UDC 517.9 + 531.19

D. Ya. Petrina (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

BCS MODEL HAMILTONIAN OF THE THEORY OF SUPERCONDUCTIVITY AS A QUADRATIC FORM*

МОДЕЛЬНИЙ ГАМІЛЬТОНІАН БКШ ТЕОРІЇ НАДПРОВІДНОСТІ ЯК КВАДРАТИЧНА ФОРМА

N. N. Bogolyubov proved that the average energies (per unit volume) of the ground states for the BCS Hamiltonian and the approximating Hamiltonian asymptotically coincide in the thermodynamic limit. In the present paper, we show that this result is also true for all excited states. We also establish that, in the thermodynamic limit, the BCS Hamiltonian and the approximating Hamiltonian asymptotically coincide as quadratic forms.

М. М. Боголюбов довів, що середні енергії (на одиницю об'єму) основних станів для гамільтоніана БКШ та апроксимуючого гамільтоніана у термодинамічній границі асимптотично збігаються. У даній роботі показано, що цей результат має місце і для усіх збуджених станів. Водночас встановлено, що гамільтоніан БКШ та апроксимуючий гамільтоніан у термодинамічній границі асимптотично збігаються як квадратичні форми.

Introduction. Consider the model BCS Hamiltonian [1] for a system of electrons located in a cube Λ centered at the origin of coordinates with periodic boundary conditions:

$$H_\Lambda = \sum_{\bar p} \overset{*}{\psi}_{\bar p} \varepsilon_p \psi_{\bar p} + \frac{1}{V} g \sum_{p,p'} \phi(p,p') \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'} \psi_{p'} = H_{0,\Lambda} + H_{I,\Lambda}. \qquad (1)$$

Here, p is a quasimomentum, $p = \frac{2\pi}{L} n$, $n = (n_1, n_2, n_3)$, $n_i \in Z$, $i = 1, 2, 3$, Z is the set of all integer numbers, $\varepsilon_p = \frac{p^2}{2m} - \mu$, m is the mass of an electron, μ is the chemical potential, L is the length of the edge of the cube, $V = L^3$ is the volume of the cube Λ, $\phi(p,p')$ is the interaction potential, and $\overset{*}{\psi}_{\bar p}$ and $\psi_{\bar p}$ denote the operators of creation and annihilation of an electron with momentum p and spin $\sigma = \pm 1$, $\bar p = (p, \sigma)$. For simplicity, in the interaction Hamiltonian $H_{I,\Lambda}$ we use the notation p, p' for $(p, 1)$ and $(p', 1)$, $-p$ and $-p'$ for $(-p, -1)$, $(-p', -1)$.

The potential $\phi(p,p')$ is a piecewise-continuous function of (p,p'), concentrated in a layer near the Fermi surface $\left|\frac{p^2}{2m} - \mu\right| < w$, $\left|\frac{p'^2}{2m} - \mu\right| < w$, $w > 0$.

Consider the approximating Hamiltonian introduced by Bogolyubov [2]:

$$H_{\mathrm{appr},\Lambda} = \sum_{\bar p} \overset{*}{\psi}_{\bar p} \varepsilon_p \psi_{\bar p} + \sum_p c_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} + \sum_p c_p \psi_{-p} \psi_p - c \qquad (2)$$

where

$$c_p = \frac{1}{V} \sum_{p'} \phi(p,p') A_{p'}, \qquad c = \frac{1}{V} \sum_{p,p'} \phi(p,p') A_p A_{p'},$$

$$A_p = (\phi_0^a, \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \phi_0^a) = (\phi_0^a, \psi_{-p} \psi_p \phi_0^a) = u_p v_p, \qquad (3)$$

$$\phi_0^a = \prod_k (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}) |0\rangle,$$

* Financial support of INTAS Project N° 2000-15 is gratefully acknowledged.

$$u_k = \sqrt{\frac{1}{2}\Big(1 - \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + c_k^2}}\Big)}, \qquad v_k = \sqrt{\frac{1}{2}\Big(1 + \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + c_k^2}}\Big)},$$

$|0\rangle$ is a vacuum, $\psi_{\pm k}|0\rangle = 0$, and ϕ_0^a is the ground state of the approximating Hamiltonian.

The self-consistency equation

$$c_p = \frac{1}{V}\sum_{p'} \phi(p,p') A_{p'} = \frac{1}{V}\sum_{p'} \phi(p,p') u_{p'} v_{p'} \tag{4}$$

where u_k, v_k are defined according to (3), coincides with the condition of the minimum of energy for the ground state of the Hamiltonian $H_{a,\Lambda} = H_{\text{appr},\Lambda}$

$$(\phi_0^a, H_{a,\Lambda}\phi_0^a) = 2\sum_p v_p^2 \varepsilon_p + \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} \tag{5}$$

with unknown u_p, v_p such that $u_p^2 + v_p^2 = 1$. It is known that the minimum of functional (5) is realized on u_k, v_k defined by (3), and c_k are a solution of equation (4).

Denote by ϕ_0 the state

$$\phi_0 = \prod_k (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle, \tag{6}$$

with unknown u_k, v_k such that $u_k^2 + v_k^2 = 1$. Let us calculate the energy of H_Λ on ϕ_0. We have

$$(\phi_0, H_\Lambda \phi_0) = 2\sum_p v_p^2 \varepsilon_p + \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} + \frac{1}{V}\sum_p \phi(p,p) v_p^4. \tag{7}$$

Note that, in the classical work [1] (see also [3]), the last (third) term $\frac{1}{V}\sum_p \phi(p,p) v_p^4$ is absent.

The calculation of a condition for the minimum of functional (6) for $(\phi_0, H_\Lambda \phi_0)$ reduces to the solution of an equation of the fourth order with respect to v_p^2. The latter problem turned out to be so complicated that the author refused to solve it. If we assume that the potential satisfies the condition $\phi(p,p) = 0$, then functional (7) coincides with functional (5), i.e., $(\phi_0, H_\Lambda \phi_0) = (\phi_0^a, H_{a,\Lambda}\phi_0^a)$, and the conditions for the minimum of functionals (5) and (7) coincide and, hence, $\phi_0^a = \phi_0$. Note that the condition $\phi(p,p) = 0$ is not restrictive from the physical point of view because $\int \phi(p,p')dpdp'$ does not depend on the behavior of $\phi(p,p')$ on the hypersurface $p = p'$ of lower dimension.

Bogolyubov [2] proved the fundamental equality

$$\lim_{V\to\infty} \frac{1}{V}\Big[(\phi_0, H_\Lambda \phi_0) - (\phi_0^a, H_{a,\Lambda}\phi_0^a)\Big] = 0, \tag{8}$$

without explicit calculation of ϕ_0 and even without the assumption that ϕ_0 is sought in the form (6). (We have solid reasons to assert that, in [2], ϕ_0 is given in the form (6), and u_k, v_k are determined from the condition of the minimum of functional (7) with $\phi(p,p) \neq 0$, though we failed to prove the existence of the minimum.) If $\phi(p,p) = 0$, then $\phi_0 = \phi_0^a$ and

$$(\phi_0, H_\Lambda \phi_0) = (\phi_0^a, H_\Lambda \phi_0^a) = (\phi_0^a, H_{a,\Lambda}\phi_0^a) \tag{9}$$

for any Λ.

According to the arguments presented in [4 – 6], the ground state ϕ_0^a is the exact ground state of the BCS Hamiltonian H_Λ in the limit as $V \to \infty$.

In [7 – 10], we proved that ϕ_0^a and the excited states are asymptotically exact eigenvectors of H_Λ. Moreover, we established the existence of one more ground state of H_Λ and its excitations. We used in [7 – 10] certain Hilbert space that turns into the Hilbert space of translation invariant functions, in the thermodynamic limit.We have been able to prove existence not only averages of H_Λ over the ground and excited states but also existence of the ground and excited states in the thermodynamic limit. In given paper we use the usual Hilbert space and prove only existence of averages of H_Λ.

We introduce the operators

$$\overset{*}{\alpha}_p = u_p \overset{*}{\psi}_p - v_p \psi_{-p}, \qquad \alpha_p = u_p \psi_p - v_p \overset{*}{\psi}_{-p}, \qquad p = (p, 1), \quad -p = (-p, -1),$$
$$\overset{*}{\alpha}_{-p} = u_p \overset{*}{\psi}_{-p} + v_p \psi_p, \quad \alpha_{-p} = u_p \psi_{-p} + v_p \overset{*}{\psi}_p, \tag{10}$$

where u_p, v_p are determined according to (3). These operators $\overset{*}{\alpha}$, α, together with the operators $\overset{*}{\psi}, \psi$, satisfy canonical anticommutation conditions. The state ϕ_0^a is a vacuum for α_p, α_{-p}, i.e., $\alpha_{\pm p} \phi_0^a = 0$. We form the excited states

$$\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a \tag{11}$$

where, among $(p_i)_{i=1}^{n}$, there may be operators with spin -1. We establish the following theorem.

Theorem 1. *The equalities*

$$\lim_{V\to\infty} \frac{1}{V} \Big(\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a, (H_\Lambda - H_{a,\Lambda}) \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a \Big) =$$
$$= \lim_{V\to\infty} \frac{1}{V} \Bigg[2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$
$$+ \frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$
$$+ \frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{(p) \neq p_n} \phi(p, p) \tilde{v}_p^4 \Bigg] = 0 \tag{12}$$

hold for arbitrary finite n and m, and for n and m that tend to ∞ together with V but so that $\lim_{V\to\infty} \frac{(n+2m)^2}{V^2} = 0$. Here, $\tilde{v}_p = v_p$ for $p \neq (p)_{j=n+1}^{n+m}$ and $\tilde{v}_{p_j} = -u_{p_j}$ for $n + 1 \leq j \leq n + m$.

If $\phi(p, p) = 0$, then equality (12) *holds even without the factor $\frac{1}{V}$ for arbitrary finite n and m, and for n and m that tend to ∞ together with V but so that $\lim_{V\to\infty} \frac{(n+2m)^2}{V} = 0$.*

The following formula, which determines the energy of excited states, is true:

$$\left(\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a, H_\Lambda \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a \right) - (\phi_0^a, H_\Lambda \phi_0^a) =$$

$$= \sum_{i=1}^{n} E_{p_i} + 2\sum_{j=n+1}^{n+m} E_{p_j} + 2\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=n+1}^{n+m}\phi(p_i,p_i)u_{p_i}^4 - \frac{1}{V}\sum_{i=1}^{n+m}\phi(p_i,p_i)v_{p_i}^4. \quad (13)$$

It follows from (13) *that the energy of excited states asymptotically tends to* $\sum_{i=1}^{n} E_{p_i} + 2\sum_{j=n+1}^{n+m} E_{p_j}$ *as* $V \to \infty$.

The following theorem is true.

Theorem 2. *The states*

$$\phi_1 = \prod_{i=1}^{n_1}\overset{*}{\alpha}_{p_i^1}\prod_{j=n_1+1}^{n_1+m_1}\overset{*}{\alpha}_{p_j^1}\overset{*}{\alpha}_{-p_j^1}\phi_0^a,$$

$$\phi_2 = \prod_{i=1}^{n_2}\overset{*}{\alpha}_{p_i^2}\prod_{j=n_2+1}^{n_2+m_2}\overset{*}{\alpha}_{p_j^2}\overset{*}{\alpha}_{-p_j^2}\phi_0^a,$$

where $n_1 \neq n_2$, $m_1 \neq m_2$, *or* $n_1 \neq n_2$, $m_1 = m_2$, *or* $n_1 = n_2$, $m_1 \neq m_2$, $m_1 + m_2 \geq 3$, *remain orthogonal after the action of the Hamiltonian* H_Λ *on one of them, i.e.,*

$$(\phi_1, H_\Lambda\phi_2) = 0. \quad (14)$$

If $m_1 = m_2$, $n_1 = n_2$, *but* $(p_i^1)_{n_1} \neq (p_i^2)_{n_2}$ *or* $(p_j^1)_{m_1} \neq (p_j^2)_{m_2}$, $m_1 + m_2 \geq 4$, *then equality* (14) *is true. In the case where* $n_1 = n_2$, $(p_i^1)_{n_1} = (p_i^2)_{n_1}$, $m_1 + m_2 = 2$, $(p_j^1)_{m_1} \neq (p_j^2)_{m_2}$, *we have* $\lim_{V\to\infty}(\phi_1, H_\Lambda\phi_2) = 0$; *if* $m_1 + m_2 = 1$, *then* $\lim_{V\to\infty}\frac{1}{V}(\phi_1, H_\Lambda\phi_2) = 0$.

Using Theorems 1 and 2, we prove the following main theorem.

Theorem 3. *The following equality is true:*

$$\lim_{V\to\infty}\frac{1}{V}\Bigg(\sum_{k,l}c_{kl}\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i^k}\prod_{j=1}^{m}\overset{*}{\alpha}_{p_j^l}\overset{*}{\alpha}_{-p_j^l}\phi_0^a, (H_\Lambda - H_{a,\Lambda})\times$$

$$\times\sum_{r,s}c'_{rs}\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i^r}\prod_{j=1}^{m}\overset{*}{\alpha}_{p_j^s}\overset{*}{\alpha}_{-p_j^s}\phi_0^a\Bigg) =$$

$$= \frac{1}{V}\sum_{k,l}\bar{c}_{kl}c'_{kl}\Bigg[2\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} + \frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)\tilde{v}_p^4\Bigg] +$$

$$+\frac{1}{V}\delta_{m,1}\sum_{k,l}\sum_{r,s}\bar{c}_{kl}c'_{rs}\delta_{(p^k)_n,(p^r)_n}\frac{1}{V}\phi(p_1,p_2)\left(u^2_{p^l_1}u^2_{p^s_2}+v^2_{p^l_1}v^2_{p^s_2}\right)=0, \qquad (15)$$

where

$$\sum_{k,l}|c_{kl}|^2<\infty, \quad \sum_{k,l}|c'_{kl}|^2<\infty, \quad \sum_{k,l}|c_{kl}|<\infty, \quad \sum_{r,s}|c'_{rs}|<\infty.$$

It follows from Theorem 3 *that, for* $\phi(p,p)=0$, *equality* (15) *holds even without the factor* $\frac{1}{V}$. *This means that, as* $V\to\infty$, *the Hamiltonians* H_Λ *and* $H_{a,\Lambda}$ *asymptotically coincide as quadratic forms for fixed* n *and* m.

Theorem 3 can also be generalized to the case where states with different n, m and n', m' are present on the left-hand side and the right-hand side of (15) [see formulas (3.21)–(3.23)].

Finally, we describe the structure of the present work.

In Section 1, we give known facts about the ground state and excited states of H_Λ that are necessary for what follows. We also present expressions that were not taken into account before in [1, 3].

In Section 2, we prove that, as $V\to\infty$, the approximating Hamiltonian and the BCS model Hamiltonian asymptotically coincide as quadratic forms on the ground state and excited states.

In Section 3, we present complicated calculations necessary for the proof of the main theorems.

1. BCS model Hamiltonian of the theory of superconductivity. *1.1. BCS Hamiltonian.* Consider the Hamiltonian of system of electrons enclosed in the cube Λ with the center in the origin of coordinates with periodic boundary condition

$$H_\Lambda=\sum_{p,\sigma}\overset{*}{\psi}_{p,\sigma}\varepsilon_p\psi_{p,\sigma}+\frac{1}{2V}\sum_{p_1,\sigma_1,p_2,\sigma_2,p_3,\sigma_3,p_4,\sigma_4}\delta_{p_1+p_2,p_3+p_4}\phi(p_2,p_3)\times$$

$$\times\overset{*}{\psi}_{p_1,\sigma_1}\overset{*}{\psi}_{p_2,\sigma_2}\psi_{p_3,\sigma_3}\psi_{p_4,\sigma_4}, \qquad \varepsilon_p=\frac{p^2}{2m}-\mu, \qquad (1.1)$$

where $\sigma=\pm1$ is the spin of electrons, μ is the chemical potential, and m is the mass of electrons, p is discrete quasimomenta $p=\frac{2\pi}{L}(n_1,n_2,n_3)$, $n_i\subset Z$, $i=1,2,3$, L is the length of the edge of the cube Λ, ϕ is potential.

Note that the operators $\psi_{p,\sigma}$, $\overset{*}{\psi}_{p,\sigma}$ satisfy the canonical anticommutation relations

$$\{\psi_{p_1,\sigma_1},\overset{*}{\psi}_{p_2,\sigma_2}\}=\delta_{p_1,p_2}\delta_{\sigma_1,\sigma_2}, \qquad \{\psi_{p_1,\sigma_1},\psi_{p_2,\sigma_2}\}=\{\overset{*}{\psi}_{p_1,\sigma_1},\overset{*}{\psi}_{p_2,\sigma_2}\}=0. \quad (1.2)$$

The model Bardeen–Cooper–Schrieffer (BCS) Hamiltonian [1] can be obtained from the general Hamiltonian (1.1) if one substitutes the following product of two Kronecker symbols $\delta_{p_1+p_2,0}\delta_{p_3+p_4,0}$ instead of the Kronecker symbol $\delta_{p_1+p_2,p_3+p_4}$ and also substitutes the Kronecker symbols $2\delta_{\sigma_1,1}\delta_{\sigma_2,-1}\delta_{\sigma_3,-1}\delta_{\sigma_4,1}$ under the sign of summation over spins $\sigma_1,\sigma_2,\sigma_3,\sigma_4$. Then the BCS Hamiltonian has the following form

$$H_\Lambda=\sum_{p,\sigma}\overset{*}{\psi}_{p,\sigma}\varepsilon_p\psi_{p,\sigma}+$$

$$+\frac{1}{V}\sum_{p_1,\sigma_1,p_2,\sigma_2,p_3,\sigma_3,p_4,\sigma_4}\delta_{p_1+p_2,0}\delta_{p_3+p_4,0}\delta_{\sigma_1,1}\delta_{\sigma_2,-1}\delta_{\sigma_3,-1}\delta_{\sigma_4,1}\times$$

$$\times\phi(p_2,p_3)\overset{*}{\psi}_{p_1,\sigma_1}\overset{*}{\psi}_{p_2,\sigma_2}\psi_{p_3,\sigma_3}\psi_{p_4,\sigma_4}=$$

$$= \sum_{\bar{p}} \overset{*}{\psi}_{\bar{p}} \varepsilon_p \psi_{\bar{p}} + \frac{1}{V} \sum_{p,p'} \phi(p,p') \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'} \psi_{p'} = H_{0,\Lambda} + H_{I,\Lambda}, \qquad (1.3)$$

where we use the following denotation

$$\bar{p} = (p, \sigma), \quad p = (p, 1), \quad -p = (-p, -1)$$

and summation over $\bar{p}$ means summation over the quasimomenta $p = \frac{2\pi}{L}(n_1, n_2, n_3)$ and spin $\sigma = \pm 1$.

In the Hamiltonian BCS only electrons with opposite momenta and spin interact.

1.2. Ground state of the BCS Hamiltonian and variational method. Now we proceed to construction of the ground state ϕ_0^a of the BCS Hamiltonian. We choose ϕ_0^a in the following form

$$\phi_0^a = \prod_k \left(1 + f_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \sum_{n=0}^{\infty} \sum_{k_1 \neq \ldots \neq k_n}' f_{k_1} \ldots f_{k_n} \overset{*}{\psi}_{k_1} \overset{*}{\psi}_{-k_1} \ldots \overset{*}{\psi}_{k_n} \overset{*}{\psi}_{-k_n} |0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \neq \ldots \neq k_n} f_{k_1} \ldots f_{k_n} \overset{*}{\psi}_{k_1} \overset{*}{\psi}_{-k_1} \ldots \overset{*}{\psi}_{k_n} \overset{*}{\psi}_{-k_n} |0\rangle \qquad (1.4)$$

where $f(k)$ is real unknown function, in $\sum_{k_1 \neq \ldots \neq k_n}' \cdots$ the summation is carried out over all $k_1 \neq \ldots \neq k_n$, and the points $k_1 \neq \ldots \neq k_n$ that differ only by permutation are indetified. The quasimomenta k in (1.4) takes arbitrary values $\frac{2\pi}{L}(n_1, n_2, n_3)$, $n_i \subset Z$, $i = 1, 2, 3$, $|0\rangle$ is the vacuum.

Calculate (ϕ_0^a, ϕ_0^a) by using anticommutation relation (1.2)

$$(\phi_0^a, \phi_0^a) = \langle 0| \prod_{k'} \left(1 + f_{k'} \psi_{-k'} \psi_{k'}\right) \prod_k \left(1 + f_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \langle 0| \prod_k \left(1 + f_k \psi_{-k} \psi_k\right) \left(1 + f_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \langle 0| \prod_k \left(1 + f_k \psi_{-k} \psi_k + f_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k} + f_k^2 \psi_{-k} \psi_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \prod_k \left(1 + f_k^2\right) = \sum_{n=0}^{\infty} \sum_{k_1 \neq \ldots \neq k_n}' f_{k_1}^2 \ldots f_{k_n}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \neq \ldots \neq k_n} f_{k_1}^2 \ldots f_{k_n}^2. \quad (1.5)$$

In (1.5) we used the following formulas

$$\left[\psi_{-k'} \psi_{k'}, \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right] = \delta_{k,k'} - \overset{*}{\psi}_k \psi_k \delta_{k,k'} - \overset{*}{\psi}_{-k} \psi_{-k} \delta_{k,k'},$$
$$\langle 0|\psi_{-k} \psi_k|0\rangle = 0, \qquad \langle 0|\overset{*}{\psi}_k \overset{*}{\psi}_{-k}|0\rangle = 0 \qquad (1.6)$$

that follows from anticommutation relations (1.2) and from the definition of vacuum $\psi_k|0\rangle = 0$, $\psi_{-k}|0\rangle = 0$.

In what follows we will use normalized to unity ϕ_0^a

$$\frac{\phi_0^a}{(\phi_0^a,\phi_0^a)^{\frac{1}{2}}} = \frac{\prod\limits_k \left(1 + f_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle}{\prod\limits_k \left(1+f_k^2\right)^{\frac{1}{2}}} = \prod_k \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle, \tag{1.7}$$

where

$$u_k = \frac{1}{\sqrt{1+f_k^2}}, \qquad v_k = \frac{f_k}{\sqrt{1+f_k^2}}, \quad u_k^2 + v_k^2 = 1. \tag{1.8}$$

For the sake of simplicity we will preserve for normalized ground state (1.7) the same denotation ϕ_0^a, namely

$$\phi_0^a = \prod_k \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle. \tag{1.9}$$

Now calculate the average energy $(\phi_0^a, H_\Lambda \phi_0^a)$ of ground state ϕ_0^a (1.9). Calculate first $(\phi_0^a, H_{0,\Lambda}\phi_0^a)$. By using formulas (1.6) one obtains

$$\begin{gathered}\left(\phi_0^a, H_{0,\Lambda}\phi_0^a\right) = \langle 0| \prod_{k'} \left(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\right) \sum_p \varepsilon_p \left(\overset{*}{\psi}_p \psi_p + \overset{*}{\psi}_{-p}\psi_{-p}\right), \\ \prod_k \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle = \sum_p \langle 0| \prod_{k'\neq p} \left(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\right) \prod_{k\neq p} \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right), \\ 2v_p\varepsilon_p \left(u_p + v_p \psi_{-p}\psi_p\right) \overset{*}{\psi}_p \overset{*}{\psi}_{-p}|0\rangle = \sum_p \prod_{k\neq p} (u_k^2+v_k^2) 2v_p\varepsilon_p, \\ \langle 0| v_p \psi_{-p}\psi_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p}|0\rangle = \sum_p 2v_p^2 \varepsilon_p.\end{gathered} \tag{1.10}$$

Consider the following part of the Hamiltonian of interaction H_Λ

$$\frac{1}{V} \sum_{p\neq p'} \phi(p,p') \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'}\psi_{p'}$$

and calculate its average over ϕ_0^a. One obtains

$$\begin{gathered}\left(\phi_0^a, \frac{1}{V}\sum_{p\neq p'} \phi(p,p')\overset{*}{\psi}_p \overset{*}{\psi}_{-p}\psi_{-p'}\psi_{p'}\phi_0^a\right) = \\ = \langle 0| \prod_{k'} \left(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\right) \frac{1}{V} \sum_{p\neq p'} \phi(p,p') \overset{*}{\psi}_p \overset{*}{\psi}_{-p}\psi_{-p'}\psi_{p'} \times \\ \times \prod_k \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle = \\ = \frac{1}{V}\sum_{p\neq p'} \phi(p,p') v_p v_{p'} \langle 0| \prod_{k'\neq p} \left(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\right) \prod_{k\neq p'} \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\right)|0\rangle = \\ = \frac{1}{V}\sum_{p\neq p'} \phi(p,p') v_p v_{p'} \prod_{k\neq p\neq p'} \left(u_k^2+v_k^2\right) \langle 0| \left(u_{p'} + v_{p'}\psi_{-p'}\psi_{p'}\right)\left(u_p + v_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p}\right)|0\rangle = \\ = \frac{1}{V}\sum_{p\neq p'} \phi(p,p') v_p v_{p'} u_p u_{p'}.\end{gathered} \tag{1.11}$$

And finally calculate the average

$$\sum_p \left(\phi_0^a, \frac{1}{V}\phi(p,p)\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\psi_{-p}\psi_p\phi_0^a\right) =$$

$$= \sum_p \langle 0| \prod_{k'} \left(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\right)\frac{1}{V}\phi(p,p)\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\psi_{-p}\psi_p \times$$

$$\times \prod_k \left(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle = \frac{1}{V}\sum_p \phi(p,p)\prod_{k\neq p}\left(u_k^2+v_k^2\right)\times$$

$$\times\langle 0|\left(u_p + v_p\psi_{-p}\psi_p\right)\left(\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\psi_{-p}\psi_p\right)\left(u_p + v_p\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\right)|0\rangle =$$

$$= \frac{1}{V}\sum_p \phi(p,p)v_p^2. \tag{1.12}$$

Summing up expression (1.10) – (1.12) one obtains

$$(\phi_0^a, H_\Lambda\phi_0^a) = 2\sum_p v_p^2\varepsilon_p + \frac{1}{V}\sum_{p\neq p'}\phi(p,p')u_pu_{p'}v_pv_{p'} + \frac{1}{V}\sum_p\phi(p,p)v_p^2 =$$

$$= 2\sum_p v_p^2\varepsilon_p + \frac{1}{V}\sum_{p,p'}\phi(p,p')u_pu_{p'}v_pv_{p'} + \frac{1}{V}\sum_p\phi(p,p)v_p^4. \tag{1.13}$$

Note that the first two terms in (1.13) diverge as V when the volume V tends to infinity because

$$\lim_{V\to\infty}\left[2\frac{1}{V}\sum_p v_p^2\varepsilon_p + \frac{1}{V^2}\sum_{p,p'}\phi(p,p')u_pu_{p'}v_pv_{p'}\right] =$$

$$= 2\int v^2(p)\left(\frac{p^2}{2m}-\mu\right)dp + \int\phi(p,p')u(p)u(p')v(p)v(p')dpdp' \tag{1.14}$$

for smooth $u(p,)v(p)$, $\phi(p,p')$. The last term in (1.13) tends to $\int\phi(p,p)v^4(p)dp$ and is finite. (Note that for continuous $p\in R^3$ we use the denotation $u(p)$, $v(p)$.)

Thus the average (1.13) is expressed in terms of functions u_k and v_k that satisfy condition $u_k^2+v_k^2=1$. We determine them from the condition of minimum of the functional of the average energy (1.13). (Note that in the classical BCS paper [1] the last third term has been omited at the very beginning without explanation.)

To minimize expression (1.13), we represent it in terms of independent variables, say, in terms of v_k. We have

$$(\phi_0, H_\Lambda\phi) = \sum_p 2\varepsilon_p v_p^2 + \frac{1}{V}\sum_{p,p'}\phi(p,p')\sqrt{1-v_p^2}\sqrt{1-v_{p'}^2}v_pv_{p'} + \frac{1}{V}\sum_p\phi(p,p)v_p^4. \tag{1.15}$$

By differentiating (1.15) with respect to v_p, we obtain the condition of minimum of the average energy

$$4\varepsilon_pv_p + 2\frac{1}{V}\sum_{p'}\phi(p,p')\sqrt{1-v_p^2}\sqrt{1-v_{p'}^2}v_{p'} -$$

$$-2\frac{1}{V}\sum_{p'}\phi(p,p')\frac{v_p}{\sqrt{1-v_p^2}}\sqrt{1-v_{p'}^2}v_pv_{p'} + \frac{4}{V}\phi(p,p)v_p^3 = 0.$$

We omit the last fourth term that tends to zero as $V\to\infty$ (see discussion in the end of this subsection).

After evident simplification, this equation takes the form (without the last term)

$$\varepsilon_p v_p \sqrt{1-v_p^2} + \left(\frac{1}{2} - v_p^2\right) \frac{1}{V} \sum_{p'} \phi(p,p') \sqrt{1-v_{p'}^2}\, v_{p'} = 0. \tag{1.16}$$

Denote by c_p the following function

$$c_p = \frac{1}{V} \sum_{p'} \phi(p,p') \sqrt{1-v_{p'}^2}\, v_{p'} \tag{1.17}$$

then equation (1.16) is reduced to algebraic equation for v_p

$$\varepsilon_p v_p \sqrt{1-v_p^2} + \left(\frac{1}{2} - v_p^2\right) c_p = 0. \tag{1.16$'$}$$

This equation has the following solution

$$v_p = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + c_p^2}}\right)}, \qquad u_p = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + c_p^2}}\right)} \tag{1.18}$$

with does not exceed one. Note that expression (1.15) and equation (1.16) are invariant with respect to the replacement $v_p \to -v_p$. Equation (1.16$'$) is invariant with respect to the replacement $v_p \to -v_p$, $c_p \to -c_p$.

Now investigate the exact equation with the last term.

It follows from (1.13) the following equation of minimum of $(\phi_0^a, H_\Lambda \phi_0^a)$ (without neglecting of the term $\frac{1}{V}\sum_p \phi(p,p) v_p^4$)

$$4\varepsilon_p v_p + 2\frac{1}{V}\sum_{p'} \phi(p,p')\sqrt{1-v_p^2}\sqrt{1-v_{p'}^2}\, v_{p'} -$$

$$- 2\frac{1}{V}\sum_{p'} \phi(p,p') \frac{v_p}{\sqrt{1-v_p^2}} \sqrt{1-v_{p'}^2}\, v_p v_{p'} + \frac{4}{V}\phi(p,p) v_p^3 = 0.$$

After obvious simplification this equation takes form

$$\varepsilon_p v_p \sqrt{1-v_p^2} + \left(\frac{1}{2} - v_p^2\right) \frac{1}{V} \sum_{p'} \phi(p,p') \sqrt{1-v_{p'}^2}\, v_{p'} + \frac{1}{V}\phi(p,p) v_p^3 \sqrt{1-v_p^2} = 0.$$

Denote, as above by c_p the following expression (see (1.17))

$$c_p = \frac{1}{V} \sum_{p'} \phi(p,p') \sqrt{1-v_{p'}^2}\, v_{p'}$$

then equation of minimum is reduced to the equation

$$\varepsilon_p v_p \sqrt{1-v_p^2} + \left(\frac{1}{2} - v_p^2\right) c_p + \frac{1}{V}\phi(p,p) v_p^3 \sqrt{1-v_p^2} = 0$$

that is equation of fourth order with respect to v_p^2 and with small parameter $\frac{1}{V^2}$ in a coefficient of v_p^8. On formal level last equation is reduced to equation (3.16$'$) as $V \to \to \infty$. But a rigorous proof of this assertion is connected with very cumbersome calculation.

We will not give very cumbersome proof of above formulated assertion because we will use not restricted, from physical point of view, the following assumption about potential $\phi(p,p)$. Namely, *we will use the following modification of our potential*

$$\phi(p, p') = (1 - \delta_{p,p'}) w_p w_{p'} \tag{1.19}$$

or a linear combination of terms (1.19). With this assumption all the terms with $\phi(p, p)$ will disappear and we will have the classical BCS theory with respect of the ground state ϕ_0^a. Above mentioned assumption (1.19) has been also used in [5].

1.3. Equation for c_p. Inserting expression (1.18) in (1.17) one obtains the following equation for c_p

$$c_p = -\frac{1}{V} \sum_{p'} \phi(p, p') \frac{1}{2} \frac{c_{p'}}{\sqrt{\varepsilon_{p'}^2 + c_{p'}^2}} \tag{1.20}$$

$\left(\text{obtaining equation (1.20), we put } \sqrt{c_{p_1}^2} = -c_{p'}\right)$.

In (1.20) we pass to the limit $V \to \infty$ and replace summation over p' by integration. As a result we obtain

$$c(p) = -\int \phi(p, p') \frac{1}{2(2\pi)^3} \frac{c(p')}{\sqrt{\varepsilon^2(p') + c^2(p')}} dp' \tag{1.21}$$

where, us usual, the functions $\varepsilon(p), c(p)$ depend on continuous momenta p. We obtained a nonlinear integral equation for the function $c(p)$. This function is known as the gap. An explanation of this terminology will be given in the next subsection.

The nonlinear integral equation for the gap has a unique different from zero solution for a general potential $\phi(p, p')$ satisfying certain condition [11]. We will construct some approximate exact solution of equation (1.21). Namely consider potential $\phi(p, p')$ which is constant in certain layer of the Fermi sphere

$$\phi(p, p') = \begin{cases} gW < 0,\ g < 0, W > 0, \quad \text{if} \quad \left|\dfrac{p^2}{2m} - \mu\right| < \omega,\ \left|\dfrac{p'^2}{2m} - \mu\right| < \omega,\ \omega > 0, \\ 0 \quad \text{if} \quad p \text{ or } p' \text{ do not belong to these layer, and } g \text{ is coupling constant.} \end{cases}$$

Show that equation (1.21) has a solution independent on momentum $c(p) = c$. The constant c satisfies equation

$$c = -gw \int_{-\omega}^{\omega} \frac{c d\varepsilon}{2\sqrt{\varepsilon^2 + c^2}} \tag{1.22}$$

where a new constant w contains all numbers that are in (1.21) and appear after integration over the spherical variables. Note that in (1.22) $|p'|$ was replaced by its mean value in the layer $\left|\dfrac{p'^2}{2m} - \mu\right| < \omega$ and this mean value was included in w.

Integrating equation (1.22) one obtains

$$-\frac{1}{gw} = Arsh \frac{\omega}{c} \qquad \text{and} \qquad c = \frac{\omega}{sh\left(-\dfrac{1}{gw}\right)}. \tag{1.23}$$

It follows from (1.23) that the gap c is nonanalitical function of coupling constant g, and has a singularity at $g = 0$. (Remark that if one puts $\sqrt{c_{p'}^2} = +c_{p'}$ then equation

$$c_p = \frac{1}{V} \sum_{p'} \phi(p, p') \frac{1}{2} \frac{c_{p'}}{\sqrt{\varepsilon_{p'}^2 + c_{p'}^2}}$$

has not solution for considered potential $gw < 0$.)

1.4. Excited states. Consider a state

$$\phi^a_{p_1} = \frac{\overset{*}{\psi}_{p_1}}{u_{p_1}}\phi^a_0 = \prod_{k\neq p_1}\left(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)\overset{*}{\psi}_{p_1}|0\rangle. \tag{1.24}$$

In this state all electrons with momenta $k \neq p_1$ are in pairs with momenta $\pm k$ and opposite spin and the electron with momenta $-p_1$ and spin $-\frac{1}{2}$ is taken away. This state is called a one-particle excited state. It is easy to check, repeating the calculation from the Subsection 1.2 that the state $\phi^a_{p_1}$ is normalized to unity, and orthogonal to ϕ^a_0, i.e. $(\phi^a_{p_1}, \phi^a_{p_1}) = 1$, $(\phi^a_0, \phi^a_{p_1}) = 0$.

Calculate the average energy of the excited state ϕ_p. Repeating the calculation performed for $(\phi^a_0, H_\Lambda\phi^a_0)$ one obtains

$$(\phi^a_{p_1}, H_\Lambda\phi^a_{p_1}) = \varepsilon_{p_1} + \sum_{p\neq p_1} 2\varepsilon_p v_p^2 +$$

$$+\frac{1}{V}\sum_{\substack{p\neq p_1\\ p'\neq p_1}} \phi(p,p')u_p u_{p'} v_p v_{p'} + \frac{1}{V}\sum_{p\neq p_1}\phi(p,p)v_p^4. \tag{1.25}$$

This formula has obvious physical meaning. The average energy of the one-particle excited state $\phi^a_{p_1}$ consists of the kinetic energy of a free electron with momenta p_1, the kinetic energy of the pairs with momenta $p \neq p_1$, and the potential energy of the pairs whose momenta are not equal to p_1.

Consider the difference between the average energies of excited and ground-states, i.e. $(\phi^a_{p_1}, H_\Lambda\phi^a_{p_1}) - (\phi^a_0, H_\Lambda\phi^a_0)$.

It follows from (1.13) and (1.25) that

$$(\phi^a_{p_1}, H_\Lambda\phi^a_{p_1}) - (\phi^a_0, H_\Lambda\phi^a_0) = \varepsilon_{p_1}(1-2v_{p_1}^2) - \frac{2}{V}\sum_p \phi(p_1,p)u_{p_1}u_p v_{p_1} v_p +$$

$$+\frac{1}{V}\phi(p_1,p_1)v_{p_1}^2(1-2v_{p_1}^2) \cong \varepsilon_{p_1}(1-2v_{p_1}^2) - 2c_{p_1}u_{p_1}v_{p_1}. \tag{1.26}$$

Note that we used definition (1.17) of c_p, put $\sqrt{c^2_{p_1}} = -c_{p_1}$ and neglect the last third term, that is zero for $\phi(p,p) = 0$.

The first two terms in (1.26) are finite in the termodynamic limit $V \to \infty$ and the second term becomes an finite integral. The third term tends to zero as $V \to \infty$ and we neglect it (or it is equal to zero if $\phi(p,p) = 0$).

Substituting expressions (1.18) for v_p and u_p into the two first terms in (1.26) and neglecting the last third term one obtains

$$(\phi^a_{p_1}, H_\Lambda\phi^a_{p_1}) - (\phi^a_0, H_\Lambda\phi^a_0) \cong \sqrt{\varepsilon^2_{p_1} + c^2_{p_1}} = E_{p_1}. \tag{1.27}$$

Expression (1.27) gives the disintegration energy of the pair with momentum p_1. It follows from (1.27) that this energy cannot be less than $|c_p|$ that is greater than zero for all p_1 from layer $\left|\frac{p_1^2}{2m} - \mu\right| < \omega$. This means that average energy of the excited state $\phi^a_{p_1}$ is separated from the average energy of the ground state by the gap $|c_{p_1}|$.

One can also obtain excited state by applying to ϕ^a_0 the operator of annihilation. For example,

$$\phi^a_{-p_1} = \frac{\psi_{p_1}}{v_{p_1}}\phi^a_0 = \prod_{k\neq p_1}\left(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)\overset{*}{\psi}_{-p_1}|0\rangle.$$

One can easily repeat the calculation performed for $\phi^a_{p_1}$. Note that states ϕ^a_p with different p are orthogonal (it will be shown in Section 3).

Introduce the many-particle excited states

$$\frac{\overset{*}{\psi}_{p_1}}{u_{p_1}}\cdots\frac{\overset{*}{\psi}_{p_m}}{u_{p_m}}\frac{\overset{*}{\psi}_{-p_{m+1}}}{u_{-p_{m+1}}}\cdots\frac{\overset{*}{\psi}_{-p_{m+n}}}{u_{-p_{m+n}}}\phi^a_0 =$$
$$= \overset{*}{\psi}_{p_1}\cdots\overset{*}{\psi}_{p_m}\overset{*}{\psi}_{-p_{m+1}}\cdots\overset{*}{\psi}_{-p_{m+n}}\prod_{k\neq p_1\neq\ldots\neq -p_{m+n}}\left(u_k+v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle \quad (1.28)$$

with $p_i\neq p_j$, $1\leq i\leq m$, $m+1\leq j\leq m+n$.

These states are orthogonal and normalized to unity. The difference between the average energy of these states and the energy of the ground state is equal to $\sum_{i=1}^{m+n}E_{p_i}$. We omit the corresponding calculation absolutely analogic to the one performed above for $\phi^a_{p_1}$ and remark that the terms that tend to zero as $V\to\infty$ are neglected (see for details Section 3, formula (3.4)).

Note that the excited state with pairs of the operators of creation of electrons with opposite momenta and spin are not orthogonal to the ground state. For example for the state

$$\phi^a_{p_1,-p_1}=\frac{\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1}\phi^a_0}{u_{p_1}}=\prod_{k\neq p_1}\left(u_k+v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1}|0\rangle$$

one has

$$(\phi^a_{p_1,-p_1},\phi^a_0)=v_{p_1}.$$

We will introduce proper excited states of pairs in the next section.

2. Approximating Hamiltonian. *2.1. Ground state as a vacuum for operators of annihilation of quasiparticles.* Introduce the following operators

$$\begin{aligned}\overset{*}{\alpha}_p&=u_p\overset{*}{\psi}_p-v_p\psi_{-p}, &\qquad \alpha_p&=u_p\psi_p-v_p\overset{*}{\psi}_{-p},\\ \overset{*}{\alpha}_{-p}&=u_p\overset{*}{\psi}_{-p}+v_p\psi_p, &\qquad \alpha_{-p}&=u_p\psi_{-p}+v_p\overset{*}{\psi}_p\end{aligned}\quad (2.1)$$

where the functions u_p, v_p were defined by formulas (1.18), and we use denotation $p=(p,1)$, $-p=(-p,-1)$. Note that the functions u_p, v_p do not depend on spin.

It is easy to show that the operators $\overset{*}{\alpha}_p$, α_p, $\overset{*}{\alpha}_{-p}$, α_{-p} satisfy the following canonical anticommutation relations

$$\begin{aligned}\{\alpha_{p_1},\overset{*}{\alpha}_{p_2}\}&=\delta_{p_1,p_2}, &\qquad \{\alpha_{p_1},\alpha_{p_2}\}&=\{\overset{*}{\alpha}_{p_1},\overset{*}{\alpha}_{p_2}\}=0,\\ \{\alpha_{-p_1},\overset{*}{\alpha}_{-p_2}\}&=\delta_{p_1,p_2}, &\qquad \{\alpha_{-p_1},\alpha_{-p_2}\}&=\{\overset{*}{\alpha}_{-p_1},\overset{*}{\alpha}_{-p_2}\}=0\end{aligned}\quad (2.2)$$

and the operators α_p, $\overset{*}{\alpha}_p$ anticommute with α_{-p}, $\overset{*}{\alpha}_{-p}$, i.e., the operators with opposite spin anticommute.

The operators (2.1) are linear combination of the operators of annihilation and creation of electrons and they are known as the operators of annihilation and creation of quasiparticles.

Let us show that the ground state ϕ^a_0 is a vacuum for the annihilation operators α_p, α_{-p}. Consider $\alpha_p\phi^a_0$, $\alpha_{-p}\phi^a_0$. We have

$$\alpha_p \phi_0^a \Big(u_p \psi_p - v_p \overset{*}{\psi}_{-p}\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \Big(u_p v_p \overset{*}{\psi}_{-p} - v_p u_p \overset{*}{\psi}_{-p}\Big) \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = 0, \tag{2.3}$$

$$\alpha_{-p} \phi_0^a = \Big(-u_p v_p \overset{*}{\psi}_p + u_p v_p \overset{*}{\psi}_p\Big) \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = 0.$$

Now calculate $\overset{*}{\alpha}_p \phi_0^a$, $\overset{*}{\alpha}_{-p} \phi_0^a$. We have

$$\overset{*}{\alpha}_p \phi_0^a = \Big(u_p \overset{*}{\psi}_p - v_p \psi_{-p}\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \Big(u_p^2 + v_p^2\Big) \overset{*}{\psi}_p \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = \overset{*}{\psi}_p \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big),$$

$$\overset{*}{\alpha}_{-p} \phi_0^a = \Big(u_p \overset{*}{\psi}_{-p} + v_p \psi_p\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = \tag{2.4}$$

$$= \Big(u_p^2 + v_p^2\Big) \overset{*}{\psi}_{-p} \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = \overset{*}{\psi}_{-p} \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle.$$

The state $\overset{*}{\alpha}_p \phi_0^a$, $\overset{*}{\alpha}_{-p} \phi_0^a$ are normalized to unity.

Obviously that the states $\overset{*}{\alpha}_p \phi_0^a$, $\overset{*}{\alpha}_{-p} \phi_0^a$ with different momenta or spin are orthogonal. This property can be proved using representation (2.4) or using anticommutation relations (2.2) and the fact that ϕ_0^a is the vacuum for α_p, α_{-p}. For example,

$$(\overset{*}{\alpha}_{p_1} \phi_0^a, \overset{*}{\alpha}_{p_2} \phi_0^a) = (\phi_0^a, \alpha_{p_1} \overset{*}{\alpha}_{p_2} \phi_0^a) = -(\phi_0^a, \overset{*}{\alpha}_{p_2} \alpha_{p_1} \phi_0^a) = 0.$$

Consider the following state

$$\overset{*}{\alpha}_p \overset{*}{\alpha}_{-p} \phi_0^a = \Big(u_p \overset{*}{\psi}_p - v_p \psi_{-p}\Big) \Big(u_p \overset{*}{\psi}_{-p} + v_p \psi_p\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \Big(u_p \overset{*}{\psi}_p - v_p \psi_{-p}\Big) \overset{*}{\psi}_{-p} \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \Big(u_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} - v_p\Big) \prod_{k \neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle. \tag{2.5}$$

Now calculate $(\overset{*}{\alpha}_p \overset{*}{\alpha}_{-p} \phi_0^a, \phi_0^a)$ in terms of the operators ψ_k, $\overset{*}{\psi}_k$, ψ_{-k}, $\overset{*}{\psi}_{-k}$. We have

$$(\overset{*}{\alpha}_p \overset{*}{\alpha}_{-p} \phi_0^a, \phi_0^a\Big) =$$

$$= \langle 0| \Big(u_p \psi_{-p} \psi_p - v_p\Big) \prod_{k' \neq p} \Big(u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \langle 0| \prod_{k' \neq p} \Big(u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}\Big) \Big(u_p \psi_{-p} \psi_p - v_p\Big) \prod_k \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle =$$

$$= \langle 0| \prod_{k' \neq p} \Big(u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}\Big) (u_p v_p - v_p u_p) \prod_{k \neq p} \Big(u_k + v_k) \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) |0\rangle = 0.$$

In terms of the operators α_k, $\overset{*}{\alpha}_k$, α_{-k}, $\overset{*}{\alpha}_{-k}$ this calculation is a trivial one

$$(\overset{*}{\alpha}_p \overset{*}{\alpha}_{-p}\phi_0^a, \phi_0^a) = (\phi_0^a, \alpha_{-p}\alpha_p\phi_0^a) = 0.$$

By using the operators of creation of the quasiparticles $\overset{*}{\alpha}_p \overset{*}{\alpha}_{-p}$ one can construct the orthonormal basis of excited states

$$\overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_m} \overset{*}{\alpha}_{-p_{m+1}} \ldots \overset{*}{\alpha}_{-p_{m+n}} \phi_0^a \tag{2.6}$$

without any restriction on momenta. The property of orthogonality can be easy checked as above.

In what follows for the sake of simplicity we will suppose that potential $\phi(p, p')$ is separated kernel

$$\phi(p, p') = g w_p w_{p'} \tag{2.7}$$

where the function $w_p > 0$ is a continuous function with support in layer of the Fermi sphere $\left|\frac{p^2}{2m} - \mu\right| < \omega$, $\omega > 0$, and g is coupling constant $g < 0$. Note that it is easy to generalize all results for potential $\phi(p, p') = g w_p w_{p'}(1 - \delta_{p,p'})$.

2.2. Approximating Hamiltonian. Consider the following Hamiltonian

$$H_{a,\Lambda} = \sum_{\bar{p}} \overset{*}{\psi}_{\bar{p}} \varepsilon_p \psi_{\bar{p}} + c \sum_p w_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} + c \sum_p w_p \psi_{-p}\psi_p - g^{-1}c^2 V I \tag{2.8}$$

where constant c will be defined later from the condition of minimum of the energy of ground state of the Hamiltonian $H_{a,\Lambda}$.

This Hamiltonian has been proposed by Bogolyubov [2] and is known as the approximating Hamiltonian.

We proceed to diagonalization of the approximating Hamiltonian. Recall that a Hamiltonian is called diagonal one if it consists of the sums of products of creation and annihilation operators with the same momenta and spins. To do this we apply the canonical transformation of the operators $\overset{*}{\psi}_p$, ψ_p, $\overset{*}{\psi}_{-p}$, ψ_{-p} and express them through the operators $\overset{*}{\alpha}_p$, α_p, $\overset{*}{\alpha}_{-p}$, α_{-p} given by formulas (2.1) but with unknown real u_p and v_p which satisfy the condition $u_p^2 + v_p^2 = 1$.

From (2.1) we obtain

$$\begin{aligned} \overset{*}{\psi}_p &= u_p \overset{*}{\alpha}_p + v_p \alpha_{-p}, & \psi_p &= u_p \alpha_p + v_p \overset{*}{\alpha}_{-p}, \\ \overset{*}{\psi}_{-p} &= u_p \overset{*}{\alpha}_{-p} - v_p \alpha_p, & \psi_{-p} &= u_p \alpha_{-p} - v_p \overset{*}{\alpha}_p. \end{aligned} \tag{2.9}$$

It is easy to check that the operators $\overset{*}{\psi}_p$, ψ_p, $\overset{*}{\psi}_{-p}$, ψ_{-p} satisfy canonical anticommutation relations.

Substituting (2.9) in $H_{a,\Lambda}$ (2.8), after elementary but cumbersome computation , one obtains

$$\begin{aligned} H_{a,\Lambda} = \sum_{\bar{p}} \Big\{ & \overset{*}{\alpha}_{\bar{p}} \alpha_{\bar{p}} \Big[\varepsilon_p (u_p^2 - v_p^2) + 2c w_p u_p v_p\Big] + \\ & + \overset{*}{\alpha}_{\bar{p}} \overset{*}{\alpha}_{-\bar{p}} \Big[-\varepsilon_p u_p v_p + \frac{c}{2} w_p (u_p^2 - v_p^2)\Big] + \\ & + \alpha_{-\bar{p}} \alpha_{\bar{p}} \Big[-\varepsilon_p u_p v_p + \frac{c}{2} w_p (u_p^2 - v_p^2)\Big] \Big\} + \\ & + \sum_{\bar{p}} (\varepsilon_p v_p^2 - c w_p u_p v_p) - g^{-1} c^2 V. \end{aligned} \tag{2.10}$$

The Hamiltonian $H_{a,\Lambda}$ will be diagonalized if the coefficients of the operators $\overset{*}{\alpha}_{\bar p}\overset{*}{\alpha}_{-\bar p}$ and $\alpha_{-\bar p}\alpha_{\bar p}$ in (2.10) are equal to zero:

$$-\varepsilon_p u_p v_p + \frac{c}{2} w_p (u_p^2 - v_p^2) = 0, \qquad u_p^2 + v_p^2 = 1. \tag{2.11}$$

The last equation coincide with equation (1.16′) with $-cw_p$ instead of c_p and it has solution (1.18) with $c_p = -cw_p$.

Substituting (1.18) in (2.10) one has, as in (1.25),

$$H_{a,\Lambda} = \sum_{\bar p} E_p \overset{*}{\alpha}_{\bar p}\alpha_{\bar p} + V\left[\frac{1}{V}\sum_p \left[\varepsilon_p - E_p\right] - g^{-1}c^2\right] = \sum_{\bar p} E_p \overset{*}{\alpha}_{\bar p}\alpha_{\bar p} + C(c)V,$$

$$E_p = \sqrt{\varepsilon_p^2 + c^2 w_p^2}. \tag{2.12}$$

Note that ground state for the approximating Hamiltonian $H_{a,\Lambda}$ is the vacuum for the operator $\alpha_{\bar p}$, i.e. ϕ_0^a, and the energy of the ground state, according to (2.12) is $C(c)V$.

2.3. Coincidence of the energies of ground states of the model BCS and the approximating Bogolyubov Hamiltonians. It follows from (2.8), (2.12) that the energy of the ground state ϕ_0^a of the approximating Hamiltonian

$$H_{a,\Lambda} = \sum_{\bar p} \overset{*}{\psi}_{\bar p}\varepsilon_p\psi_{\bar p} + c\sum_p w_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} + c\sum_p w_p \psi_{-p}\psi_p - g^{-1}c^2 VI =$$

$$= \sum_{\bar p} E_p \overset{*}{\alpha}_{\bar p}\alpha_{\bar p} + C(c)VI \tag{2.13}$$

is equal to

$$C(c)V = V\left[\frac{1}{V}\sum_p(\varepsilon_p - E_p) - g^{-1}c^2\right], \quad E_p = \sqrt{\varepsilon_p^2 + c^2 w_p^2},$$

and constant c should be determined from condition of minimum of $C(c)$ with respect to c^2. This condition is $\left(\text{for } w_p = w = \text{const for } \left|\frac{p^2}{2m} - \mu\right| < \omega\right)$

$$1 = -\frac{1}{2V} g \sum_p \frac{w^2}{\sqrt{\varepsilon_p^2 + c^2 w^2}} \tag{2.14}$$

or in the thermodynamic limit

$$1 = -\frac{g}{2} \int\limits_{\left|\frac{p^2}{2m} - \mu\right| < \omega} \frac{w^2}{\sqrt{\varepsilon_p^2 + c^2 w^2}} dp.$$

For some conditions imposed on w, g equation (2.14) has a unique solution [11].

Now consider the energy of the ground state ϕ_0^a for the model BCS Hamiltonian H_Λ. According to (1.15)–(1.20) we have

$$(\phi_0^a, H_\Lambda \phi_0^a) = 2\sum_p \varepsilon_p v_p^2 + \frac{1}{V} g \sum_{p,p'} w_p w_{p'} u_p v_p u_{p'} v_{p'} + \frac{1}{V} g \sum_p w_p w_p v_p^4 \cong$$

$$\cong 2\sum_p \varepsilon_p v_p^2 + \sum_p c_p u_p v_p =$$

$$= \sum_p \left(\varepsilon_p - \frac{\varepsilon_p^2}{\sqrt{\varepsilon_p^2 + c_p^2}} - \frac{c_p^2}{\sqrt{\varepsilon_p^2 + c_p^2}} \right) + \frac{1}{2} \sum \frac{c_p^2}{\sqrt{\varepsilon_p^2 + c_p^2}} =$$

$$= \sum_p (\varepsilon_p - E_p) + \frac{1}{2} \sum_p \frac{c_p^2}{\sqrt{\varepsilon_p^2 + c_p^2}} \tag{2.15}$$

$\left(\text{we neglect the term } \frac{1}{V} g \sum_p w_p w_p v_p^4 \right)$.

Now take into account that c_p does not depend on p, i.e. $c_p = -cw$ and satisfies equation (1.20)

$$cw = -g \frac{1}{V} \sum_p w^2 \frac{1}{2} \frac{cw}{\sqrt{\varepsilon_p^2 + c^2 w^2}}$$

which coincides with condition of minimum of $C(c)$ (2.14).

Taking into account (1.20) and (2.15) we finally obtain

$$(\phi_0^a, H_\Lambda \phi_0^a) = \sum_p (\varepsilon_p - E_p) - g^{-1} c^2 V \left(-\frac{1}{2V} g \sum_p \frac{w^2}{\sqrt{\varepsilon_p^2 + c^2 w^2}} \right) =$$

$$= \sum_p (\varepsilon_p - E_p) - g^{-1} c^2 V. \tag{2.16}$$

Thus the energies of the ground state ϕ_0^a for the model Hamiltonian BSC H_Λ and the approximating Hamiltonian $H_{a,\Lambda}$ coincide if one neglects the term $\frac{1}{V} g \sum w_p w_p v_p^4$.

2.4. Approximating Hamiltonian as quadratic form of model Hamiltonian BCS. Formulation of results. Consider the following approximating Hamiltonian

$$H_{\text{appr},\Lambda} = \sum_{\bar{p}} \overset{*}{\psi}_{\bar{p}} \varepsilon_p \psi_{\bar{p}} + \sum_p c_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} + \sum_p c_p \psi_{-p} \psi_p - g^{-1} c^2 V I \tag{2.17}$$

where

$$A_p = (\phi_0^a, \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \phi_0^a) = (\phi_0^a, \psi_{-p} \psi_p \phi_0^a) = u_p v_p,$$
$$c_p = \frac{1}{V} \sum_{p'} \phi(p, p') A_{p'}, \tag{2.18}$$
$$g^{-1} c^2 V I = \frac{1}{V} \sum_{p,p'} \phi(p, p') A_p A_{p'}, \phi(p, p') = g w_p w_{p'}.$$

There is the following motivation of definition of approximating Hamiltonian. Consider again the model BCS Hamiltonian (1.3) and represent it in the following identical form

$$H_\Lambda = H_{\text{appr},\Lambda} + \frac{1}{V} \sum_{p,p'} \phi(p, p') \left(\overset{*}{\psi}_p \overset{*}{\psi}_{-p} - A_p \right) \left(\psi_{-p'} \psi_{p'} - A_{p'} \right). \tag{2.19}$$

It follows from (2.19) that $H_{\text{appr},\Lambda} = H_\Lambda$ if one neglects the last term in (2.19). The approximating Hamiltonian is obtained from the model BCS Hamiltonian if one replaces consequently one of the operators $\overset{*}{\psi}_p \overset{*}{\psi}_{-p}$, $\psi_{-p} \psi_p$ by the operator $A_p I$, where A_p is equal to the average of these operators over the ground state, and adds the operator $-g^{-1} c^2 V I$. Note that c_p, defined according to (2.18), coincides with c_p in the previous subsection, defined according to (1.17) and in both cases they are determined from the condition of minimum of $(\phi_0^a, H_\Lambda \phi_0^a)$.

Now calculate the average of H_{appr} over ϕ_0^a. Repeating the calculation performed for $(\phi_0^a, H_\Lambda \phi_0^a)$ one obtains

$$(\phi_0^a, H_{\text{appr},\Lambda} \phi_0^a) = 2\sum_p v_p^2 \varepsilon_p + 2\sum_p c_p u_p v_p - g^{-1} c^2 V I =$$
$$= 2\sum_p v_p^2 \varepsilon_p + \frac{2}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} - \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} =$$
$$= 2\sum_p v_p^2 \varepsilon_p + \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} = (\phi_0^a, H_\Lambda \phi_0^a) - \frac{1}{V}\sum_p \phi(p,p) v_p^4, \tag{2.20}$$
$$(\phi_0^a, H_\Lambda \phi_0^a) = 2\sum_p v_p^2 \varepsilon_p + \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} + \frac{1}{V}\sum_p \phi(p,p) v_p^4 =$$
$$= C(c)V + \frac{1}{V}\sum_p \phi(p,p) v_p^4.$$

Thus the averages of $H_{a,\Lambda}$ and H_Λ over the ground state ϕ_0^a coincide in the following sense

$$\lim_{V\to\infty} \frac{1}{V}\Big[(\phi_0^a, H_\Lambda \phi_0^a) - (\phi_0^a, H_{a,\Lambda} \phi_0^a)\Big] = \lim_{V\to\infty} \frac{1}{V^2}\sum_p \phi(p,p) v^4(p) = 0. \tag{2.21}$$

In (2.21) we used that $\lim\limits_{V\to\infty} \frac{1}{V}\sum\limits_p \phi(p,p) v_p^4 = \int \phi(p,p) v^4(p)\,dp$ is finite for potential $\phi(p,p')$ with compact support. We used the factor $\frac{1}{V}$ in (2.21) because the both terms diverge as V when $V \to \infty$. But one can cancel the equal divergent terms and use the factor $\frac{1}{V^\delta}$ with arbitrary $\delta > 0$, $\delta < 1$ and therefore

$$\lim_{V\to\infty} \frac{1}{V^\delta}\Big[(\phi_0^a, H_\Lambda \phi_0^a) - (\phi_0^a, H_{a,\Lambda} \phi_0^a)\Big] = \lim_{V\to\infty} \frac{1}{V^\delta}\Big[\frac{1}{V}\sum_p \phi(p,p') v^4(p)\Big] = 0.$$

Remark. Bogolyubov has proved the following fundamental equality

$$\lim_{V\to\infty} \frac{1}{V}\Big[(\phi_0, H_\Lambda \phi_0) - (\phi_0^a, H_{a,\Lambda} \phi_0^a)\Big] = 0, \tag{2.22}$$

where ϕ_0 is the ground state of the model BCS Hamiltonian H_Λ. He has been able to prove this equality without determination exactly ϕ_0. (We have reason to believe that ϕ_0 is equal to (1.9) with u_k, v_k that satisfy equation (1.16) but with the term $\frac{4}{V}\phi(p,p) v_p^3$.)

We have proved that ϕ_0^a (1.12), with u_k, v_k (1.18) and c_p (1.19), (1.20), is the ground state of H_Λ only for states (1.5)–(1.9), i.e. coherent states, and for the potential $\phi(p,p')$ such that $\phi(p,p) = 0$. In this sense we have repeated the Bogolubov result (2.22).

We are able to generalize this equality to general excited states. Namely let

$$\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=1}^{m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a =$$

$$= \prod_{i=1}^{n} \overset{*}{\psi}_{p_i} \prod_{j=1}^{m} \left(u_{p_j} \overset{*}{\psi}_{p_j} \overset{*}{\psi}_{-p_j} - v_{p_i} \right) \prod_{k \neq (p)_n \neq (p)_m} \left(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k} \right) |0\rangle$$

be a general excited state with n excited electrons and m excited pairs then

$$\lim_{V\to\infty} \frac{1}{V} \left[\left(\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=1}^{m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a, (H_\Lambda - H_{a,\Lambda}) \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=1}^{m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a \right) \right] = 0 \tag{2.23}$$

for arbitrary finite n and m. Proof of (2.22) is presented in Section 3.

In Section 3 it is also proved that different excited states are asymptotically orthogonal to the action of the Hamiltonian H_Λ on them. This means that equality (2.22) is also true for finite linear combination of the exited states. Namely

$$\lim_{V\to\infty} \frac{1}{V} \left(\sum_{k,l} c_{kl} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^k} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^l} \overset{*}{\alpha}_{-p_j^l} \phi_0^a, (H_\Lambda - H_{a,\Lambda}) \times \right.$$

$$\left. \times \sum_{r,s} c_{rs} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^r} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^s} \overset{*}{\alpha}_{-p_j^s} \phi_0^a \right) = 0, \tag{2.24}$$

where summation with respect k, l and r, s is carried out over finite numbers.

3. Approximating Hamiltonian as quadratic form of model BCS Hamiltonian. *3.1. Asymptotic coincidence of averages of model and approximating Hamiltonian.* ***Proof of Theorem* 1.** Consider a sequence of excited states $\overset{*}{\alpha}_{p_1} \phi_0^a, \ldots, \overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a, \ldots$, and calculate the averages of $H_{a,\Lambda}$ over these states

$$(\overset{*}{\alpha}_{p_1} \phi_0^a, H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \phi_0^a), \ldots, (\overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a, H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a), \ldots .$$

The excited states are eigenvectors of $H_{a,\Lambda}$

$$H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \phi_0^a = (E_{p_1} + C(c)V) \overset{*}{\alpha}_{p_1} \phi_0^a, \ldots, H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a =$$

$$= \left(E_{p_1} + \ldots + E_{p_n} + C(c)V \right) \overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a, \ldots,$$

$$C(c)V = \sum_p 2\varepsilon_p v_p^2 + \frac{1}{V} \sum_{p,p'} \phi(p,p') u_p u_p' v_p v_{p'}$$

(see Subsection 2.4) and therefore

$$\left(\overset{*}{\alpha}_{p_1} \phi_0^a, H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \phi_0^a \right) = E_{p_1} + C(c)V, \ldots, \left(\overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a, H_{a,\Lambda} \overset{*}{\alpha}_{p_1} \ldots \overset{*}{\alpha}_{p_n} \phi_0^a \right) =$$

$$= E_{p_1} + \ldots + E_{p_n} + C(c)V.$$

Our aim is to calculate the averages of the model BCS Hamiltonian H_Λ over the excited states and to compare them with the corresponding averages of $H_{a,\Lambda}$.

We have already calculated the average (see (1.25))

$$\left(\overset{*}{\alpha}_{p_1} \phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \phi_0^a \right) =$$

$$= \varepsilon_{p_1} - 2\varepsilon_{p_1} v_{p_1}^2 - 2c_{p_1} u_{p_1} v_{p_1} + C(c)V +$$

$$+ \frac{1}{V} \sum_{p \neq p_1} \phi(p,p) v^4(p) + \frac{1}{V} \phi(p_1,p_1) u_{p_1}^2 v_{p_1}^2 =$$

$$= E_{p_1} + C(c)V + \frac{1}{V}\sum_{p\neq p_1}\phi(p,p)v^4(p) + \frac{1}{V}\phi(p_1,p_1)u_{p_1}^2v_{p_1}^2.$$

We have

$$\lim_{V\to\infty}\frac{1}{V}\left[\left(\overset{*}{\alpha}_{p_1}\phi_0^a,(H_\Lambda - H_{a,\Lambda})\overset{*}{\alpha}_{p_1}\phi_0^a\right)\right] =$$

$$= \lim_{V\to\infty}\frac{1}{V}\left\{\frac{1}{V}\sum_{p\neq p_1}\phi(p,p)v^4(p) + \frac{1}{V}\phi(p_1,p_1)u_{p_1}^2v_{p_1}^2\right\} = 0 \qquad (3.1$$

because $\lim_{V\to\infty}\frac{1}{V^2}\sum_{p\neq p_1}\phi(p,p)v^4(p) = 0$. We can also put in (3.1) factor $\frac{1}{V^\delta}$ with arbitrary $\delta > 0$ instead of $\frac{1}{V}$. Note that both terms in (3.1) are equal to zero i $\phi(p,p) = 0$.

Now consider the general case of excited states $\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a$ $(p_i \neq -p_j)$ an calculate the following average

$$\left(\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a, H_\Lambda\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a\right) =$$

$$= \langle 0|\overset{*}{\psi}_{p_1}\ldots\overset{*}{\psi}_{p_n}\prod_{k'\neq(p)_n}\left(u_{k'} + v_{k'}\overset{*}{\psi}_{k'}\overset{*}{\psi}_{-k'}\right), \qquad (3.2$$

$$\left(\sum_{\bar p}\varepsilon_p\overset{*}{\psi}_{\bar p}\psi_{\bar p} + \frac{1}{V}\sum\phi(p,p')\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\psi_{-p'}\psi_{p'}\right)\overset{*}{\psi}_{p_1}\ldots\overset{*}{\psi}_{p_n}\times$$

$$\times\prod_{k\neq(p)_n}\left(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \sum_{i=1}^n\varepsilon_{p_i} + \sum_{p\neq(p)_n}2v_p^2\varepsilon_p +$$

$$+\frac{1}{V}\sum_{p\neq p'\neq(p)_n}\phi(p,p')u_pv_pu_{p'}v_{p'} + \frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)v_p^2 =$$

$$= \sum_{i=1}^n\varepsilon_{p_i}(1-2v_{p_i}^2) + \sum_p 2v_p^2\varepsilon_p -$$

$$-2\sum_{i=1}^n\frac{1}{V}\sum_p\phi(p_i,p)u_{p_i}v_{p_i}u_pv_p + \frac{1}{V}\sum_{p,p'}\phi(p,p')u_pv_pu_{p'}v_{p'} +$$

$$+\frac{1}{V}\sum_{i=1}^n\sum_{j=1}^n\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} + \frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)v_p^4 =$$

$$= \sum_{i=1}^n\left[\varepsilon(p_i)(1-2v^2(p_i)) - 2c_{p_i}u_{p_i}v_{p_i}\right] + C(c)V + \frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)v_p^4 +$$

$$+\frac{1}{V}\sum_{i=1}^n\sum_{j=1}^n\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j} =$$

$$= \sum_{i=1}^n E_{p_i} + C(c)V + \frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)v_p^4 + \frac{1}{V}\sum_{i=1}^n\sum_{j=1}^n\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}.$$

If follows from (3.2) that

$$\lim_{V\to\infty}\frac{1}{V}\left[\left(\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a,(H_\Lambda-H_{a,\Lambda})\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a\right)\right]=$$

$$=\lim_{V\to\infty}\frac{1}{V}\left[\frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)v_p^4+\frac{1}{V}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}\right]=0 \quad (3.3)$$

for arbitrary finite n and even for n such that $\lim\limits_{V\to\infty}\dfrac{n^2}{V^2}=0$. Note that for arbitrary finite n equality (3.3) also holds if instead of factor $\dfrac{1}{V}$ one puts $\dfrac{1}{V^\delta}$ with arbitrary $0<\delta<1$ $\left(\text{and even for } n \text{ such that } \lim\limits_{V\to\infty}\dfrac{n^2}{V^\delta}=0\right)$.

It follows from (1.13) and (3.2) that

$$\left(\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a,H_\Lambda\overset{*}{\alpha}_{p_1}\ldots\overset{*}{\alpha}_{p_n}\phi_0^a\right)-(\phi_0^a,H_\Lambda\phi_0^a)=$$

$$=\sum_{i=1}^{n}E_{p_i}-\frac{1}{V}\sum_{i=1}^{n}\phi(p_i,p_i)v_{p_i}^4+\frac{1}{V}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}. \quad (3.4)$$

The right-hand side of (3.4) tends to $\sum\limits_{i=1}^{n}E_{p_i}$ as $V\to\infty$.

Consider a general exited state of n pairs

$$\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\ldots\overset{*}{\alpha}_{p_n}\overset{*}{\alpha}_{-p_n}\phi_0^a=$$

$$=\prod_{i=1}^{n}\left(u_{p_i}\overset{*}{\psi}_{p_i}\overset{*}{\psi}_{-p_i}-v_{p_i}\right)\prod_{k\neq(p)_n}\left(u_k+v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle=$$

$$=\prod_{k}\left(\tilde u_k+\tilde v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle, \quad (3.5)$$

where in (3.5) $\tilde u_k=u_k$, $\tilde v_k=v_k$ for $k\neq(p)_n$, $\tilde u_{p_i}=-v_{p_i}$, $\tilde v_{p_i}=u_{p_i}$, $i=1,\ldots,n$. One sees from (3.5) that the excited state with n pairs is a "ground" state with functions $\tilde u_k,\tilde v_k$. One can use the result obtained for ground state (1.13)

$$\left(\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\ldots\overset{*}{\alpha}_{p_n}\overset{*}{\alpha}_{-p_n}\phi_0^a,H_\Lambda\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\ldots\overset{*}{\alpha}_{p_n}\overset{*}{\alpha}_{-p_n}\phi_0^a\right)=$$

$$=\sum_{p}2\varepsilon_p\tilde v_p^2+\frac{1}{V}\sum_{p,p'}\phi(p,p')\tilde u_p\tilde v_p\tilde u_{p'}\tilde v_{p'}+\frac{1}{V}\sum_{p}\phi(p,p)\tilde v^4(p)=$$

$$=\sum_{p\neq(p)_n}2\varepsilon_pv_p^2+\sum_{i=1}^{n}2\varepsilon_{p_i}u_{p_i}^2+\frac{1}{V}\sum_{\substack{p\neq(p)_n\\ p'\neq(p)_n}}\phi(p,p')u_pv_pu_{p'}v_{p'}+$$

$$+\sum_{i=1}^{n}\frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p_i)u_pv_p(-u_{p_i}v_{p_i})+$$

$$+\sum_{i=1}^{n}\frac{1}{V}\sum_{p'\neq(p)_n}\phi(p_i,p')(-u_{p_i}v_{p_i})u_pv_p+$$

$$+\frac{1}{V}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+\frac{1}{V}\sum_{p}\phi(p,p)\tilde v_p^4=$$

$$= \sum_p 2\varepsilon_p v_p^2 + \frac{1}{V}\sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} +$$

$$+ \sum_{i=1}^{n} 2\varepsilon_{p_i}(u_{p_i}^2 - v_{p_i}^2) - 2\sum_{i=1}^{n}\frac{1}{V}\sum_p \phi(p,p_i) u_p v_p u_{p_i} v_{p_i} -$$

$$- 2\sum_{i=1}^{n}\frac{1}{V}\sum_{p'} \phi(p_i,p') u_{p_i} v_{p_i} u_{p'} v_{p'} +$$

$$+ \frac{4}{V}\sum_{i=1}^{n}\sum_{j=1}^{n} \phi(p_i,p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V}\sum_p \phi(p,p)\tilde{v}_p^4 =$$

$$= C(c)V + 2\sum_{i=1}^{n} \varepsilon_{p_i}(1 - 2v_{p_i}^2) - 4\sum_{i=1}^{n} c(p_i) u_{p_i} v_{p_i} +$$

$$+ \frac{4}{V}\sum_{i=1}^{n}\sum_{j=1}^{n} \phi(p_i,p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V}\sum_p \phi(p,p)\tilde{v}_p^4 =$$

$$= C(c)V + 2\sum_{i=1}^{n} E_{p_i} + \frac{4}{V}\sum_{i=1}^{n}\sum_{j=1}^{n} \phi(p_i,p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+ \frac{1}{V}\sum_p \phi(p,p)\tilde{v}_p^4. \tag{3.6}$$

If follows from (3.6) that

$$\lim_{V\to\infty}\frac{1}{V}\left[\left(\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\ldots\overset{*}{\alpha}_{p_n}\overset{*}{\alpha}_{-p_n}\phi_0^a, (H_\Lambda - H_{a,\Lambda})\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\ldots\overset{*}{\alpha}_{p_n}\overset{*}{\alpha}_{-p_n}\phi_0^a\right)\right] =$$

$$= \lim_{V\to\infty}\frac{1}{V}\left[\frac{4}{V}\sum_{i=1}^{n}\sum_{j=1}^{n} \phi(p_i,p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V}\sum_p \phi(p,p)\tilde{v}_p^4\right] = 0 \tag{3.7}$$

for arbitrary fixed n and even for n such that $\lim_{V\to\infty}\frac{n^2}{V^2} = 0$. Note that equality (3.7) also holds if instead of factor $\frac{1}{V}$ one puts $\frac{1}{V^\delta}$, $0<\delta<1$.

Now consider the generalest state of n electrons and m pairs

$$\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a = \overset{*}{\psi}_{p_1}\ldots\overset{*}{\psi}_{p_n}\prod_{k\neq(p)_n}\left(\tilde{u}_k + \tilde{v}_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle,$$

$$(p)_n = (p_1,\ldots,p_n),$$

$$\tilde{u}_k = u_k, \quad \tilde{v}_k = v_k \quad \text{for} \quad k \neq (p_{n+1},\ldots,p_{n+m}),$$

$$\tilde{v}_k = -v_k, \quad \tilde{v}_k = u_k \quad \text{for} \quad k = p_j, \quad n+1 \le j \le n+m,$$

and calculate the average

$$\left(\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a, H_\Lambda\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a\right) =$$

$$= \langle 0|\psi_{p_1}\ldots\psi_{p_n}\prod_{k'\neq(p)_n}\left(\tilde{u}_{k'} + \tilde{v}_{k'}\psi_{-k'}\psi_{k'}\right)\times$$

$$\times H_\Lambda\overset{*}{\psi}_{p_1}\ldots\overset{*}{\psi}_{p_n}\prod_{k\neq(p)_n}\left(\tilde{u}_k + \tilde{v}_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle =$$

$$= \sum_{i=1}^{n} \varepsilon_{p_i} + \sum_{p \neq (p)_n} 2\tilde{v}_p^2 \varepsilon_p +$$

$$+\frac{1}{V} \sum_{(p,p') \neq (p)_n} \phi(p,p') \tilde{u}_p \tilde{v}_p \tilde{u}_{p'} \tilde{v}_{p'} + \frac{1}{V} \sum_{p \neq (p)_n} \phi(p,p) \tilde{v}_p^4 =$$

$$= \sum_{i=1}^{n} \varepsilon_{p_i} (1 - 2v_{p_i}^2) + 2 \sum_{j=n+1}^{n+m} \varepsilon_{p_j} (1 - 2v_{p_j}^2) +$$

$$+2 \sum_{p} \varepsilon_p v_p^2 + \frac{1}{V} \sum_{p,p'} \phi(p,p') u_p v_p u_{p'} v_{p'} -$$

$$-2 \sum_{i=1}^{n+m} \frac{1}{V} \sum_{p} \phi(p_i, p) u_{p_i} v_{p_i} u_p v_p - 2 \sum_{i=n+1}^{n+m} \frac{1}{V} \sum_{p} \phi(p_i, p) u_{p_i} v_{p_i} u_p v_p +$$

$$+2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{p \neq (p)_n} \phi(p,p) \tilde{v}_p^4 =$$

$$= \sum_{i=1}^{n} \varepsilon_{p_i} (1 - 2v_{p_i}^2) + 2 \sum_{j=n+1}^{n+m} \varepsilon_{p_j} (1 - 2v_{p_j}^2) + C(c)V - 2 \sum_{i=1}^{n+m} c(p_i) u_{p_i} v_{p_i} -$$

$$-2 \sum_{i=n+1}^{n+m} c(p_i) u_{p_i} v_{p_i} + 2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V} \sum_{p \neq (p)_n} \phi(p,p) \tilde{v}_p^4 = \sum_{i=1}^{n} E_{p_i} + 2 \sum_{j=n+1}^{n+m} E_{p_j} + C(c)V +$$

$$+2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{p \neq (p)_n} \phi(p,p) \tilde{v}_p^4. \qquad (3.8)$$

It follows from (3.8) that

$$\Big(\prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a, H_\Lambda \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j} \overset{*}{\alpha}_{-p_j} \phi_0^a \Big) - (\phi_0^a, H_\Lambda \phi_0^a) =$$

$$= \sum_{i=1}^{n} E_{p_i} + 2 \sum_{j=n+1}^{n+m} E_{p_j} + 2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+\frac{1}{V}\sum_{i=n+1}^{n+m}\phi(p_i,p_i)u_{p_i}^4-\frac{1}{V}\sum_{i=1}^{n+m}\phi(p_i,p_i)v_{p_i}^4. \tag{3.8'}$$

It also follows from (3.8) that

$$\lim_{V\to\infty}\frac{1}{V}\Big[\Big(\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a,(H_\Lambda-H_{a,\Lambda})\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a\Big)\Big]=$$

$$=\lim_{V\to\infty}\frac{1}{V}\Big[2\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+$$

$$+\frac{1}{V}\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+$$

$$+\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+\frac{1}{V}\sum_{p\neq(p)_n}\phi(p,p)\tilde{v}_p^4\Big]=0 \tag{3.9}$$

for arbitrary fixed n, m and even for n, m such that $\lim\limits_{V\to\infty}\dfrac{(n+2m)^2}{V^2}=0$. The last equality also holds if instead of factor $\dfrac{1}{V}$ one puts $\dfrac{1}{V^\delta}$, $0<\delta<1$, and $\dfrac{(n+2m)^2}{V^{1+\delta}}\to$ $\to 0$ as $V\to\infty$.

If $\phi(p,p)=0$ then equalities (3.3), (3.7) and (3.9) hold even without the factor $\dfrac{1}{V}$, namely for arbitrary finite $n\geq 0$, $m\geq 0$

$$\lim_{V\to\infty}\left[\left(\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a,(H_\Lambda-H_{a,\Lambda})\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}\phi_0^a\right)\right]=$$

$$=\lim_{V\to\infty}\left[2\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+\right.$$

$$+\frac{1}{V}\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}+$$

$$\left.+\frac{1}{V}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m}\phi(p_i,p_j)u_{p_i}v_{p_i}u_{p_j}v_{p_j}\right]=0. \tag{3.10}$$

Equality (3.10) also holds if n,m tend to ∞ together with V in such a way that $\lim\limits_{V\to\infty}\dfrac{(n+2m)^2}{V}=0$.

3.2. Orthogonality of excited states to another excited states after action of model Hamiltonian. Proof of Theorem 2. We have shown that excited states are eigenvectors of the approximating Hamiltonian and therefore excited states after action of the approximating Hamiltonian are again orthogonal to all different excited states. We are going to prove that this property of excited states is still true in asymptotic sense for the model Hamiltonian.

First consider the following different excited states $\overset{*}{\alpha}_{p_2}\phi_0^a$, $\overset{*}{\alpha}_{p_1}\phi_0^a$ with $p_1\neq p_2$ and, for the sake of simplicity, with the same spin $+1$. Let show that states $\overset{*}{\alpha}_{p_2}\phi_0^a$, $H_\Lambda\overset{*}{\alpha}_{p_1}\phi_0^a$ are orthogonal as well as $\overset{*}{\alpha}_{p_2}\phi_0^a$, $\overset{*}{\alpha}_{p_1}\phi_0^a$.

Calculate the average

$$\left(\overset{*}{\alpha}_{p_2}\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1}\phi_0^a\right) = \langle 0|\psi_{p_2} \prod_{k'\neq p_2} (u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}) \times$$

$$\times\Big[\sum_{\bar{p}} \overset{*}{\psi}_{\bar{p}}\varepsilon_p\psi_{\bar{p}} + \frac{1}{V}\sum_{p,p'}\phi(p,p')\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\psi_{-p'}\psi_{p'}\Big]\overset{*}{\psi}_{p_1}\prod_{k\neq p_1}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle =$$

$$= \langle 0|\psi_{p_2}\prod_{k'\neq p_2}(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'})\Big[\varepsilon_{p_1}\overset{*}{\psi}_{p_1}\prod_{k\neq p_1}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle +$$

$$+ \sum_{p\neq p_1} 2\varepsilon_p v_p \overset{*}{\psi}_{p_1}\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\prod_{k\neq p_1\neq p}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle\Big] + \frac{1}{V}\sum_{\substack{p\neq p_2\\ p'\neq p_1}}\phi(p,p')v_p v_{p'} \times$$

$$\times\langle 0|\psi_{p_2}\prod_{k'\neq p_2\neq p}(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'})\overset{*}{\psi}_{p_1}\prod_{k\neq p_1\neq p'}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle =$$

$$= \langle 0|\psi_{p_2}v_{p_1}\psi_{-p_1}\prod_{k'\neq p_2\neq p_1}(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'})\varepsilon_{p_1}\prod_{k\neq p_1}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle +$$

$$+\langle 0|\psi_{p_2}\prod_{k'\neq p_2\neq p_1}(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'})v_{p_1}\psi_{-p_1}\sum_{p\neq p_1}2\varepsilon_p v_p\overset{*}{\psi}_p\overset{*}{\psi}_{-p}\times$$

$$\times\prod_{k\neq p_1\neq p}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle + \frac{1}{V}\sum_{\substack{p\neq p_2\\ p'\neq p_1}}\phi(p,p')v_p v_{p'}\langle 0|\psi_{p_2}\times$$

$$\times\prod_{k'\neq p_2\neq p_1\neq p}(u_{k'} + v_{k'}\overset{*}{\psi}_{k'}\psi_{-k'})v_{p_1}\psi_{-p_1}\prod_{k\neq p_1\neq p'}(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle = 0 \quad (3.11)$$

because $\psi_{-p_1}\prod_{k\neq p_1}(u_k + v_k)\overset{*}{\psi}_k\overset{*}{\psi}_{-k})|0\rangle = 0$. Using the same arguments one can prove that $(\overset{*}{\alpha}_{p_2}\phi_0^a, \overset{*}{\alpha}_{p_1}\phi_0^a) = 0$ if $p_1 \neq p_2$.

Note that the average (3.11) is also equal to zero if some $u_{k'}, v_{k'}$ and u_k, v_k are replaced by $\tilde{u}_{k'}, \tilde{v}_{k'}, \tilde{u}_k, \tilde{v}_k$. This means that the equality (3.11) is true if one consider the following excited states

$$\overset{*}{\alpha}_{p_2}\prod_{i=1}^{m_1}\overset{*}{\alpha}_{p'_i}\alpha_{p'_{-i}}\phi_0^a, \qquad \overset{*}{\alpha}_{p_1}\prod_{j=2}^{m^2}\overset{*}{\alpha}_{p_i}\overset{*}{\alpha}_{-p_i}\phi_0^a$$

with $m_1 \geq 0$, $m_2 \geq 0$.

If is obvious that the equality is still true if instead of $\overset{*}{\alpha}_{p_2}$ and $\overset{*}{\alpha}_{p_1}$ one puts some products of operators of creations $\prod_{i=1}^{n_1}\overset{*}{\alpha}_{p_i}$ and $\prod_{j=1}^{n_2}\overset{*}{\alpha}_{p_j}$ and the sets $(p)_{n_1}$ and $(p)_{n_2}$ do not coincide.

Calculate the following average with $p_1 \neq p_2$

$$\left(\overset{*}{\alpha}_{p_2}\overset{*}{\alpha}_{-p_2}\phi_0^a, H_\Lambda\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\phi_0^a\right) =$$

$$= \langle 0|(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2})\prod_{k'\neq p_2}(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'})\times$$

$$\times(H_{0,\Lambda} + H_{I,\Lambda})\left(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\right)\prod_{k\neq p_1}\left(u_k + v_k\overset{*}{\psi}_k\overset{*}{\psi}_{-k}\right)|0\rangle .$$

Show first that the term with $H_{0,\Lambda}$ is equal to zero. It is equal to the following expression

$$\langle 0|(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2}) \prod_{k'\neq p_2} (u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}) \times$$

$$\times\Big\{ 2\varepsilon_{p_1} u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} \prod_{k\neq p_1} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) +$$

$$+ \sum_{p\neq p_1} \Big(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\Big) 2\varepsilon_p v_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \prod_{k\neq p_1\neq p} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big)\Big\}|0\rangle =$$

$$= \langle 0|(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2}) \prod_{k'\neq p_2} (u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}) \times$$

$$\times\Big\{ \Big(u_{p_2} + v_{p_2}\overset{*}{\psi}_{p_2}\overset{*}{\psi}_{-p_2}\Big) 2\varepsilon_{p_1} u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} \prod_{k\neq p_1\neq p_2} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) +$$

$$+ \sum_{p\neq p_1\neq p_2} \Big(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\Big) 2\varepsilon_p v_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \Big(u_{p_2} + v_{p_2}\overset{*}{\psi}_{p_2}\overset{*}{\psi}_{-p_2}\Big) \times$$

$$\times \prod_{k\neq p_1\neq p_2} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big)\Big\}|0\rangle + \langle 0|(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2}) \times$$

$$\times \prod_{k'\neq p_2} (u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}) 2\varepsilon_{p_2} v_{p_2} \overset{*}{\psi}_{p_2} \overset{*}{\psi}_{-p_2} (u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}) \times$$

$$\times \prod_{k\neq p_1\neq p_2} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big)|0\rangle = 0. \qquad (\text{:}$$

The first two terms are equal to zero because

$$\langle 0|\Big(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2}\Big)\Big(u_{p_2} + v_{p_2}\overset{*}{\psi}_{p_2}\psi_{-p_2}\Big)|0\rangle = 0,$$

and the last third term is equal to zero because

$$\langle 0|\Big(u_{p_1} + v_{p_1}\psi_{-p_1}\psi_{p_1}\Big)\Big(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\Big)|0\rangle = 0.$$

Now, for the sake of simplicity we put $\phi(p, p') = w(p)w(p')$, and calculate

$$\Big(\overset{*}{\alpha}_{p_2}\overset{*}{\alpha}_{-p_2}\phi_0^a, H_{I,\Lambda}\overset{*}{\alpha}_{p_1}\overset{*}{\alpha}_{-p_1}\phi_0^a\Big) =$$

$$= \langle 0|\Big(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2}\Big) \prod_{k'\neq p_2} \Big(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\Big) \times$$

$$\times \frac{1}{V}\sum_{p,p'} w_p w_{p'} \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'}\psi_{p'} \Big(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\Big) \prod_{k\neq p_1} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big)|0\rangle =$$

$$= \langle 0|\Big[u_{p_2}\frac{1}{V} w_{p_2} \prod_{k'\neq p_2} \Big(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\Big) +$$

$$+ \frac{1}{V}\sum_{p\neq p_2} w_p v_p \prod_{k'\neq p_2\neq p} \Big(u_{k'} + v_{k'}\psi_{-k'}\psi_{k'}\Big)(u_{p_2}\psi_{-p_2}\psi_{p_2} - v_{p_2})\Big] \times$$

$$\times\Big[w_{p_1} u_{p_1} \prod_{k\neq p_1} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big) +$$

$$+ \Big(u_{p_1}\overset{*}{\psi}_{p_1}\overset{*}{\psi}_{-p_1} - v_{p_1}\Big) \sum_{p'\neq p_1} w_{p'} v_{p'} \prod_{k\neq p_1\neq p'} \Big(u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k}\Big)\Big]|0\rangle =$$

$$= \frac{1}{V} w_{p_1} w_{p_2} u_{p_1}^2 u_{p_2}^2 + \frac{1}{V} w_{p_1} w_{p_2} v_{p_1}^2 v_{p_2}^2. \tag{3.13}$$

Note that we used the same equality as in (3.12).

Finally, we have,

$$\left(\overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \phi_0^a\right) = \frac{1}{V} w_{p_1} w_{p_2} u_{p_1}^2 u_{p_2}^2 + \frac{1}{V} w_{p_1} w_{p_2} v_{p_1}^2 v_{p_2}^2.$$

This means that $\overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \phi_0^a$, $H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \phi_0^a$ are asymptotically orthogonal as $V \to \infty$.

Show that the states ϕ_0^a, $H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \phi_0^a$ are also asymptotically orthogonal as $V \to \infty$. We have

$$(\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \phi_0^a) = \langle 0| \prod_{k'} (u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}) \times$$

$$\times \left[\sum_{\bar{p}} \overset{*}{\psi}_{\bar{p}} \varepsilon_p \psi_{\bar{p}} + \frac{1}{V} \sum_{p,p'} w_p w_{p'} \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'} \psi_{p'} \right] \times$$

$$\times (u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1})(u_{p_2} \overset{*}{\psi}_{p_2} \overset{*}{\psi}_{-p_2} - v_{p_2}) \prod_{k \neq p_1 \neq p_2} (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle =$$

$$= -\frac{1}{V} (w_{p_2} v_{p_2}^2 w_{p_1} u_{p_1}^2 + w_{p_2} u_{p_2}^2 w_{p_1} v_{p_1}^2). \tag{3.14}$$

We have omited a calculation analogic to (3.13), use that contribution of $H_{0,\Lambda}$ is zero, and the equality

$$\langle 0| (v_{p_i} \psi_{-p_i} \psi_{p_i} + u_{p_i})(u_{p_i} \overset{*}{\psi}_{p_i} \psi_{-p_i} - v_{p_i})|0\rangle = 0.$$

And at last consider the states $\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \phi_0^a$. We have

$$(\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \phi_0^a) = \langle 0| \prod_{k'} (u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}) \times$$

$$\times \Big[2u_{p_1} \varepsilon_{p_1} \overset{*}{\psi}_{p_i} \psi_{-p_i} \prod_{k \neq p_1} (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle +$$

$$+ \sum_{p \neq p_1} 2v_p \varepsilon_p \overset{*}{\psi}_p \overset{*}{\psi}_{-p} (u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1}) \prod_{p \neq p_1} (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle +$$

$$+ \frac{1}{V} \sum_{p,p' \neq p_1} \phi(p,p') \overset{*}{\psi}_p \overset{*}{\psi}_{-p} v_{p'} (u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1}) \prod_{p \neq p_1} (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle +$$

$$+ \frac{1}{V} \sum_p \phi(p, p_1) \overset{*}{\psi}_p \overset{*}{\psi}_{-p} u_{p_1} \prod_{k \neq p_1} (u_k + v_k \overset{*}{\psi}_k \overset{*}{\psi}_{-k})|0\rangle \Big] =$$

$$= 2\varepsilon_{p_1} u_{p_1} v_{p_1} + \frac{1}{V} \sum_{p' \neq p_1} \phi(p_1, p')(-v_{p_1}) v_{p'} v_{p_1} u_{p'} +$$

$$+ \frac{1}{V} \sum_p \phi(p, p_1) u_{p_1} v_p u_{p_1} u_p. \tag{3.15}$$

If follows from (2.14) that

$$\lim_{V \to \infty} \frac{1}{V} (\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \phi_0^a) = 0. \tag{3.16}$$

Consider states $\prod_{j=1}^{m_1} \overset{*}{\alpha}_{p_j^1} \overset{*}{\alpha}_{-p_j^1} \phi_0^a$, $\prod_{j=1}^{m_2} \overset{*}{\alpha}_{p_j^2} \overset{*}{\alpha}_{-p_j^2} \phi_0^a$ where some $p_{j_0}^1$ coincide with $p_{j_0}^2$. We connect the pairs with p_{j_0} to ϕ_0^a, as a result, one obtains a new state ϕ_0^a in which instead of $u_{p_{j_0}^1}$, $v_{p_{j_0}^1}$ one puts $\tilde{u}_{p_{j_0}^1} = -v_{p_{j_0}^1}$, $\tilde{v}_{p_{j_0}^1} = u_{p_{j_0}^1}$. Consider the states

$$\prod_{j=1, j\neq j_0}^{m_1} \overset{*}{\alpha}_{p_j{}^1} \overset{*}{\alpha}_{-p_j{}^1} \tilde{\phi}_0^a, \qquad \prod_{j=1, j\neq j_0}^{m_2} \overset{*}{\alpha}_{p_j^2} \overset{*}{\alpha}_{-p_j^2} \tilde{\phi}_0^a$$

with different $(p_j^1)_{j=1, j\neq j_0}^{m_1}$, $(p_j^2)_{j=1, j\neq j_0}^{m_2}$. For these states all previous results are true. (Of course, there can be more than one such j_0.)

Show that the average of H_Λ over ϕ_0^a with three or more pairs of operators $\overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1}$, $\overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2}$, $\overset{*}{\alpha}_{p_3} \overset{*}{\alpha}_{-p_3}$ are equal to zero. For example, consider

$$\left(\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \overset{*}{\alpha}_{p_3} \overset{*}{\alpha}_{-p_3} \phi_0^a\right) =$$

$$= \langle 0| \prod_{k'} (u_{k'} + v_{k'} \psi_{-k'} \psi_{k'}) \Big[\sum_{\bar{p}} \varepsilon_p \overset{*}{\psi}_{\bar{p}} \psi_{\bar{p}} + \frac{1}{V} \sum_{p,p'} w_p w_{p'} \overset{*}{\psi}_p \overset{*}{\psi}_{-p} \psi_{-p'} \psi_{p'} \Big] \times$$

$$\times \left(u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1}\right)\left(u_{p_2} \overset{*}{\psi}_{p_2} \overset{*}{\psi}_{-p_2} - v_{p_2}\right)\left(u_{p_3} \overset{*}{\psi}_{p_3} \overset{*}{\psi}_{-p_3} - v_{p_3}\right) \times$$

$$\times \prod_{k\neq p_1 \neq p_2 \neq p_3} \left(u_k + v_k \overset{*}{\psi}_k \psi_{-k}\right) |0\rangle. \tag{3.17}$$

Note that $H_{0,\Lambda}$ can act and change no more than one of the operators $(u_{p_i} \overset{*}{\psi}_{p_i} \overset{*}{\psi}_{-p_i} - - v_{p_i})$, $i = 1, 2, 3$. The operator H_Λ can act and change no more than two of the operators $(u_{p_i} \overset{*}{\psi}_{p_i} \overset{*}{\psi}_{-p_i} - v_{p_i})$, $i = 1, 2, 3$. Then one of the unchanged operators, say, $\left(u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1}\right)$ together with the operator $(u_{p_1} + v_{p_1} \psi_{-p_1} \psi_{p_1})$ on the left-hand side of (3.17) will be equal to zero according to the identity

$$\langle 0| \left(u_{p_1} + v_{p_1} \psi_{-p_1} \psi_{p_1}\right) \left(u_{p_1} \overset{*}{\psi}_{p_1} \overset{*}{\psi}_{-p_1} - v_{p_1}\right) |0\rangle = 0,$$

and we have

$$\left(\phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \overset{*}{\alpha}_{p_3} \overset{*}{\alpha}_{-p_3} \phi_0^a\right) = 0. \tag{3.18}$$

The same arguments give us that

$$\left(\overset{*}{\alpha}_{p_3} \overset{*}{\alpha}_{-p_3} \phi_0^a, H_\Lambda \overset{*}{\alpha}_{p_1} \overset{*}{\alpha}_{-p_1} \overset{*}{\alpha}_{p_2} \overset{*}{\alpha}_{-p_2} \phi_0^a\right) = 0. \tag{3.18$'$}$$

Recall that equalities (3.18), (3.18′) is still true if on the left- and right-hand side of it the products of the operators of creations $\prod_{i=1}^{n_1'} \overset{*}{\alpha}_{p_i}$, $\prod_{j=1}^{n_2'} \overset{*}{\alpha}_{p_j}$ with different $(p_i)_{i=1}^{n_1}$, $(p_j)_{j=1}^{n_2}$ are present. The proof was given at the very beginning of this section for arbitrary numbers of pairs an the left- and right-hand side of (3.18), (3.18′).

If $n_1 = n_2$ and $(p_i)_{i=1}^{n_1}$ are equal to $(p_j)_{j=1}^{n_2}$ but $m_1 + m_2 \geq 3$ then the orthogonality follows from (3.17)–(3.18′). If $1 < m_1 + m_2 \leq 2$ then these states are asymptotically orthogonal as it follows from (3.13, (3.14). For $m_1 + m_2 = 1$ we have formula (3.16).

Thus we have proved that different orthogonal excited states of the ground state ϕ_0^a are asymptotically orthogonal if the Hamiltonian H_Λ acts on one of them and formulas (3.11)–(3.18′) are true.

3.3. Estimate of average of $H_\Lambda - H_{a,\Lambda}$ over general excited states. Proof of Theorem **3.** For arbitrary fixed n, m define vectors

$$\phi_1^{n,m} = \sum_{k,l} c_{kl} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^k} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^l} \overset{*}{\alpha}_{-p_j^l} \phi_0^a,$$

$$\phi_2^{n,m} = \sum_{r,s} c'_{rs} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^r} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^s} \overset{*}{\alpha}_{-p_j^s} \phi_0^a$$

and calculate the following average using (3.9) and (3.11)–(3.18′)

$$\frac{1}{V}(\phi_1^{n,m}(H_\Lambda - H_{a,\Lambda})\phi_2^{n,m}) =$$

$$= \frac{1}{V}\Bigg(\sum_{k,l} c_{kl} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^k} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^l} \overset{*}{\alpha}_{-p_j^l} \phi_0^a, (H_\Lambda - H_{a,\Lambda}) \times$$

$$\times \sum_{r,s} c'_{rs} \prod_{i=1}^{n} \overset{*}{\alpha}_{p_i^r} \prod_{j=n+1}^{n+m} \overset{*}{\alpha}_{p_j^s} \overset{*}{\alpha}_{-p_j^s} \phi_0^a \Bigg) =$$

$$= \frac{1}{V} \sum_{k,l} \bar{c}_{kl} c'_{kl} \Bigg[2\frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+ \frac{1}{V} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} +$$

$$+ \frac{1}{V} \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \phi(p_i, p_j) u_{p_i} v_{p_i} u_{p_j} v_{p_j} + \frac{1}{V} \sum_{p \neq (p)_n} \phi(p,p) \tilde{v}_p^4 \Bigg] +$$

$$+ \frac{1}{V} \delta_{m,1} \sum_{k,l} \sum_{r,s} \bar{c}_{kl} c'_{rs} \delta_{(p^k)_n (p^r)_n} \frac{1}{V} \phi(p_1, p_2)(u_{p_1^l}^2 u_{p_2^s}^2 + v_{p_1^l}^2 v_{p_2^s}^2) = I. \quad (3.19)$$

(If some $p_{j_0}^l$ coincide with $p_{j_0}^s$ then, as in Subsection 3.2, we have to consider $\tilde{\phi}_0^a$ instead of ϕ_0^a and $(p_j^i)_{j=1, j\neq j_0}^m$ do not coincide with $(p_j^s)_{j=1, j\neq j_0}^m$.)

We omited index k for p_i, p_j with $1 \le i \le n$, $1 \le j \le n$ and index l for p_i, p_j with $n+1 \le i \le n+m$, $n+1 \le j \le n+m$ in the first term.

In proving (3.19) we used (3.8), (3.9) and (3.11)–(3.13).

Now estimate I, for the case $\phi(p,p') = w_p w_{p'}$, $|w_p| < w$. From (3.19) one obtains

$$|I| \le \frac{1}{V^2} \Bigg| \sum_{k,l} \bar{c}_{kl} c'_{kl} \Bigg| \Bigg[2w^2(n+2m)^2 + \frac{1}{V^2} \sum_p w_p^2 \Bigg] +$$

$$+ \frac{1}{V^2} \delta_{m,1} \Bigg| \sum_{k,l} \sum_{r,s} \bar{c}_{kl} c'_{rs} \Bigg| 2w^2 \le$$

$$\le \frac{1}{V^2} \Bigg[2w^2(n+2m)^2 + \frac{1}{V^2} \sum_p w_p^2 \Bigg] \Bigg\{ \sum_{k,l} |c_{kl}|^2 \Bigg\}^{\frac{1}{2}} \Bigg\{ \sum_{k,l} |c'_{kl}|^2 \Bigg\}^{\frac{1}{2}} +$$

$$+ \frac{1}{V^2} \delta_{m,1} 2w^2 \sum_{k,l} |c_{kl}| \sum_{r,s} |c'_{rs}|. \quad (3.20)$$

It follows from (3.20) that

$$\lim_{V\to\infty} I = 0$$

if

$$\sum_{k,l}|c_{kl}|^2<\infty,\quad \sum_{k,l}|c'_{kl}|^2<\infty,\quad \sum_{k,l}|c_{kl}|<\infty,\quad \sum_{r,s}|c'_{rs}|<\infty$$

for arbitrary finite n and m and even for infinite n and m such that $\lim\limits_{V\to\infty}\dfrac{(n+2m)^2}{V^2}= = 0$. $\Big($For general potential one has to put in (3.20) $\sup\limits_{(p,p')}|\phi(p,p')|$ instead of w^2 and $\dfrac{1}{V^2}\sum\limits_p|\phi(p,p)|$ instead of $\dfrac{1}{V^2}\sum w_p^2.\Big)$

Remark. Suppose that $\phi(p,p)=0$. It is not restricted condition imposed on potential $\phi(p,p')$. Indeed, the integral $\int\phi(p,p')dpdp'$ does not change if $\phi(p,p)=0$ because the hyperplane $p=p'$ is of lower dimension in $\mathcal{R}^3\times\mathcal{R}^3$. For such potential the term $\dfrac{1}{V}\sum\limits_{p\neq(p)_n}\phi(p,p)\tilde{v}_p^4$ in (3.19) is equal to zero. The rest of terms in $(\phi_1,(H_\Lambda-H_{a,\Lambda})\phi_2)$ tend to zero as $V\to\infty$ for arbitrary finite m and n even without the factor $\dfrac{1}{V}$.

According to (3.19), (3.20) we have for finite m and n

$$\lim_{V\to\infty}\left(\phi_1^{n,m},(H_\Lambda-H_{a,\Lambda})\phi_2^{n,m}\right)=\lim_{V\to\infty}\left(\sum_{k,l}c_{kl}\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i^k}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j^l}\overset{*}{\alpha}_{-p_j^l}\phi_0^a,\right.$$
$$\left.(H_\Lambda-H_{a,\Lambda})\sum_{r,s}c'_{rs}\prod_{i=1}^{n}\overset{*}{\alpha}_{p_i^r}\prod_{j=n+1}^{n+m}\overset{*}{\alpha}_{p_j^s}\overset{*}{\alpha}_{-p_j^s}\phi_0^a\right)=0. \tag{3.21}$$

Note that (3.21) is still true if n and m tend to infinity together with V in such a way that $\lim\limits_{V\to\infty}\dfrac{(n+2m)^2}{V}=0$.

In (3.19) we considered states with equal number n of the operators $\overset{*}{\alpha}_{p_i}$ and equal number m of the pairs of operators $\overset{*}{\alpha}_{p_j}\overset{*}{\alpha}_{-p_j}$. Now consider the following states

$$\phi_1^{n_1,m_1}=\sum_{k,l}c_{kl}\prod_{i=1}^{n_1}\overset{*}{\alpha}_{p_i^k}\prod_{j=n_1+1}^{n_1+m_1}\overset{*}{\alpha}_{p_j^l}\overset{*}{\alpha}_{-p_j^l}\phi_0^a,$$
$$\phi_2^{n_2,m_2}=\sum_{r,s}c'_{r,s}\prod_{i=1}^{n_2}\overset{*}{\alpha}_{p_i^r}\prod_{j=n_2+1}^{n_2+m_2}\overset{*}{\alpha}_{p_j^s}\overset{*}{\alpha}_{-p_j^s}\phi_0^a \tag{3.22}$$

with $n_1\neq n_2$, $m_1\neq m_2$, or $n_1\neq n_2$, $m_1=m_2$, or $n_1=n_2, m_1\neq m_2$, $m_1+m_2\geq$ ≥ 3, and $\sum\limits_{k,l}|c_{kl}|^2<\infty$, $\sum\limits_{r,s}|c'_{rs}|^2<\infty$.

The average of $H_\Lambda-H_{a,\Lambda}$ over these states are equal to zero in the limit $V\to\infty$. Indeed, all the terms in the states $\phi_1^{n_1,m_1}$ and $\phi_2^{n_2,m_2}$ are orthogonal eigenvectors of $H_{a,\Lambda}$ and therefore

$$(\phi_1^{n_1,m_1},H_{a,\Lambda}\phi_2^{n_2,m_2})=0.$$

We have also that

$$(\phi_1^{n_1,m_1},H_\Lambda\phi_2^{n_2,m_2})=0.$$

The last equality has been proved in Subsection 3.2.

In the case $n_1 = n_2$, $(p)_{n_1} = (p)_{n_2}, m_1 + m_2 = 2$ we have

$$\lim_{V \to \infty} (\phi_1^{n_1, m_1}, H_\Lambda \phi_2^{n_2, m_2}) = 0$$

according to (3.13)–(3.14).

Thus for general states $\phi_1^{n_1, m_1}$ and $\phi_2^{n_2, m_2}$ (3.22) with $1 < m_1 + m_2$ we have

$$\lim_{V \to \infty} \left(\phi_1^{n_1, m_1}, (H_\Lambda - H_{a,\Lambda}) \phi_2^{n_2, m_2} \right) = 0.$$

If in $\phi_1^{n_1, m_1}$ or $\phi_2^{n_2, m_2}$ one has $m_1 + m_2 = 1$, then, according to (3.16)

$$\lim_{V \to \infty} \frac{1}{V} \left(\phi_1^{n_1, m_1}, (H_\Lambda - H_{a,\Lambda}) \phi_2^{n_2, m_2} \right) = 0.$$

It is easy to generalize above obtained results for linear combination of states $\phi_1^{n_1, m_1}$ and $\phi_2^{n_2, m_2}$.

1. *Bardeen J., Cooper L. N., Schrieffer J. R.* Theory of superconductivity // Phys. Rev. – 1957. – **108**. – P. 1175–1204.
2. *Bogolyubov N. N.* On the model Hamiltonian in the theory of superconductivity // Selected paper of N. N. Bogolyubov. – Kiev: Nauk. Dumka, 1970. – Vol. 3. – P. 110–173.
3. *Schrieffer J. R.* Theory of superconductivity. – Moscow: Nauka, 1970. – 312 p. (in Russian).
4. *Bardeen J., Rickayzen G.* Ground-state energy and Green's function for reduced Hamiltonian for superconductivity // Phys. Rev. – 1960. – **118**. – P. 936–937.
5. *Matias D. C., Lieb E.* Exact wave functions in superconductivity // J. Math. Phys. – 1961. – **2**. – P. 600–602.
6. *Muheschlegel B.* Asymptotic expansion of the Bardeen–Cooper–Schrieffer partition function by means of the functional method // J. Math. Phys. – 1962. – **3**. – P. 522–530.
7. *Petrina D. Ya.* Spectrum and states of the BCS Hamiltonian in a finite domain. I. Spectrum // Ukr. Math. J. – 2000. – **52**, № 5. – P. 667–690.
8. *Petrina D. Ya.* Spectrum and states of the BCS Hamiltonian in a finite domain. II. Spectra of excitations // Ibid. – 2001. – **53**, № 12. – P. 1290–1315.
9. *Petrina D. Ya.* Spectrum and states of the BCS Hamiltonian in a finite domain. III. The BCS Hamiltonian with mean-field interaction // Ibid. – 2002. – **54**, № 11. – P. 1486–1504.
10. *Petrina D. Ya.* Model BCS Hamiltonian and approximating Hamiltonian for a infinite volume. IV. Two branches of their common spectra and states // Ibid. – 2003. – **55**, № 2. – P. 174–197.
11. *Petrina D. Ya.* Mathematical foundation of quantum statistical mechanics. Continuous system. – Dordrecht: Kluwer, 1995. – 444 p.

Received 10.06.2003