pair of them coincide, then this pair elastically collides but the vector that determines the elastic collision is a random one and uniformly distributed on unit sphere. After collision particles move freely until the next collision. Point-particles interact on time interval $[0, t]$ only if their phase points belong to hyperplane V_{ij} where the vector of difference of their positions is parallel to the vector of their momenta, i.e., $q_i - q_j = \tau(p_i - p_j), 0 \leq \tau \leq t$.

In connection with this circumstance the following problem occurs: how to define correlation functions and averages of observables because this stochastic dynamics differs from free one only on the hyperplanes V_{ij} where particles interact. In the standard statistical mechanics in which particles interact through short-range potential the sets of lower dimension are neglected because correlation functions and averages of observables are determined by the Lebesgue integrals.

It turns out that in this case it is necessary to take into account contributions from the hyperplanes V_{ij} where point-particles interact [3, 4]. It was a great surprise that in solutions of the Boltzmann equation and hierarchy, represented by series of iterations, the contributions from the hyperplanes V_{ij} have been taken into account and in the like manner [5]. Thus, a new conception of correlation functions, that took into account the contributions from hyperplanes where point-particles interact, has been proposed.

For these correlation functions the hierarchy has been derived, it has been named the stochastic Boltzmann hierarchy, and it differs from the standard Boltzmann hierarchy by certain terms with δ-functions and the boundary condition. Solutions of the stochastic Boltzmann hierarchy satisfy the chaos condition (or the condition of propagation of chaos), namely, all the correlation functions $F_s(t, x_1, \dots, x_s) = F_1(t, x_1) \dots F_1(t, x_s)$ outside all the hyperplanes V_{ij}, $1 \leq i < j \leq s$, if the initial correlation functions satisfy the condition of chaos. The one-particle correlation function is solution of the Boltzmann equation [3–5].

In the present paper we established that Kac's results concerning spatially homogeneous case of systems of hard spheres directly follows from the above described our results [3–5].

Namely, it was established that the evolution operator in spatially homogeneous case can be obtained from the evolution operator of the stochastic dynamics in phase space by means of specific averaging over the space of positions. It was shown that in the framework of the stochastic dynamics the functional-average of spatially homogeneous observables over distribution functions with spatially homogeneous initial functions diverges as volume of system tends to infinity. After some specific averaging over the space of positions we obtained the functional-average of spatially homogeneous observables over spatially homogeneous distribution functions. This functional defines the operator of evolution of spatially homogeneous distribution functions and the infinitesimal generator of the obtained operator of evolution coincides with that proposed by Kac.

We showed that this infinitesimal generator of the operator of evolution in the spatially homogeneous case can also be obtained from the infinitesimal generator (in phase space) of the evolution operator of the stochastic dynamics by the same specific averaging over the space of positions. The equation for spatially homogeneous distribution function was derived. It was shown that Kac's results in mean field approximation can be obtained by simple modification from our functional-average.

By using the equation for spatially homogeneous distribution functions, the hierarchy for spatially homogeneous correlation functions was derived. In mean field approximation obtained hierarchy coincides with that derived by Kac. The operator that determines the spatially homogeneous hierarchy can be obtained by averaging over the space of positions from the corresponding operator of the stochastic spatially inhomogeneous Boltzmann hierarchy.

Solutions of the spatially inhomogeneous stochastic Boltzmann hierarchy and the spatially homogeneous hierarchy can be represented by series of iterations. For the stochastic Boltzmann hierarchy (spatially inhomogeneous) the series are uniformly convergent, on compacts in the phase space and on finite time interval for initial data that belong to the space of sequences of functions bounded with respect to positions and exponentially decreasing with respect to momenta.

The series of iterations of the spatially homogeneous hierarchy can be obtained by simple replacement of the stochastic evolution operators by the spatially homogeneous evolution operator. From these representations one can see that spatially inhomogeneous and homogeneous correlation functions do not possess the chaos property, i.e., are not products of one-particle correlation function, even if the initial correlation functions satisfy condition of chaos. But the one-particle spatially inhomogeneous and homogeneous correlation functions are solutions of the spatially inhomogeneous and homogeneous Boltzmann equations respectively.

We have also the chaos property or propagation of chaos in the following sense. If one considers solutions of the stochastic spatially inhomogeneous Boltzmann hierarchy with spatially homogeneous initial correlation functions that satisfy the condition of chaos $F_s(0, x_1, \ldots, x_s) = F_s(0, p_1, \ldots, p_s) = F_1(0, p_1) \ldots F_1(0, p_s)$ then outside all hyperplanes V_{ij}, $1 \leq i < j \leq s$, the correlation functions do not depend on positions

$F_s(t, x_1, \ldots, x_s) = F_s(t, p_1, \ldots, p_s)$ and satisfy the condition of chaos (or propagation of chaos), i.e.

$$F_s(t, p_1, \ldots, p_s) = F_1(t, p_1) \ldots F_s(t, p_s).$$

The one-particle correlation function $F_s(t, p_s)$ satisfies the spatially homogeneous Boltzmann equation (Sect. IV).

In this meaning the mean field approximation is obtained by neglecting the fact that the correlation functions $F_s(t, x_1, \ldots, x_s)$ depend on positions on the hyperplanes $V_{ij}(q_i - q_j = \tau(p_i - p_j), 0 \leq \tau \leq t)$, $1 \leq i < j \leq s$. For arbitrary fixed $(p_1, \ldots, p_s)$ we identify them on the hyperplanes V_{ij} with their value outside hyperplanes V_{ij} where $F_s(t, x_1, \ldots, x_s), s \geq 1$, do not depend on positions, i.e. $F_s(t, x_1, \ldots, x_s) = F_s(t, p_1, \ldots \ldots, p_s)$ and coincide with solutions of the spatially homogeneous hierarchy in the mean-field approximation.

I. Stochastic dynamics for the space-homogeneous stochastic Boltzmann hierarchy.

1. ***System of N particles.*** Consider N particles with unit mass in three-dimensional space R^3 and denote by $x_1 = (q_1, p_1), \ldots, x_N = (q_N, p_N)$ their phase points. The stochastic dynamics of this system is defined as follows: Particles move as free ones until the positions of two arbitrary particles with numbers i and j coincide at time $-\tau$: $q_i - p_i\tau = q_j - p_j\tau$. Then these two particles collide, their momenta become

$$p_i^* = p_i - \eta_{ij}\,\eta_{ij}\cdot(p_i - p_j), \quad p_j^* = p_j + \eta_{ij}\,\eta_{ij}\cdot(p_i - p_j),$$
$$|\eta_{ij}| = 1, \quad \eta_{ij}\cdot(p_i - p_j) \geq 0,$$

at time $-t$ their phase points are $x_i(-t) = (q_i - p_i\tau - p_i^*(t - \tau), p_i^*)$, $x_j(-t) = (q_j - p_j\tau - p_j^*(t - \tau), p_j^*)$, and they move freely until the next collision. The vectors η_{ij} are random and uniformly distributed on the sphere $|\eta_{ij}| = 1$. If $\eta_{ij} \in S_2^-$, $\eta_{ij}\cdot(p_i - p_j) \leq 0$, then particles continue move freely. We neglect the case when three or more particles collide at the same point.

Consider an infinitesimal time $-\Delta t$ and introduce the following functional [3, 4]

$$(S_N(-\Delta t) f_N, \phi_N) =$$
$$= \int dx_1 \ldots dx_N f_N(q_1 - p_1\Delta t, p_1, \ldots, q_N - p_N\Delta t, p_N)\phi_N(q_1, p_1, \ldots, q_N, p_N) +$$
$$+ \sum_{i<j=1}^{N} \int dx_1 \ldots dx_N \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j)\, \delta(q_i - p_i\tau - q_j + p_j\tau) \times$$
$$\times \Big[f_N(q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \ldots$$
$$\ldots, q_j - p_j\tau - p_j^*(\Delta t - \tau), p_j^*, \ldots, q_N - p_N\Delta t, p_N) -$$
$$- f_N(q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, p_i \ldots$$
$$\ldots, q_j - p_j\Delta t, p_j, \ldots, q_N - p_N\Delta t, p_N) \Big] \phi_N(q_1, p_1, \ldots, q_N, p_N) \qquad (1.1)$$

that is equal to the average of the observable $\phi_N(x_1, \ldots, x_N)$ over the state

$$S_N(-\Delta t) f_N(x_1, \ldots, x_N) = f_N(x_1(-\Delta t), \ldots, x_N(-\Delta t)).$$

As usual, it is supposed that f_N is real symmetric differentiable normalized function and ϕ_N is real symmetric test function.

Note that in the average (functional) (1.1) *the contributions from the hypersurfaces* $q_i - q_j = \tau(p_i - p_j)$, $0 \leq \tau \leq \Delta t$, $1 \leq i < j \leq N$, *of lower dimension, where the* *stochastic particles interact, are taken into account. These contributions are equal to the second term in the right-hand side of* (1.1).

Now consider the case when the functions f_N and ϕ_N do not depend on positions

$$f_N(q_1,p_1,\ldots,q_N,p_N) = f_N(p_1,\ldots,p_N),$$
$$\phi_N(q_1,p_1,\ldots,q_N,p_N) = \phi_N(p_1,\ldots,p_N),$$
$$f_N(q_1 - p_1\Delta t, p_1,\ldots,q_N - p_N\Delta t, p_N) = f_N(p_1,\ldots,p_N),$$
$$f_N(q_1 - p_1\Delta t, p_1,\ldots,q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*,\ldots$$
$$\ldots, q_j - p_j\tau - p_j^*(\Delta t - \tau), p_j^*,\ldots,q_N - p_N\Delta t, p_N) =$$
$$= f_N(p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots p_N).$$

In this case the functional (1.1) is divergent, the first and second terms are proportional to V^N and V^{N-1} respectively (V is the volume of R^3). Instead of functional (1.1) we introduce the following functional

$$(\tilde{S}_N(-\Delta t)f_N,\phi_N) =$$
$$= \lim_{V\to\infty}\frac{1}{V^N}\int_\Lambda dq_1\int dp_1\cdots\int_\Lambda dq_N\int dp_N f_N(p_1,\ldots,p_N)\phi_N(p_1,\ldots,p_N) +$$
$$+\lim_{V\to\infty}\frac{1}{V^{N-1}}\sum_{i<j=1}^{N}\int_\Lambda dq_1\int dp_1\cdots\int_{\Lambda_i} dq_i\int dp_i\ldots$$
$$\cdots\int_{\Lambda_j} dq_j\int dp_j\cdots\int_\Lambda dq_N\int dp_N\int_0^{\Delta t} d\tau\,\times$$
$$\times\int_{S_2^+} d\eta_{ij}\,\eta_{ij}\cdot(p_i - p_j)\,\delta(q_i - p_i\tau - q_j + p_j\tau)\,\times$$
$$\times\Big[f_N(p_1,\ldots p_i^*,\ldots p_j^*,\ldots,p_N) - f_N(p_1,\ldots,p_i\ldots,p_j,\ldots p_N)\Big]\phi_N(p_1,\ldots p_N) =$$
$$= \int dp_1\ldots dp_N f_N(p_1,\ldots p_N)\phi_N(p_1,\ldots,p_N) +$$
$$+\Delta t\sum_{i<j=1}^{N}\int dp_1\ldots dp_N\int_{S_2^+} d\eta_{ij}\,\eta_{ij}\cdot(p_i - p_j)\,\times$$
$$\times\Big[f_N(p_1,\ldots p_i^*,\ldots p_j^*,\ldots,p_N) - f_N(p_1,\ldots,p_i\ldots,p_j,\ldots p_N)\Big]\phi_N(p_1,\ldots p_N). \tag{1.2}$$

We denote by Λ_i and Λ_j the spheres with centers in the points $-p_i\tau$, $-p_j\tau$ respectively and with volumes $V(\Lambda_i) = V(\Lambda_j) = V$. The sphere Λ has the center in the origin, $V(\Lambda) = V$. Functional (1.2) was obtained from functional (1.1) by averaging over space of positions (configurational space) and is average of the state $\tilde{S}^N(-\Delta t)f_N(p_1,\ldots,p_N)$ over the observables $\phi_N(p_1,\ldots,p_N)$. Formula (1.2) defines the operator of evolution $\tilde{S}_N(-\Delta t)$ of states $f_N(p_1,\ldots,p_N)$ in spatially homogeneous case. The second term is associated with the contribution from the hypersurface $q_i - q_j = \tau(p_i - p_j)$, $1 \le i < j \le$ $\le N$, where the stochastic particles interact.

It follows from (1.2) that the operator of evolution $\tilde{S}_N(-\Delta t)$ of the state of $N-$ particle system in the spatially homogeneous case is defined by the following formula for an infinitesimal time $-\Delta t$

$$\tilde{S}_N(-\Delta t) f_N(p_1, \dots, p_N) = f_N(\Delta t, p_1, \dots, p_N) =$$
$$= f_N(p_1, \dots, p_N) + \Delta t \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j) \times$$
$$\times \Big[f_N(p_1, \dots p_i^*, \dots p_j^*, \dots, p_N) - f_N(p_1, \dots, p_i \dots, p_j, \dots p_N) \Big]. \qquad (1.3)$$

In (1.3) the state $\tilde{S}_N(-\Delta t) f_N(p_1, \dots, p_N)$ is averaged over the random vectors η_{ij}, $i \le j \le N$. For the fixed random vectors η_{ij} we have

$$\tilde{S}_N(-\Delta t) f_N(p_1, \dots, p_N) = f_N(\Delta t, p_1, \dots, p_N) =$$
$$= f_N(p_1, \dots, p_N) + \Delta t \sum_{i<j=1}^{N} \eta_{ij}(p_i - p_j) \Theta\left((\eta_{ij}\cdot(p_i - p_j)\right) \times$$
$$\times \Big[f_N(p_1, \dots p_i^*, \dots p_j^*, \dots, p_N) - f_N(p_1, \dots, p_i \dots, p_j, \dots p_N) \Big], \qquad (1.4)$$

where $\Theta(\alpha) = 1, \alpha > 0, \Theta(\alpha) = 0, \alpha < 0$. In (1.4) we used the same denotation for the states $f_N(\Delta t, p_1, \dots, p_N)$ and operator $\tilde{S}_N(-\Delta t)$ with fixed η_{ij} as for those averaged with respect to η_{ij}.

The probabilistic interpretation of formulae (1.3)–(1.4) is obvious.

We do not know how to construct the functional $\left(\tilde{S}_N(-t) f_N, \phi_N\right)$ directly for arbitrary time t and $N \ge 3$. Therefore we will define this functional using the following procedure. We suppose that function $f_N(t, p_1, \dots, p_N) = \tilde{S}_N(-t) f_N(p_1, \dots, p_N)$ is already defined. We also suppose that the operator $\tilde{S}_N(-t)$ satisfies group property and is defined formally as follows: $\tilde{S}_N(-t) = \lim_{n\to\infty} \prod_{i=1}^{n} \tilde{S}_N(-\Delta t_i), \sum_{i=1}^{n} \Delta t_i = t$ where $\tilde{S}_N(-\Delta t_i)$ for infinitesimal Δt_i is already determined by (1.3). By using the group property of the operator of evolution

$$\tilde{S}_N(-t - \Delta t) = \tilde{S}_N(-\Delta t)\tilde{S}_N(-t) = \tilde{S}_N(-t)\tilde{S}_N(-\Delta t)$$

one can define function

$$f_N(t + \Delta t, p_1, \dots, p_N) = \tilde{S}_N(-\Delta t) f_N(t, p_1, \dots, p_N)$$

and obtain the functional equal to average of the state $\tilde{S}_N(-\Delta t) f_N(t, p_1, \dots, p_N)$, over the observable $\phi_N(p_1, \dots, p_N)$ with respect to the random vectors η_{ij} that appear on interval $(-t, -t - \Delta t)$ with an infinitesimal Δt.

The average $\left(\tilde{S}_N(-\Delta t) f_N(t), \phi_N\right)$ of the state $\tilde{S}_N(-\Delta t) f_N(t, p_1, \dots, p_N)$ over the observable $\phi_N(p_1, \dots, p_N)$ is determined by formula (1.2) if one puts the function $f_N(t, p_1, \dots, p_N)$ instead of $f_N(p_1, \dots, p_N)$

$$(\tilde{S}_N(-\Delta t) f_N(t), \phi_N) = \int dp_1 \dots dp_N\, f_N(t, p_1, \dots p_N) \phi_N(p_1, \dots, p_N) +$$
$$+ \Delta t \sum_{i<j=1}^{N} \int dp_1 \dots dp_N \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j) \times$$
$$\times \Big[f_N(t, p_1, \dots p_i^*, \dots p_j^*, \dots, p_N) - f_N(t, p_1, \dots, p_i \dots, p_j, \dots p_N) \Big] \phi_N(p_1, \dots p_N). \qquad (1.5)$$

It follows from (1.5) that

$$\tilde{S}_N(-\Delta t) f_N(t, p_1, \ldots, p_N) = \tilde{S}_N(-\Delta t)\tilde{S}_N(-t) f_N(p_1, \ldots, p_N) =$$

$$= f_N(t + \Delta t, p_1, \ldots p_N) = f_N(t, p_1, \ldots p_N) + \Delta t \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j) \times$$

$$\times \Big[f_N(t, p_1, \ldots p_i^*, \ldots p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i \ldots, p_j, \ldots p_N) \Big]. \tag{1.6}$$

2. *Equation for $f_N(t, p_1, \ldots, p_N)$.* We obtain from (1.6) the following equation for the function $f_N(t, p_1, \ldots p_N)$

$$\frac{\partial f_N(t, p_1, \ldots p_N)}{\partial t} = \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j) \times$$

$$\times \Big[f_N(t, p_1, \ldots p_i^*, \ldots p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i \ldots, p_j, \ldots p_N) \Big], \tag{1.7}$$

$$f_N(t, p_1, \ldots, p_N)\Big|_{t=0} = f_N(p_1, \ldots, p_N).$$

In equation (1.7) the right-hand side is averaged with respect to the random unit vectors $\eta_{ij}, 1 \le i < j \le N$, that appear on infinitesimal interval $(t, t + \Delta t)$.

From (1.7) it follows the equation for the state $f_N(t, p_1, \ldots, p_N)$ with fixed random vectors η_{ij}

$$\frac{\partial f_N(t, p_1, \ldots p_N)}{\partial t} = \sum_{i<j=1}^{N} \eta_{ij}\cdot(p_i - p_j)\, \Theta\left(\eta_{ij}\cdot(p_i - p_j)\right) \times$$

$$\times \Big[f_N(t, p_1, \ldots p_i^*, \ldots p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i \ldots, p_j, \ldots p_N) \Big], \tag{1.8}$$

$$f_N(t, p_1, \ldots, p_N)|_{t=0} = f_N(p_1, \ldots, p_N).$$

It can be obtained if in functionals (1.1), (1.2), (1.5) integration with respect to the random vectors η_{ij} is omitted and consequently is omitted in (1.6), (1.7).

The function $f_N(t, p_1, \ldots, p_N)$ from equation (1.7) does not depend on any random vectors if the initial function $f_N(p_1, \ldots, p_N)$ does not depend on them, while the function from equation (1.8) depend on random vectors $\eta_{ij}, 1 \le i < j \le N$. We preserve for both functions the same denotation.

Thus we have obtained equation (1.7), (1.8) *for evolution of state that does not depend on position — in the case of spatially homogeneous state — starting from the stochastic dynamics in phase space and by averaging the functional* (1.1) *over the configurational space.*

Many authors [6–8] postulate analogical equation for explanation of phenomenon of propagation of chaos but with mean field multipliers $1/N$ in the right-hand side. Their equation

$$\frac{\partial f_N(t, p_1, \ldots p_N)}{\partial t} = \frac{1}{N} \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij}\, \eta_{ij}\cdot(p_i - p_j) \times$$

$$\times \Big[f_N(t, p_1, \ldots p_i^*, \ldots p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i \ldots, p_j, \ldots p_N) \Big], \tag{1.9}$$

$$f_N(t, p_1, \ldots, p_N)\Big|_{t=0} = f_N(p_1, \ldots, p_N),$$

can also be obtained from the stochastic dynamics if one puts the multiplier $1/N$ in the second term of functionals (1.1), (1.2), (1.5).

We will also need the operators $\tilde{S}_N(t)$ and equation (1.7) for positive t. To obtain them it is sufficient to replace in (1.6) and (1.7) S_2^+ by S_2^- $(\eta_{ij}\,|\eta_{ij}\cdot(p_i - p_j) \le 0)$ (details about the stochastic dynamics and the evolution operators $S_N(t)$ for $t > 0$ can be found in [3, 4]).

II. Derivation of the spatially homogeneous hierarchy.

1. ***Spatially homogeneous hierarchy in framework of canonical and great canonical ensemble.*** Consider N-particle system with normalized state $f_N(t, p_1, \ldots, p_N)$ that satisfies equation (1.7) and introduce the following sequence of reduced correlation functions

$$F_s^{(N)}(t, (p)_s) = F_s^{(N)}(t, p_1, \ldots, p_s) =$$

$$= \frac{N!}{(N-s)!} \int dp_{s+1} \ldots dp_N f_N(t, p_1, \ldots, p_s, p_{s+1}, \ldots, p_N), \quad 1 \le s \le N-1, \tag{2.1}$$

$$F_N^{(N)}(t, (p)_N) = F_N^{(N)}(t, p_1, \ldots, p_N) = f_N^{(N)}(t, p_1, \ldots, p_N)(p)_s = (p_1, \ldots, p_s).$$

By integrating the left and the right-hand side of (1.7) over momenta $p_{s+1}, \ldots, p_N$, taking into account that $f_N(t, p_1, \ldots, p_N)$ is symmetric with respect to $p_1, \ldots, p_N$, and Jacobian of transformation $(p_i, p_j) \to (p_i^*, p_j^*)$ is equal to one, we obtain the following hierarchy (see derivation of spatially inhomogeneous hierarchy in [1, 2]).

$$\frac{\partial F_s^{(N)}(t, p_1, \ldots p_s)}{\partial t} = \sum_{i<j=1}^{s} \int_{S_2^+} d\eta_{ij} \eta_{ij} \cdot (p_i - p_j) \times$$

$$\times \Big[F_s^{(N)}(t, p_1, \ldots p_i^*, \ldots p_j^*, \ldots, p_s) - F_s^{(N)}(t, p_1, \ldots, p_i \ldots, p_j, \ldots p_s) \Big] +$$

$$+ \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$

$$\times \Big[F_{s+1}^{(N)}(t, p_1, \ldots p_i^*, \ldots p_s, \ldots, p_{s+1}^*) - F_{s+1}^{(N)}(t, p_1, \ldots, p_i \ldots, p_s, p_{s+1}) \Big], \tag{2.2}$$

$$1 \le s \le N-1, \quad F_s^{(N)}(t, p_1, \ldots, p_s)\big|_{t=0} = F_s^{(N)}(p_1, \ldots, p_s).$$

Note that $F_N^{(N)}(t, p_1, \ldots, p_N)$ satisfies equation (1.7).

As known, in great canonical ensemble system can be in states $f_N(t, p_1, \ldots, p_N)$ with arbitrary $N = 0, 1 \ldots$, with certain probability. Functions $f_N(t, p_1, \ldots, p_N)$ are not normalized in this case.

The infinite sequence of reduced correlation functions are defined as follows

$$F_s(t, (p)_s) = \frac{1}{\Xi} \sum_{n=0}^{\infty} \frac{1}{n!} \int dp_{s+1} \ldots dp_{s+n}\, f_{s+n}(t, p_1, \ldots p_s, p_{s+1}, \ldots, p_{s+n}), \quad s \ge 1, \tag{2.3}$$

where grand partition function Ξ is equal to

$$\Xi = \sum_{n=0}^{\infty} \frac{1}{n!} \int dp_1 \ldots dp_n\, f_n(t, p_1, \ldots, p_n) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dp_1 \ldots dp_n\, f_n(p_1, \ldots, p_n), \quad f_0 = 1.$$

In the last equality we used the fact that

$$\int dp_1 \ldots dp_n\, f_n(t, p_1, \ldots, p_n) = \int dp_1 \ldots dp_n\, f_n(p_1, \ldots, p_n)$$

that follows directly from (1.7). It is easy to check that sequence (2.3) satisfies hierarchy (2.2) with $1 \leq s < \infty$ (see analogical calculation for spatially inhomogeneous case in [1, 2]).

In order to derive the corresponding hierarchy in the "mean field" case one should use equation (1.9) and the functions $F_s(t, p_1, \ldots, p_s)$ defined according to (2.1) but without factors $\dfrac{N!}{(N-s)!}$. The corresponding hierarchy looks as follows

$$\frac{\partial F_s^{(N)}(t, p_1, \ldots p_s)}{\partial t} = \frac{1}{N} \sum_{i<j=1}^{s} \int_{S_2^+} d\eta_{ij}\, \eta_{ij} \cdot (p_i - p_j) \times$$

$$\times \Big[F_s^{(N)}(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_s) - F_s^{(N)}(t, p_1, \ldots, p_i, \ldots, p_j, \ldots, p_s) \Big] +$$

$$+ \frac{N-s}{N} \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$

$$\times \Big[F_{s+1}^{(N)}(t, p_1, \ldots, p_i^*, \ldots, p_s, \ldots, p_{s+1}^*) - F_{s+1}^{(N)}(t, p_1, \ldots, p_i, \ldots, p_s, p_{s+1}) \Big], \quad (2.4)$$

$$1 \leq s \leq N, \quad F_s^{(N)}(t, p_1, \ldots, p_s)\big|_{t=0} = F_s^{(N)}(p_1, \ldots, p_s).$$

Performing formally the thermodynamical limit $N \to \infty$ in (2.4) and taking into account that the first term on the right-hand side of (2.4) tends to zero as $N \to \infty$, one gets the limiting hierarchy

$$\frac{\partial F_s(t, p_1, \ldots p_s)}{\partial t} = \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1} \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$

$$\times \Big[F_{s+1}(t, p_1, \ldots, p_i^*, \ldots, p_s, \ldots, p_{s+1}^*) - F_{s+1}(t, p_1, \ldots, p_i, \ldots, p_s, p_{s+1}) \Big], \quad (2.4')$$

$$s \geq 1, \quad F_s(t, p_1, \ldots, p_s)\big|_{t=0} = F_s(p_1, \ldots, p_s).$$

(The corresponding limit hierarchy for (2.2) is the same as (2.2), but $s \geq 1$ and instead of $F_s^{(N)}(t, p_1, \ldots p_s)$ one should put $F_s(t, p_1, \ldots p_N) = \lim_{N\to\infty} F_s^{(N)}(t, p_1, \ldots p_s)$.)

The "mean field" hierarchy (2.4′) has the following characteristic property: it preserves the chaos property, i.e. if the initial functions

$$F_s(p_1, \ldots, p_s) = F_1(p_1) \ldots F_1(p_s) \quad (2.5)$$

have the chaos property then functions $F_s(t, p_1, \ldots p_s)$ have also the chaos property

$$F_s(t, p_1, \ldots, p_s) = F_1(t, p_1) \ldots F_1(t, p_s) \quad (2.6)$$

and function $F_1(t, p_1)$ satisfies the nonlinear Boltzmann equation

$$\frac{\partial F_1(t, p_1)}{\partial t} = \int dp_2 \int_{S_2^+} d\eta\, \eta \cdot (p_1 - p_2) \Big[F_1(t, p_1^*) F_1(t, p_2^*) - F_1(t, p_1) F_1(t, p_2) \Big]. \quad (2.7)$$

This property follows directly from hierarchy (2.4′) because it permits the separation of variables if initial data satisfy (2.5). It is also supposed that solution of equation (2.7) exists.

2. ***Hierarchy with fixed random vectors.*** In hierarchy (2.2) the functions $F_s^{(N)}(p_1, \ldots, p_N)$ do not depend on any random vectors. This hierarchy was derived from equation (1.7). We will also need the hierarchy with fixed random vectors. It can be derived from the state $f_N(t, p_1, \ldots, p_N)$ that satisfies equation (1.8) and for the sequence of the reduced correlation functions defined as follows

$$F_s^{(N)}(t,(p)_s) = F_s^{(N)}(t, p_1, \ldots, p_s) = \frac{N!}{(N-s)!} \int dp_{s+1} \ldots dp_N \times$$
$$\times \prod_{i<j=1}^{N\ (s)} \int_{S_2^+} d\eta_{ij} f_N(t, p_1, \ldots, p_s, p_{s+1}, \ldots, p_N),$$
$$F_N^{(N)}(t,(p)_N) = F_N^{(N)}(t, p_1, \ldots, p_N) = f_N(t, p_1, \ldots, p_N), \tag{2.7'}$$

where $\prod_{i<j=1}^{N\ (s)}$ means that the pairs (i, j) with $1 \le i \le s, 1 \le j \le s$ are excluded. Note that only these random vectors η_{ij} that appears on interval $[t,\ t + \Delta t]$ and are present in equation (1.8) are considered.

As before, we preserve for functions (2.7) the same denotation as for functions (2.1).

It follows directly from equation (1.8) that sequence (2.7) satisfies the following hierarchy

$$\frac{\partial F_s^{(N)}(t, p_1, \ldots p_s)}{\partial t} = \sum_{i<j=1}^{s} \eta_{ij} \cdot (p_i - p_j)\, \Theta(\eta_{ij} \cdot (p_i - p_j)) \times$$
$$\times \Big[F_s^{(N)}(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_s) - F_s^{(N)}(t, p_1, \ldots, p_i, \ldots, p_j, \ldots, p_s)\Big] +$$
$$+ \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$
$$\times \Big[F_{s+1}^{(N)}(t, p_1, \ldots, p_i^*, \ldots, p_s, p_{s+1}^*) - F_{s+1}^{(N)}(t, p_1, \ldots, p_i, \ldots, p_s, p_{s+1})\Big], \tag{2.8}$$
$$1 \le s < N-1, \quad F_s^{(N)}(t, p_1, \ldots, p_s)\big|_{t=0} = F_s^{(N)}(p_1, \ldots, p_s).$$

Obviously that function $F_N^{(N)}(t, p_1, \ldots, p_N)$ satisfies equation (1.8).

We did not indicate that functions (2.7) depend on random vector. Later we will discuss how they depend on random vectors.

The "mean field" version of hierarchy (2.8) can be easy obtained if one omits the sign of integration with respect to $\eta_{i,j}$ in the first term on the right-hand side of (2.4). The limit "mean field" hierarchy is the same as (2.4′).

III. Representation of solutions of the spatially homogeneous hierarchy.

1. ***Representation of solutions of the spatially homogeneous hierarchy through series of iterations.*** Represent hierarchy (2.8) with $1 \le s < \infty$ and omitted (N) in abstract form. Denote by $F(t)$ sequence of functions $F_s(t, p_1, \ldots, p_s)$

$$F(t) = \Big(F_1(t, p_1), \ldots, F_s(t, p_1, \ldots, p_s), \ldots\Big) \tag{3.1}$$

and by $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{A}}$ the operators defined by the first and correspondingly second term of the right-hand side of (2.8)

$$\left(\tilde{\mathcal{H}}F(t)\right)_s(p_1,\ldots,p_s) = \tilde{\mathcal{H}}_s F_s(t,p_1,\ldots,p_s) = \sum_{i<j=1}^{s} \eta_{ij}\cdot(p_i-p_j)\,\Theta(\eta_{ij}\cdot(p_i-p_j))\times$$
$$\times\Big[F_s(t,p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots,p_s) - F_s(t,p_1,\ldots,p_i,\ldots,p_j,\ldots,p_s)\Big], \tag{3.2}$$
$$\left(\tilde{\mathcal{A}}F(t)\right)_s(p_1,\ldots,p_s) = (\tilde{\mathcal{A}}_s F(t))_s(p_1,\ldots,p_s) =$$
$$= \sum_{i=1}^{s}\int dp_{s+1}\int_{S_2^+} d\eta_{i,s+1}\,\eta_{i,s+1}\cdot(p_i-p_{s+1})\times$$
$$\times\Big[F_{s+1}(t,p_1,\ldots,p_i^*,\ldots,p_s,p_{s+1}^*) - F_{s+1}(t,p_1,\ldots,p_i,\ldots,p_s,p_{s+1})\Big].$$

Then hierarchy (2.8) has the following abstract form

$$\frac{dF(t)}{dt} = \tilde{\mathcal{H}}F(t) + \tilde{\mathcal{A}}F(t) = \tilde{\mathcal{L}}F(t), \quad F(t)|_{t=0} = F(0). \tag{3.3}$$

Recall that the operator $\tilde{\mathcal{H}}_s$ is the infinitesimal generator of the operator $\tilde{\mathcal{S}}_s(-t)$ that for an arbitrary t is defined formally as follows $\tilde{\mathcal{S}}_s(-t) = \prod_{i=1}^{n}\tilde{\mathcal{S}}_s(-\Delta t_i)$, $\sum_{i=1}^{n}\Delta t_i = t$ and for infinitesimal Δt_i, $\tilde{\mathcal{S}}_s(-\Delta t_i)$ is defined by (1.4).

As known solution of (3.3) can be represented by the following series of iterations

$$F(t) = \sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n\tilde{\mathcal{S}}(-t)\tilde{\mathcal{S}}(t_1)\tilde{\mathcal{A}}\tilde{\mathcal{S}}(-t_1)\ldots\tilde{\mathcal{S}}(t_n)\tilde{\mathcal{A}}\tilde{\mathcal{S}}(-t_n)F(0) \tag{3.4}$$

where $\tilde{\mathcal{S}}(-t)$ is the direct sum of the operators $\tilde{\mathcal{S}}_s(-t)$

$$\tilde{\mathcal{S}}(-t) = \sum_{s=1}^{\infty}\bigoplus\tilde{\mathcal{S}}_s(-t).$$

From (3.4) one gets

$$F_s(t,(p)_s) = \sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n\tilde{\mathcal{S}}_s(-t,(p)_s)\tilde{\mathcal{S}}_s(t_1,(p)_s)\times$$
$$\times\sum_{i=1}^{s}\int dp_{s+1}\int_{S_2^+} d\eta_{i,s+1}\,\eta_{i,s+1}\cdot(p_i-p_{s+1})\times$$
$$\times\Big[\tilde{\mathcal{S}}_{s+1}(-t_1,(p)_{s+1}^*) - \tilde{\mathcal{S}}_{s+1}(-t_1,(p)_{s+1})\Big]\ldots$$
$$\ldots\tilde{\mathcal{S}}_{s+n-1}(t_{n-1},(p)_{s+n-1})\sum_{i=1}^{s+n-1}\int dp_{s+n}\int_{S_2^+} d\eta_{i,s+n}\eta_{i,s+n}\cdot(p_i-p_{s+n})\times$$
$$\times\Big[\tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n}^*) - \tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n})\Big]F_{s+n}(0,(p)_{s+n}) \tag{3.4'}$$

where for the sake of simplicity we use the denotation

$$\Big[\tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n}^*) - \tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n})\Big]F_{s+n}(0,(p)_{s+n}) \equiv$$
$$\equiv \tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n}^*)F_{s+n}(0,(p)_{s+n}^*) - \tilde{\mathcal{S}}_{s+n}(-t_n,(p)_{s+n})F_{s+n}(0,(p)_{s+n}),$$

and $(p)_{s+n}^* = (p_1,\ldots,p_i^*,\ldots,p_{s+n-1},p_{s+n}^*)$ in terms with number i.

We have defined the operator $\tilde{\mathcal{S}}_N(-\Delta t)$ for infinitesimal $\Delta t > 0$ by formulae (1.2)–(1.4). For arbitrary $t > 0$ the operator $\tilde{\mathcal{S}}_N(-t)$ is formally defined by using the group property

$$\tilde{S}_N(-t) = \lim_{n\to\infty} \prod_{i=1}^{n} \tilde{S}_N(-\Delta t_i), \quad \sum_{i=1}^{n} \Delta t_i = t, \tag{3.5}$$

but we do not prove existence of this limit.

The operator $\tilde{\mathcal{S}}_N(-t)$ can be defined as the group of operators with the infinitesimal generator $\tilde{\mathcal{H}}_N$, determined by (1.7), but again only formally because we did not give rigorous meaning to $\tilde{\mathcal{H}}_N$ as an operator in some functional space.

The operator $\tilde{\mathcal{S}}_N(-t)$ can also be determined by the corresponding stochastic process $p_1(-t), \ldots, p_N(-t)$ by usual formula of the operator of shift along the "trajectory"

$$\tilde{\mathcal{S}}_N(-t) f_N(p_1, \ldots, p_N) = f_N(p_1(-t), \ldots, p_N(-t)),$$
$$p_1(t)|_{t=0} = p_1, \ldots, p_N(-t)|_{t=0} = p_N$$

where the stochastic process is such that the function

$$f_N(t, p_1, \ldots, p_N) = \tilde{\mathcal{S}}_N(-t) f_N(p_1, \ldots, p_N)$$

satisfies equation (1.8). But this approach is not elaborated yet.

To define the operator $\tilde{\mathcal{S}}_N(\Delta t)$ with $\Delta t > 0$ it is sufficient to replace S_2^+ by S_2^- in formulae (1.5), (1.6) and for arbitrary $t > 0$ to use (3.5) with $\tilde{\mathcal{S}}_N(\Delta t_i)$.

Now obtain representation for $F_s(t, (p)_s)$ using solutions of the spatially inhomogeneous hierarchy. It also can be represented in the following abstract form

$$\frac{dF(t)}{dt} = \mathcal{H}F(t) + \mathcal{A}F(t),$$
$$F(t) = \Big(F_1(t, x_1), \ldots, F_s(t, (x)_s), \ldots\Big), \quad F(t)|_{t=0} = F(0) \tag{3.6}$$

or componentwise [3, 5]

$$\frac{\partial F_s(t, x_1, \ldots x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) +$$
$$+ \sum_{i<j=1}^{s} \delta(q_i - q_j) \eta_{ij} \cdot (p_i - p_j) \Theta(\eta_{ij} \cdot (p_i - p_j)) \times$$
$$\times \Big[F_s(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_s) - F_s(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s)\Big] +$$
$$+ \sum_{i=1}^{s} \int dq_{s+1} \delta(q_i - q_{s+1}) \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$
$$\times \Big[F_{s+1}(t, x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_i, \ldots, x_s, x_{s+1})\Big], \tag{3.6'}$$
$$F_s(t, x_1, \ldots, x_s)|_{t=0} = F_s(x_1, \ldots, x_s), \quad s \geq 1.$$

The functions from (3.6′) depend on the random vectors η_{ij}, $1 \leq i < j \leq s$, where the operator $\mathcal{H}$ and $\mathcal{A}$ are defined by the first and correspondingly the second terms in the right-hand side of (3.6′). Solution of (4.6) can be represented as the following series of iterations

$$F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \mathcal{S}(-t)\mathcal{S}(t_1)\mathcal{A}\mathcal{S}(-t_1) \ldots \mathcal{S}(t_n)\mathcal{A}\mathcal{S}(-t_n)F(0) \tag{3.7}$$

or component-wise

$$F_s(t,(x)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \mathcal{S}_s(-t,(x)_s)\mathcal{S}_s(t_1,(x)_s) \times$$
$$\times \sum_{i=1}^{s} \int dx_{s+1} \delta(q_i - q_{s+1}) \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$
$$\times \Big[\mathcal{S}_{s+1}(-t_1,(x)^*_{s+1}) - \mathcal{S}_{s+1}(-t_1,(x)_{s+1})\Big] \ldots \mathcal{S}_{s+n-1}(t_{n-1},(x)_{s+n-1}) \times$$
$$\times \sum_{i=1}^{s+n-1} \int dx_{s+n} \delta(q_i - q_{s+n}) \int_{S_2^+} d\eta_{i,s+n}\, \eta_{i,s+n} \cdot (p_i - p_{s+n}) \times$$
$$\times \Big[\mathcal{S}_{s+n}(-t_n,(x)^*_{s+n}) - \mathcal{S}_{s+n}(-t_n,(x)_{s+n})\Big] F_{s+n}(0,(x)_{s+n}). \qquad (3.7')$$

Recall that $S_s(-t)$ is the operator of evolution of s stochastic particles, $\mathcal{H}_s$ is its infinitesimal generator and $\mathcal{S}(-t)$ is the direct sum of $\mathcal{S}_s(-t)$. The rest of the notation is the same as in (3.4). The operators $\mathcal{S}_s(-t)$ are rigorously defined as the operators of shift along trajectories in the phase space [3].

As known [3] series (3.7′) is uniformly convergent with respect to $(x)_s$ on finite time interval $[-t_0, t_0]$ if sequence of initial functions $F(0)$ belong to the space E_ξ with norm

$$\|F(0)\| = \sup_{s \geq 1} \frac{1}{\xi^s} e^{\beta \sum_{i=1}^{s} \frac{p_i^2}{2}} \sup_{(x)_s} |F_s(0,(x)_s)|$$

where $\xi > 0, \beta > 0$ are fixed numbers and t_0 is certain constant depending on (ξ, β).

It is an open problem whether series (3.4), (3.4′) is also uniformly convergent with respect to $(p)_s$ on the time interval $[-t_0, t_0]$ if sequence of initial functions $F_s(0,(p)_s)$, $s \geq 1$, belong to E_ξ. In this case series (3.4), (3.4′) would represent the *mild solution* of the spatially homogeneous hierarchy (3.3).

2. One-particle distribution function $F_1(t, p_1)$ is solution of the Boltzmann equation. Consider again series (3.7′) and denote by V_{ij} the set (hyperplanes) $q_i - q_j = \tau(p_i - p_j)$ with all $0 \leq \tau \leq t$. It is well known from the proof of the existence of the Boltzmann–Grad limit for system of hard spheres [2] that if phase points $x_1, \ldots, x_s$ are outside all the set V_{ij}, $1 \leq i < j \leq s$, then all the operators $S_{s+i}(\pm t)$ can be replaced by the operators of evolution of the free systems $S^0_{s+i}(\pm t)$ and representation (3.7′) is reduced to representation of solutions of the Boltzmann hierarchy

$$F_s(t,(x)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \mathcal{S}^0_s(-t,(x)_s)\mathcal{S}^0_s(t_1,(x)_s) \times$$
$$\times \sum_{i=1}^{s} \int dx_{s+1} \delta(q_i - q_{s+1}) \int_{S_2^+} d\eta_{i,s+1}\, \eta_{i,s+1} \cdot (p_i - p_{s+1}) \times$$
$$\times \Big[\mathcal{S}^0_{s+1}(-t_1,(x)^*_{s+1}) - \mathcal{S}^0_{s+1}(-t_1,(x)_{s+1})\Big] \ldots \mathcal{S}^0_{s+n-1}(t_{n-1},(x)_{s+n-1}) \times$$
$$\times \sum_{i=1}^{s+n-1} \int dx_{s+n} \delta(q_i - q_{s+n}) \int_{S_2^+} d\eta_{i,s+n}\, \eta_{i,s+n} \cdot (p_i - p_{s+n}) \times$$
$$\times \Big[\mathcal{S}^0_{s+n}(-t_n,(x)^*_{s+n}) - \mathcal{S}^0_{s+n}(-t_n,(x)_{s+n})\Big] F_{s+n}(0,(x)_{s+n}). \qquad (3.8)$$

For $s = 1$ series (3.8) represents $F_1(t, x_1)$ in the entire phase space of one particle and coincides with solution of the Boltzmann equation

$$\frac{\partial F_1(t,x_1)}{\partial t} = -p_1 \frac{\partial}{\partial q_1} F_1(t,x_1) +$$

$$+ \int dp_2 \int_{S_2^+} d\eta_{12}\, \eta_{12}\cdot(p_1 - p_2)\Big[F_1(t,x_1^*)F_1(t,x_2^*) - F_1(t,x_1)F_1(t,x_2)\Big] \qquad (3.9)$$

if initial functions $F_s(0,x_1,\ldots,x_s)$ have the chaos property

$$F_s(0,x_1,\ldots,x_s) = F_1(0,x_1)\ldots F_1(0,x_s). \qquad (3.10)$$

The functions $F_s(t,(x)_s)$ determined by series (3.8) have also the chaos property

$$F_s(t,x_1,\ldots,x_s) = F_1(t,x_1)\ldots F_1(t,x_s) \qquad (3.10')$$

for time on interval $[-t_0,t_0]$ and $F(0)\in E_\xi$. This fact is considered as a rigorous derivation of the Boltzmann equation.

Note that the functions $F_1(t,x_1)\ldots F_1(t,x_s)$ do not coincide with solutions $F_s(t,(x)_s)$ of the stochastic hierarchy (3.6), (3.6′) given by (3.7′) in the entire phase space of s particles, i.e.

$$F_s(t,x_1,\ldots,x_s) \neq F_1(t,x_1)\ldots F_1(t,x_s),\ldots, \quad s>1,$$

if $q_i - q_j = \tau(p_i - p_j)$ for some $0\le\tau\le t$ at least for one pair (i,j). It can be proved as follows. In the entire phase space of s particles representation (3.7′) is reduced to the following one

$$F_s(t,(x)_s) = \sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \mathcal{S}_s(-t,(x)_s)\mathcal{S}_s(t_1,(x)_s)\times$$

$$\times\sum_{i=1}^{s}\int dx_{s+1}\delta(q_i - q_{s+1})\int_{S_2^+} d\eta_{i,s+1}\,\eta_{i,s+1}\cdot(p_i - p_{s+1})\times$$

$$\times\Big[(\mathcal{S}_s(-t_1)\mathcal{S}_1^0(-t_1))((x)_{s+1}^*) - \mathcal{S}_s(-t_1)\mathcal{S}_1^0(-t_1))((x)_{s+1})\Big]\ldots$$

$$\ldots\Big(\mathcal{S}_s(t_{n-1})\mathcal{S}_{n-1}^0(t_{n-1})\Big)((x)_{s+n-1})\times$$

$$\times\sum_{i=1}^{s+n-1}\int dx_{s+n}\delta(q_i - q_{s+n})\int_{S_2^+} d\eta_{i,s+n}\,\eta_{i,s+n}\cdot(p_i - p_{s+n})\times$$

$$\times\Big[\Big(\mathcal{S}_s(-t_n)\mathcal{S}_n^0(-t_n)\Big)\Big((x)_{s+n}^*\Big) - (\mathcal{S}_s(-t_n)\mathcal{S}_n^0(-t_n)\Big)((x)_{s+n})\Big]F_{s+n}(0,(x)_{s+n}) \qquad (3.11)$$

where the denotation $\big(\mathcal{S}_s(-t_i)S_i^0(-t_i)\big)\big((x)_{s+i}^*\big)$ means that the operator $\mathcal{S}_s(-t_i)$ acts on the first s phase points of the set $((x)_{s+i}^*)$ and the operator $\mathcal{S}_i^0(-t_n)$ acts on the rest i phase points of the set $((x)_{s+i}^*)$. The same denotation is used for $\big(\mathcal{S}_s(-t_i)S_i^0(-t_i)\big)\big((x)_{s+i}\big)$. Recall that $\mathcal{S}_s(\pm t_i)$ is the operator of evolution of the stochastic particles and $S_i^0(\pm t_i)$ is the operator of evolution of the free particles. On the hyperplanes V_{ij} the operators $S_s(\pm t_i)$ do not coincide with the operators $S_s^0(\pm t_i)$ and, thus,

$$F_s(t,x_1,\ldots,x_s) \neq F_1(t,x_1)\ldots F_1(t,x_s), \quad s>1,$$

even if initial functions $F_s(0,x_1,\ldots,x_s)$ satisfy (4.10).

Suppose that all the initial functions $F_{s+n}(0,(x)_{s+n})$ are spatially homogeneous ones

$$F_{s+n}(0,(x)_{s+n}) = F_{s+n}(0,(p)_{s+n}).$$

Then it follows from (3.11) that the functions $F_s(t,(x)_s)$ depend on position $(q)_s$ only on hyperplanes V_{ij} because outside all the hyperplanes V_{ij}, $1 \le i < j \le s$, the operators of the stochastic and free evolution coincide $S_s(\pm t_i) = S_s^0(\pm t_i)$ and result of action of the operators $S_s^0(\pm t_i)$ on spatially homogeneous functions are again spatially homogeneous functions and $S_s^0(\pm t) = I$, $S_i^0(\pm t) = I$.

If we have $S_s(\pm t_i) S_i^0(\pm t_i)((x)_{s+i}^*)$ with some x_k^*, $1 \le k \le s$, then the corresponding V_{kl}, $1 \le l \ne k \le s$, should be excluded from consideration because p_k^* depends on certain momenta from the set $(p_{s+1}, \dots, p_{s+i})$ and one can neglect V_{kl} as hyperplanes of lower dimension in Lebesgue integrals with respect to $(p_{s+1}, \dots, p_{s+i})$.

Thus, the functions $F_s(t,(x)_s)$ depend only on momenta $(p)_s$ almost everywhere with respect to positions $(q)_s$ if the initial functions do not depend on positions. The function $F_1(t,x_1)$ does not depend on q_1, i.e. $F_1(t,x_1) = F_1(t,p_1)$.

Now consider (3.11) with $(x)_s$ outside the all V_{ij}, $1 \le i < j \le s$. Then in the spatially homogeneous case representation (3.11) is reduced to the following one

$$F_s(t,(p)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \times$$
$$\times \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{i,s+1} \eta_{i,s+1} \cdot (p_i - p_{s+1}) \Big[I((p)_{s+1}^*) - I((p)_{s+1}) \Big] \cdots$$
$$\cdots \sum_{i=1}^{s+n-1} \int dp_{s+n} \int_{S_2^+} d\eta_{i,s+n}\, \eta_{i,s+n} \cdot (p_i - p_{s+n}) \times$$
$$\times \Big[I((p)_{s+n}^*) - I((p)_{s+n}) \Big] F_{s+n}(0,(p)_{s+n}). \qquad (3.11')$$

For $F_1(t,p_1)$ one gets the following representation:

$$F_1(t,p_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{i=1}^{s} \int dp_{1+1} \int_{S_2^+} d\eta_{i,1+1}\, \eta_{i,1+1} \cdot (p_i - p_{1+1}) \times$$
$$\times \Big[I((p)_{1+1}^*) - I(p)_{1+1}) \Big] \sum_{i=1}^{1+n-1} \int dp_{1+n} \int_{S_2^+} d\eta_{i,1+n}\, \eta_{i,1+n} \cdot (p_i - p_{1+n}) \times$$
$$\times \Big[I((p)_{1+n}^*) - I((p)_{1+n}) \Big] F_{1+n}(0,(p)_{1+n}) \qquad (3.12)$$

where the following denotation is used

$$\Big[[I((p)_{s+i}^*) - I((p)_{s+i}) \Big] F_{s+i}(0,(p)_{s+i}) = F_{s+i}(0,(p)_{s+i}^*) - F_{s+i}(0,(p)_{s+i}).$$

Representation (3.11′), (3.12) follows from (3.11) because $S_s(\pm t) = S_s^0(\pm t)$ outside of V_{ij}, $i,j \in (1,\dots,s)$ and $S_{s+i}^0(\pm t) = I$ if $F_{s+n}(0,(x)_{s+n}) = F_{s+n}(0,(p)_{s+n})$. If the initial functions satisfy the chaos property (3.10) then series (3.12) represents solution of the spatially homogeneous Boltzmann equation

$$\frac{\partial F_1(t,p_1)}{\partial t} =$$
$$= \int dp_2 \int_{S_2^+} d\eta_{12}\, \eta_{12} \cdot (p_1 - p_2) \Big[F_1(t,p_1^*) - F_1(t,p_2^*) - F_1(t,p_1^*) - F_1(t,p_2^*) \Big] \qquad (3.9')$$

as it follows from (3.9).

Note that functions (3.11′) have also the chaos properties because functions (3.8) have these properties outside the all V_{ij}, $1 \le i < j \le s$.

Thus, we have the chaos property or propagation of chaos, without framework of mean field approach if we consider the functions $F_s(t,(x)_s)$ outside the hyperplanes V_{ij}, $1 \le i < j \le s$, because for spatially homogeneous initial functions $F_s(0,(x)_s)$ this property have series (3.11′).

Remark that series (3.11′), (3.12) are convergent uniformly with respect to $(p)_s$ and time $0 \le t < t_0$ if initial functions $F_s(0,(p)_s) = F_1(0,p_1)\dots F_1(0,p_s)$ belong to the space E_ξ because series (8.4) are convergent in this case [2].

There is another method of obtaining the propagation of chaos without the mean-field approach. Namely consider the following average of the functions $F_s(t,(x)_s)$ (3.11) over the space of positions

$$\lim_{V\to\infty} \frac{1}{V^s} \int_\Lambda dq_1 \cdots \int_\Lambda dq_s F_s(t, q_1, p_1, \dots, q_s, p_s) = F_s(t,(p)_s) \tag{3.13}$$

where the Lebesque integral is used and behaviour of functions $F_s(t,(x)_s)$ (3.11) on hyperplanes V_{ij}, $1 \le i < j \le s$, is neglected. It is obvious that obtained functions $F_s(t,(p)_s)$ (3.13) coincide with those defined by (3.11′) and if initial functions $F_s(0,(x)_s)$ have the chaos property (3.10) and do not depend on $(q)_s$ then functions (3.13) have also the chaos property and $F_1(t,p_1)$ satisfies the spatially homogeneous Boltzmann equation (3.9′).

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