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BERNSTEIN-TYPE THEOREMS AND UNIQUENESS THEOREMS

ТЕОРЕМИ ТИПУ БЕРНШТЕЙНА ТА ТЕОРЕМИ ПРО ЄДИНІСТЬ

Let f be an entire function of finite type with respect to finite order ρ in $\mathbb{C}^n$ and let $\mathbb{E}$ be a subset of an open cone in some n-dimensional subspace $\mathbb{R}^{2n}$ $(=\mathbb{C}^n)$ (the smaller ρ, the more sparse $\mathbb{E}$). We assume that this cone contains a ray $\left\{z = tz^0 \in \mathbb{C}^n : t > 0\right\}$. It is shown that the radial indicator $h_f(z^0)$ of f at any point $z^0 \in \mathbb{C}^n \setminus \{0\}$ may be evaluated in terms of function values at points of the discrete subset $\mathbb{E}$. Moreover, if $f \to 0$ fast enough as $z \to \infty$ over $\mathbb{E}$, this function vanishes identically. To prove these results, some special approximation technique is developed. In the last part of the paper, it is proved that, under certain conditions on ρ and $\mathbb{E}$, which are close to exact conditions, the function f bounded on $\mathbb{E}$ is bounded on the ray.

Нехай f — ціла функція скінченного типу відносно порядку ρ у $\mathbb{C}^n$, $\mathbb{E}$ — підмножина відкритого конуса (чим менше ρ, тим більш розрідженим є $\mathbb{E}$) у деякому n-вимірному підпросторі $\mathbb{R}^{2n}$ $(=\mathbb{C}^n)$. Припускається, що даний конус містить промінь $\left\{z = tz^0 \in \mathbb{C}^n : t > 0\right\}$. Показано, що радіальний індикатор $h_f(z^0)$ функції f у будь-якій точці $z^0 \in \mathbb{C}^n \setminus \{0\}$ можна оцінити через значення функції f у точках дискретної множини $\mathbb{E}$. Крім того, якщо $f \to 0$ досить швидко при $z \to \infty$ на $\mathbb{E}$, то дана функція дорівнює нулю тотожно. Для доведення цих результатів розроблено спеціальну апроксимаційну техніку. В останній частині роботи доведено, що за деяких близьких до точних умов відносно ρ і $\mathbb{E}$ функція f, обмежена на $\mathbb{E}$, буде обмеженою на всьому промені.

1. Introduction. In this paper, the authors present a new uniform approach to the problems of uniqueness, growth characteristics, and Cartwright-type theorems. This approach is based on the approximation of entire functions by other entire functions with "nice" propertiers. This approximation is the core of the present paper if we speak about entire functions, and to present it unshaded, we sometimes consider only the simple version of proven results. To extend our results on functions analytic in cone, we apply another kind of approximation, the approximation of such functions by entire ones found by Keldysh for one-dimensional case and Russakovskii for the general case.

Later on we use the standard notations of multidimensional analysis.

By c and C we denote various constants.

An entire function $f\colon \mathbb{C}^n \to \mathbb{C}$ is called a function of σ-type with respect to order $\rho \in (0, \infty)$ if

$$\limsup_{|z| \to \infty} \frac{\log|f(z)|}{|z|^\rho} = \sigma, \qquad z = (z_1, \ldots, z_n),$$

$$|z|^2 = |z_1|^2 + \ldots + |z_n|^2.$$

The same definition is used for the functions that are analytic in an open cone G provided $z \in G$ as $z \to \infty$. By $H_G(\rho, \sigma)$ we denote the class of all analytic in G functions of type at most σ with respect to order ρ. If $G = \mathbb{C}^n$, we use the notation

$H(\rho, \sigma) = H_G(\rho, \sigma)$; the dependence on n is usually implied. By $H_G(\rho)$ we denote the class

$$\bigcup_{\sigma>0} H_G(\rho, \sigma); \quad H(\rho) = H_n(\rho) = \bigcup_{\sigma>0} H_G(\rho, \sigma).$$

The (radial) indicator of $f \in H(\rho)$ at a point $z^0 \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{C}^n$, is defined as follows:

$$h_f(z^0) = \limsup_{z \to z^0} \limsup_{t \to \infty} \frac{\log|f(tz)|}{|t|^\rho}. \tag{1}$$

If $n = 1$, the first lim sup in (1), which means an upper regularization, may be omitted. Hartogs Theorem implies (see details in [1] and [2]) that definition (1) is

$$h_f(z^0) = \lim_{\delta \downarrow 0} \limsup_{t \to \infty} \frac{\log\left(\sup\left\{|f(z)| : \left|z - tz^0\right| \le t\delta\left|z^0\right|\right\}\right)}{|t|^\rho}.$$

In 1936, V. Bernstein [3] formulated the problem to describe such subsets $\mathbb{E}$ of the ray

$$l(z^0) = \{z = tz^0 \in \mathbb{C} : t > 0\}$$

for those the equality

$$h_f(z^0) = \lim_{t \to \infty} \sup_{tz^0 \in \mathbb{E}} \frac{\log\left|f(tz^0)\right|}{|t|^\rho}$$

holds for all functions $f \in H_1(\rho)$. He also obtained some sufficient conditions on these sets. Later Bernstein's result was strengthened and generalized by Pfluger [4], Levinson [5], Boas [6], Fuchs [7], Levin [8], and Malliavin [9], but all these authors considered only functions of one complex variable.

For $n > 1$ Bernstein's question may be reformulated as follows: Describe such sets $\mathbb{E} \subset \mathbb{C}^n$ that the equality

$$h_f(z^0) = h_{f, \mathbb{E}}(z^0)$$

where

$$h_{f, \mathbb{E}}(z^0) = \lim_{\delta \downarrow 0} \limsup_{t \to \infty} \frac{\log\left(\sup\left\{|f(z)| : z \in \mathbb{E}, \left|z - tz^0\right| \le t\delta\left|z^0\right|\right\}\right)}{|t|^\rho}$$

holds for all $f \in H(\rho)$. Of course, only sufficiently sparse $\mathbb{E}$ is of interest for us. It should be mentioned that $\mathbb{E}$ cannot be a subset of any finite union of $(n-1)$-dimensional complex subspaces and the intersection $\mathbb{E} \cap l(z^0)$ may be empty.

It is easy to derive some multidimensional Bernstein-type results from each of the one-dimensional theorems mentioned above. Let z^0 be a point of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{C}^n$, and let $\mathbb{F}$ be a subset of $\mathbb{S}^{n-1}$ not thin (in the sense of the pluripotential theory) at z^0. Assume that $\mathbb{E}$ is such a set in $\mathbb{C}^n$ that the intersection $\mathbb{E} \cap l(\omega)$, where

$$l(\omega) = \{z = t\omega \in \mathbb{C}^n : t > 0\},$$

satisfies the conditions of one of the one-dimensional Bernstein-type theorems for all $\omega \in \mathbb{F}$ and given ρ. Then values of any $f \in H(\rho)$ at points of $\mathbb{E}$ uniquely define $h_f(z^0)$. However, $\mathbb{E}$ is rather massive in this case.

Another straightforward multidimensional generalization is as follows: Given and $\varepsilon > 0$, let $\mathbb{E} = \{z^j\}_{j=1}^{\infty}$ be such a discrete set that the intersection of each ra $l(\omega)$ close enough to $l(z^0)$ (or at least of each ray of a union which central projectic is not thin at $z^0 / |z^0|$ for the spherical indicator not dropping down at z^0) with

$$\bigcup_{j=1}^{\infty} \left\{ z \in \mathbb{C}^n : \left| z - z^{(j)} \right| < \exp\left\{ -\left| z^{(j)} \right|^{\rho+\varepsilon} \right\} \right\},$$

satisfy the conditions of one of the Bernstein-type one-dimensional theorems for th ρ. As for each $f \in H(\rho)$ its derivative does not grow essentially faster than th function itself, by the Mean Value Theorem for Mappings f grows along this unic essentially the same as along $\mathbb{E}$. Therefore, the values of f at the points of uniquely define the value of $h_f(z^0)$. However, $\mathbb{E}$ is rather dense in this case: Its loc density in the neighborhood of tz^0 grows exponentially as $t \to \infty$ at least along som sequence.

In the present paper we'll prove some Bernstein-type multidimensional theorem for relatively sparse subsets $\mathbb{E}$ of subspaces of minimal possible dimension. In sense, these results are the sharpest possible. To formulate them, we need som definitions.

Definition 1. *Let $\mathbb{E}$ and $\mathbb{F}$ be subsets of some m-dimensional subspace* I *of $\mathbb{R}^{2n}$, and let $e_1, \ldots, e_m$ be an orthonormal basis in $\mathbb{L}$ with the correspondi coordinate functions $u_1, \ldots, u_m$. $\mathbb{E}$ is called an asymptotic net (of order 1) for if*

$$\forall u \in \mathbb{F} \ \exists w \in \mathbb{E}: |u - w| \le \varepsilon(|u|)$$

for some function $\varepsilon(R)$, $R \in \mathbb{R}_+$, monotonically decreasing to 0 as $R \to \infty$.] *is an asymptotic net order $\rho \in (0, \infty)$ for $\mathbb{F}$ if the image of $\mathbb{E}$ under the map*

$$u_j \to |u_j|^{\rho-1} u_j, \quad j = 1, \ldots, m, \qquad (2$$

is an asymptotic net (of order 1) for the image of $\mathbb{F}$ under this map.

An asymptotic net of any order may be discrete. $\mathbb{E}$ is an asymptotic net of order for $\mathbb{F}$ if, and only if, there exists such a monotonically desreasing to zero as $R \to$ function $\varepsilon(R)$, $R \in \mathbb{R}_+$, that

$$(\forall u \in \mathbb{F}: |u| > 1) \ \exists w \in \mathbb{E}: |u - w| \le |u|^{1-\rho} \varepsilon(|u|).$$

Definition 2. *Let $\mathbb{F}$ be a set in a normalized space $\mathbb{L}$, and let $\omega > 0$. point $z \in \mathbb{F} \setminus \{0\}$ is called ω-embedded in $\mathbb{F}$ if the ball $\{ y \in \mathbb{L} : \|y - z\| < \omega \|z\| \} \subset \mathbb{F}$. By $\mathbb{F}_\omega$ we denote the subset of ω-embedded points of $\mathbb{F}$.*

Definition 3. *A measurable set M of a ray $l(z^0) = \{z = tz^0 \in \mathbb{C}^n : t > 0\}$ i called relatively logarithmically dense if*

$$\text{meas}_1 (M \cap [Rz^0, ARz^0]) > \eta R$$

for some $A \in (0, \infty)$, $\eta > 0$, and all sufficiently large R.

It is evident that each relatively logarithmically dense subset of the positive ra $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ is the image of some relatively dense subset of this ray unde the map $x \mapsto \exp x$. A subset e of the ray

$$l(z^0) = \{z = tz^0 \in \mathbb{C}^n : t > 0\}$$

is called *relatively dense* (with respect to Lebesque measure) if

$$\inf\{\operatorname{meas}_1(e \cap [Rz^0, (R+A)z^0]) : R \geq 0\} = \eta$$

for some $A \in (0, \infty)$ and $\eta > 0$.

The following theorem is the main result of the paper.

Theorem 1. *Let $\mathbb{F}$ be such a subset of $\mathbb{R}^n$ that for some $\omega > 0$ the intersection $\mathbb{F}_\omega \cap l(z^0)$ is relatively logarithmically dense. Let $\mathbb{E}$ be an asymptotic net of order ρ for $\mathbb{F}$. Then an equality*

$$h_f(z^0) = \lim_{\delta \downarrow 0} \limsup_{t \to \infty} \frac{\log\sup\left\{|f(y)| : y \in \mathbb{E}, \left|y - tz^0\right| \leq t\delta\left|z^0\right|\right\}}{|t|^\rho} \tag{3}$$

holds for all $f \in H(\rho)$. (If $\mathbb{E}$ does not intersect the corresponding ball, then we define the value of the quotient as $-\infty$.)

***Remark* 1.** Theorem 1 is sharp in the following sense: The set $\mathbb{Z}$ of all integers is an asymptotic net of any order $\rho < 1$, but it is not suitable for evaluating the indicator of elements of $H_1(1)$. For instance, it does not allow us to evaluate the indicator $h_f(1)$ for $f(\zeta) = \sin \pi\zeta$.

For the subclass $H_n(\rho, \sigma)$ of $H_n(\rho)$ a certain Bernstein-type result, with an estimate from above instead of the equality, may be obtained for more sparse sets $\mathbb{E}$. We need some definitions first.

Definition 4. *Let $\mathbb{F}$ and $\mathbb{E}$ be subsets of $\mathbb{R}^n$ and let $\varepsilon > 0$. $\mathbb{E}$ is called ε-net (of order 1) for $\mathbb{F}$ if*

$$\forall z \in \mathbb{F} \ \exists \omega \in \mathbb{E}: |z - \omega| \leq \varepsilon.$$

$\mathbb{E}$ is called an ε-net of order ρ for $\mathbb{F}$ if its image under the map (2) is an ε-net for the image of $\mathbb{F}$.

Given $\varepsilon > 0$, after a suitable correction near the origin, if necessary, each asymptotic net of order ρ turns into an ε-net of this order. Each ε-net of order ρ is an asymptotic net for all smaller orders. On the other hand, $\mathbb{Z}^n$ is a $\sqrt{n}/2$-net for $\mathbb{R}^n$, but it is not an asymptotic net. It is evident to see that for each natural n and each $\rho \in (0, \infty)$ there exist such positive values $c = c(\varepsilon, n, \rho)$ and $C = C(\varepsilon, n, \rho)$ that:

1) for any ε-net $\mathbb{E}$ of order ρ for $\mathbb{F}$

$$(\forall z \in \mathbb{F}: |z| \geq 1) \quad \exists \omega \in \mathbb{E}: |z - \omega| \leq C|z|^{1-\rho};$$

2) if

$$(\forall z \in \mathbb{F}: |z| \geq 1) \quad \exists \omega \in \mathbb{E}: |z - \omega| \leq c|z|^{1-\rho},$$

then $\mathbb{E}$ is an ε-net of order ρ for $\mathbb{F}$.

Theorem 2. *Let $\mathbb{F}$ be the same as in the previous theorem. For each $\sigma \in (0, \infty)$ there exists such a number $\varepsilon_0 > 0$ and such a number $C = C(n, \rho, \sigma)$ that the estimate*

$$h_f(z^0) \le C\left|z^0\right|^\rho \limsup_{|z|\to\infty,\, z\in\mathbb{E}} \frac{\ln|f(z)|}{|z|^\rho}$$

is valid for each ε*-net* $\mathbb{E}$, $\varepsilon < \varepsilon_0$, *of order* ρ *for* $\mathbb{F}$ *and each function* $f \in H(\rho, \sigma)$.

Since the spherical indicator of a nontrivial entire function cannot be $-\infty$ at any point, Theorems 1 and 2 result in the following uniqueness theorem.

Theorem 3. *Let* $\mathbb{F}$ *and* $\mathbb{E}$ *be the same as in Theorem* 1. *If for some* $f \in H(\rho)$

$$h_{f,\mathbb{E}}(z^0) = -\infty, \tag{4}$$

then this function vanishes identically.

Theorem 4. *Let* $\mathbb{F}$ *and* $\mathbb{E}$ *be the same as in Theorem* 2. *If for some* $f \in H(\rho)$

$$\limsup_{|z|\to\infty,\, z\in\mathbb{E}} \frac{\ln|f(z)|}{|z|^\rho} < -\frac{\sigma}{C}$$

where C *is the constant defined in Theorem* 2, *then this function vanishes identically.*

Let us compare and contrast our uniqueness Theorem 3 and Theorem 4 with the results known before. Discrete (real) uniqueness sets were mainly studied for functions of exponential type. These sets were the subject of study in the series of papers by Ronkin (see [10] where there are the history of the question and detailed bibliography). Therefore, we will compare and contrast with the mentioned Ronkin's results only the particular case of exponential growth in Theorems 3 and 4. Like ours, Ronkin's uniqueness sets may be discrete subsets of open cones in $\mathbb{R}^n$ with an arbitrary small opening. The upper density of Ronkin's uniqueness sets is, generally speaking, much larger than ours. From this point of view, our theorems are stronger even for functions of exponential type. On the other hand, Ronkin does not assume any regular density, only the upper. So his theorems are not the particular cases of ours. However, the main distinction of our results from Ronkin's theorems is that we do not suppose that entire functions in questions are 0 on $\mathbb{E}$. This zero condition is essential for Ronkin's method.

The proof of Theorems 1 and 2 are based on some special approximation. A work horse of this approximation is the following lemma that, in the authors' opinion, is of interest on its own:

Lemma 1. *Given* n, ρ, σ, *and* $\Delta > 1$, *there exists such a number* $q_0 \in (1, \infty)$, *that for each triple of* $f \in H(\rho, \sigma)$, $q > q_0$, *and vector* $\xi \in \mathbb{R}^n \setminus \{0\}$ *there exists a function* $\varphi_\xi \in H(1, q^{\rho+1}|\xi|^{\rho-1})$ *with the following properties*:

1) $x \in \mathbb{R}^n, \quad |x-\xi| \le \dfrac{|\xi|}{\sqrt{q}} \Rightarrow \left|\varphi_\xi(x)\right| \le |f(x)| + C\exp\left\{-\Delta|\xi|^\rho\right\}$;

2) $x \in \mathbb{R}^n, \quad |x-\xi| \le \dfrac{|\xi|}{(2q)} \Rightarrow \left|\varphi_\xi(x)\right| \ge \dfrac{|f(x)|}{4} - C\exp\left\{-\Delta|\xi|^\rho\right\}$;

3) $x \in \mathbb{R}^n, \quad |x-\xi| \ge \dfrac{|\xi|}{\sqrt{q}} \Rightarrow \left|\varphi_\xi(x)\right| \le C\exp\left\{-\Delta|\xi|^\rho\right\}$.

Here $C = C(f) < \infty$ *does not depend on* x *and* ξ.

Note that properties 1 and 3 imply that function φ_ξ is an element of S. Bernstein's class of functions of exponential type not exceeding $q^{\rho+1}|\xi|^{\rho-1}$ that are bounded on $\mathbb{R}^n$. We will presently use this fact.

Theorem 5 (V. Bernstein [3]; see also the first chapter of [8]). *Let g be an analytic function of normal type with respect to finite order ρ in an angle $\alpha < \arg\zeta < \beta$, and let $\Theta \in (\alpha, \beta)$. For arbitrary $\varepsilon > 0$, $\delta > 0$, $0 < \eta < 1$ there exists such a sequence of intervals $(r_k \exp\{i\Theta\},\ r_k(1+\delta)\exp\{i\Theta\})$, $k \in \mathbb{N}$, $r_k \to \infty$, as $k \to \infty$, that an inequality*

$$\log|g(r\exp\{i\Theta\})| > (h_g(\Theta) - \varepsilon)r^{\rho}$$

is valid at all points of each interval $(r_k \exp\{i\Theta\},\ r_k(1+\delta)\exp\{i\Theta\})$ save some exceptional set of Lebesque measure less than $\eta\delta r_k$.

Using in our scheme more precise Bernstein-type theorems mentioned above instead of this theorem, one can easily obtain more sophisticated versions of Theorem 1 and Theorem 2.

The same approximation allows us to get some Cartwright-type results about entire functions bounded on a ray. To formulate them, we need the following definition.

Definition 5. *Let $\mathbb{F}$ be a measurable subset of an open cone $\mathbb{K}$ in some m-dimensional subspace $\mathbb{L} \subset \mathbb{R}^{2n} (= \mathbb{C}^n)$. $\mathbb{F}$ is called relatively dense in $\mathbb{K}$ if for some $L < \infty$ and $\delta > 0$*

$$\inf\{\mathrm{meas}_m\{\mathbb{F} \cap \mathbb{B}_L(y)\} : y \in \mathbb{K}\} = \delta.$$

Here

$$\mathbb{B}_L(y) = \{x \in \mathbb{L} : \|x - y\| \le L\},$$

$\mathbb{F}$ is called ρ-relatively dense in $\mathbb{K}$ if its image under the map (2) *is relatively dense (with respect to m-dimensional Lebesgue measure) in the image of $\mathbb{K}$.*

It is evident that $\mathbb{F}$ is ρ-relatively dense in $\mathbb{K}$ if, and only if, there exist such positive constants L and δ that the inequality

$$\mathrm{meas}_m(\{z \in \mathbb{L} : |z - w| \le L|w|^{1-\rho}\} \cap \mathbb{F}) \ge \delta|w|^{m(1-\rho)}$$

holds for each $w \in \mathbb{K}$, $|w| \ge 1$. L and δ are called *density characteristics* of $\mathbb{F}$.

Theorem 6. *Let $\mathbb{K}$ be an open circular cone with an axis $l(z^0)$ in $\mathbb{R}^n$, let $\mathbb{F}$ be ρ-relatively dense in $\mathbb{K}$, and let $\mathbb{E}$ be an asymptotic net of order ρ for $\mathbb{F}$. Then each function $f \in H(\rho)$ bounded on $\mathbb{E}$ is bounded on $l(z^0)$.*

For each $\sigma \in (0, \infty)$ there exists such a number $\varepsilon_0 > 0$ that each function $f \in H(\rho, \sigma)$ bounded on some ε-net $\mathbb{E}$, $\varepsilon < \varepsilon_0$, of order ρ for $\mathbb{F}$ is bounded on $l(z^0)$.

Remark 2. In the first statement of Theorem 6, any function f in question is bounded on all rays close enough to $l(z^0)$. To obtain the similar statement about the boundedness on the ray $l(z^0)$ only, one should change a circular cone $\mathbb{K}$ to a solid of revolution with the axis $l(z^0)$ and the generatrix controlled by the function $\varepsilon(R)$ defining the asymptotic net.

We'll omit the proof of this statement since it only slightly differs from the proof of Theorem 6.

The following corollary of Theorem 6 concernong functions of exponential type is, in the authors' opinion, interesting on its own.

Corollary 1. *Let $\mathbb{E}$ be an asymptotic net for an open circular cone $\mathbb{K}$ with an* *axis $l(z^0)$ in $\mathbb{R}^n$. Then each function of exponential type bounded on $\mathbb{E}$ is bounded on $l(z^0)$.*

For each $\sigma \in (0, \infty)$ there exists such a number $\varepsilon_0 > 0$ that each function $f \in H_n(1, \sigma)$ bounded on some ε-net, $\varepsilon < \varepsilon_0$, for this cone is bounded on $l(z^0)$.

The second statement of Theorem 6 was known before [11]. Unlike most known Cartwight-type theorems, the boundedness guaranteed by Theorem 6 as well as Corollary 1 is not uniform with respect to functions but individual. However, this is the case, and, generally speaking, there are no uniform estimates with the exception of the trivial case of $\mathrm{Clos}(\mathbb{E}) \supset l(z^0)$. We'll present the corresponding examples at the end of Section 3.

We generalize these theorems considering analytic functions in a cone. To derive the corresponding results, we approximate these function by entire ones with controlled growth. Such approximation for $n = 1$ was obtained by Keldysh [12]; using $\bar{\partial}$-problem technique, Russakovskii [13] proved the existence of this approximation for the general case. Our analogue of Theorem 1 is as follows.

Theorem 7. *Let $\mathbb{F}$ be such a subset of $\mathbb{R}^n$ that for some $\omega > 0$ the intersection $\mathbb{F}_\omega \cap l(z^0)$ is relatively logarithmically dense. Let $\mathbb{E}$ be an asymptotic net of order ρ for $\mathbb{F}$. Then an equality* (3) *holds for all $f \in H_G(\rho)$ provided the open cone $G \subset \mathbb{C}^n$ contains the ray $l(z^0)$.*

Speaking about uniqueness theorems, note that while the regularized indicator of an entire function of finite type σ with respect to ρ is bounded from below by $-\sigma$, it can be even equal to $-\infty$ identically for the case of a nontrivial function which is analytic only in a cone. For this reason, any analogue of Theorem 3 for functions which are analytic in a cone needs some restriction on the smallness of the cone's opening.

Theorem 8. *Let $\mathbb{F}$ be such a subset of $\mathbb{R}^n$ that for some $\omega > 0$*

$$\mathbb{F}_\omega \cap l(\mathbf{1}), \qquad l(\mathbf{1}) = \{z = t\mathbf{1} : t > 0\}, \qquad \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$$

is relatively logarithmically dense, let $\mathbb{E} \in \mathbb{R}^n$ be an asymptotic net of order $\rho > 1$ for $\mathbb{F}$, and let

$$G = \left\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : -\frac{\pi}{\tau} < \arg z_j < \frac{\pi}{\tau}, j = 1, \ldots, n\right\},$$
$$1 < \tau < \rho.$$

If equality (4) *with $z^0 = \mathbf{1}$ holds for some function $f \in H_G(\rho)$, then this function vanishes identically.*

The analogues of Theorem 2 and Theorem 4 for functions which are analytic in a cone were obtained in the joint paper of Russakovskii and one of the authors [14].

In the second section Lemma 1 is proved. Section 3 contains the proofs of Theorems [1 – 5]. The generalization will be considered in the fourth, last, section.

2. Approximation lemma. We introduce some notations first. Let $z = (z_1, \ldots$ $\ldots, z_n) \in \mathbb{C}^n$, and let $k = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$ be a multiindex. Then, by definition,

$$z^k = z^{k_1} \ldots z^{k_n}, \qquad k! = k_1! \ldots k_n!$$

$$D^k f(z) = \frac{\partial^{|k|_1} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}(z).$$

The proof of Lemma 1 is based on the property of Taylor polynomials of entire functions of finite order that contains in the following Lemma 2 (see [15]).

Lemma 2. *Given* $n \in \mathbb{N}$, $\rho \in (0, \infty)$, $\sigma \in (0, \infty)$, $\Theta \in (0, \infty)$, *let a finite number* p *satisfy an inequality*

$$p > e\gamma\rho\sigma\Theta^{\rho} \tag{5}$$

where $\gamma = \max\{1, 2^{\rho-1}\}$. *Then*

$$\forall f \in H(\rho, \sigma) \quad \forall \varepsilon > 0 \quad \exists C < \infty \quad \forall z^0 \in \mathbb{C}^n \quad \forall m > 0:$$

$$\max\left\{\left| f(z) - \sum_{|k|_1 \le pm^{\rho}} \frac{D^k f(z^0)(z-z^0)}{k!}\right| : \left|z - z^0\right| < \Theta m\right\} \le$$

$$\le C\exp\left\{\gamma(\sigma+\varepsilon)\left|z^0\right|^{\rho}\right\}\left(\frac{e\gamma(\sigma+\varepsilon)\Theta^{\rho}}{p}\right)^{pm^{\rho}}$$

and

$$\frac{\left|D^k f(z)\right|}{k!} \le C\exp\left\{\gamma(\sigma+\varepsilon)\left|z^0\right|^{\rho}\right\}\left(\frac{e\gamma(\sigma+\varepsilon)\Theta^{\rho}}{p^{\rho}|k|_1}\right)^{|k|_1/\rho}\prod_{j=1}^{n}\left(\frac{k_j}{|k|_1}\right)^{-k_j/2}$$

for any $k \in (\mathbb{Z}_+)^n \setminus \{0\}$. *Here* $C = C(f, \varepsilon) < \infty$.

***Proof of Lemma* 1.** Without loss of generality, we assume that $\sigma = 1$ — the general case may be reduced to this particular by the map $z \mapsto z/\sigma^{1/\rho}$. Let $f \in H(\rho, 1)$, and let q be a large positive constant — we'll determine its value later.

For $\xi \in \mathbb{R}^n$, $|\xi| > 1$, set $m = (q|\xi|)^{\rho}$. Define

$$\varphi_{\xi}(z) = \frac{1}{2}T_m(z,\xi)\int_{-1/2}^{1/2}\left(\left\{\frac{\sin\left(q\sqrt{\sum_{j=1}^{n}(z_j-\xi_j)^2}\Big/|\xi| - t\right)}{q\sqrt{\sum_{j=1}^{n}(z_j-\xi_j)^2}\Big/|\xi| - t}\right\}^{[m]} + \right.$$

$$\left. + \left\{\frac{\sin\left(q\sqrt{\sum_{j=1}^{n}(z_j-\xi_j)^2}\Big/|\xi| + t\right)}{q\sqrt{\sum_{j=1}^{n}(z_j-\xi_j)^2}\Big/|\xi| + t}\right\}^{[m]}\right) dt \Bigg/ \int_{-1/2}^{1/2}\left\{\frac{\sin\tau}{\tau}\right\}^{[m]} d\tau, \tag{6}$$

where

$$T_m(z,\xi) = \sum_{|k|_1 \le m}\frac{D^k f(\xi)(z-\xi)^k}{k!}$$

is a Taylor polynomial of f. Since the integrand is an entire function of exponential type $[m]q\xi$, our choice of m implies that $\varphi_{\xi} \in (1, q^{\rho+1}|\xi|^{\rho-1})$. Properties 1, 2, and 3 may be verified as follows: According to the mentioned lemma, for all q satisfying the inequality

$$2\gamma + q^{\rho}\log\frac{2e\gamma\rho}{q^{\rho}} < -\Delta, \qquad \Delta > 0$$

(it is the first restriction on the value of q) the following implication is true:

$$|z-\xi| \le |\xi| \Rightarrow |f(z)-T_m(z,\xi)| \le C\exp\{2\gamma|\xi|^\rho\}\left(\frac{2e\gamma\rho}{q^\rho}\right)^m \le$$

$$\le C\exp\{-\Delta|\xi|^\rho\}.$$

Let $\dfrac{\sin x}{x}$ is an even bell-shaped function in neighborhood of 0. Since x is sufficiently close to ξ, the absolute value of the coefficient of T_m in (6) does not exceed 1, the implication

$$|z-\xi| \le |\xi| \Rightarrow |\varphi_\xi(x)| \le |T_m(x,\xi)| \le |f(x)| + C\exp\{-\Delta|\xi|^\rho\}$$

is true. If $|u| \le 1/2$, then

$$\left|\int_{-1/2}^{1/2}\left\{\left[\frac{\sin(u-t)}{u-t}\right]\right\}^{[m]} dt\right| \ge \int_0^{1/2}\left\{\left[\frac{\sin\tau}{\tau}\right]\right\}^{[m]} d\tau = \frac{1}{2}\int_{-1/2}^{1/2}\left\{\left[\frac{\sin\tau}{\tau}\right]\right\}^{[m]} d\tau.$$

Therefore, the inequality

$$|\varphi_\xi(x)| \ge \frac{1}{4}|T_m(x,\xi)| \ge \frac{|f(x)|}{4} - C\exp\{-\Delta|\xi|^\rho\}$$

holds for $|x-\xi| \le |\xi|/(2q)$. So, property 2 is also verified. Let

$$r = |x-\xi| \ge \frac{|\xi|}{\sqrt{q}}.$$

Evaluate, to begin with, $|T_m(x,\xi)|$:

$$|T_m(x,\xi)| \le C\exp\{2\gamma|\xi|^\rho\}\sum_{|k|_1\le m}\left(\frac{2e\gamma\rho}{|k|_1}\right)^{|k|_1/\rho} \times$$

$$\times \max\left\{\frac{|x_1-\xi_1|^{k_1}|x_n-\xi_n|^{k_n}}{\prod_{j=1}^n\left(k_j/|k|_1\right)^{k_j/2}} : |x-\xi| = r\right\} \le$$

$$\le C\exp\{2\gamma|\xi|^\rho\}\left(\frac{r\sqrt{q}}{|\xi|}\right)^m\sum_{\upsilon\le m}\left(\frac{3e\gamma\rho}{\upsilon}\right)^{\upsilon/\rho}\left(\frac{|\xi|}{\sqrt{q}}\right)^\upsilon.$$

For all ξ that are remote enough from the origin the maximal term of the last sum does not exceed $\exp\left\{4\gamma\left(|\xi|/\sqrt{q}\right)^\rho\right\}$. Using this estimate, one can easily verify that the inequality

$$|T_m(x,\xi)| \le C\exp\left\{5\gamma\left(1+q^{-\rho/2}\right)|\xi|^\rho\right\}\left(\frac{\sqrt{q}|x-\xi|}{|\xi|}\right)^m$$

holds for all $\xi\in\mathbb{R}^n$, $|\xi|>1$, such that $|x-\xi| \ge |\xi|/\sqrt{q}$. Therefore,

$$|\varphi_\xi(x)| \le C\exp\{10\gamma|\xi|^\rho\}\left(\frac{\sqrt{q}|x-\xi|}{|\xi|}\right)^m\left(\frac{|\xi|}{q|x-\xi|}\right)^m\left(2\sin\frac{1}{2}\right)^{-m} \le$$

$$\le C\exp\left\{\left(10\gamma - q^\rho\log\left(2\sqrt{q}\sin\frac{1}{2}\right)\right)|\xi|^\rho\right\} \le C\exp\{-\Delta|\xi|^\rho\}$$

for these ξ and x provided that q satisfies the corresponding (second) condition. Lemma 1 is proved.

3. Radial indicator and boundedness of entire functions on a ray. ***Proofs of Theorems* 1 *and* 3.** Without loss of generality, we can assume that

$$z^0 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in \mathbb{R}^n.$$

Later on we will use the following notation: Let $\xi \in \mathbb{R}^n \setminus \{0\}$. By $\mathbb{K}(\xi, r)$ we denote the minimal open cone in $\mathbb{R}^n$ with the vertex at the origin that contains the ball $\{y \in \mathbb{R}^n : |y - \xi| < r|\xi|\}$. We shall prove that

$$h_f(z^0) = h_{f, \mathbb{E}}(z^0).$$

By definition the right-hand side of this equation does not exceed the left-hand side. Hence, to prove Theorem 1, it is enough to verify that $h_f(z^0) \le h_{f, \mathbb{E}}(z^0)$. Let $\Delta = 2\sigma$, and let q in Lemma 1 be so large that for some $\omega > 2/\sqrt{q}$ the intersection of $\mathbb{F}_\omega$ and any ray of the cone $\mathbb{K}(z^0, 1/(2q))$ is relatively logarithmically dense. Let $\xi \in \mathbb{F}_\omega \cap \mathbb{K}(z^0, 1/(2q))$, and let $\varphi_\xi \in H(1, q^{\rho+1}|\xi|^{\rho-1})$ be a function defined in Lemma 1 for these $f, \Delta, q,$ and ξ. According to statements 1 and 3 of this lemma,

$$|x - \xi| \le |\xi|/\sqrt{q} \Rightarrow \left|\varphi_\xi(x)\right| \le |f(x)| + C\exp\left\{-\Delta|\xi|^\rho\right\},$$

and

$$|x - \xi| \ge \frac{|\xi|}{\sqrt{q}} \Rightarrow \left|\varphi_\xi(x)\right| \le C\exp\left\{-\Delta|\xi|^\rho\right\}.$$

Therefore, φ_ξ is bounded on the real hyperplane. Assume, for the sake of simplicity, that the value

$$M_\xi = \sup\{|\varphi_\xi(x)| : x \in \mathbb{R}^n\}$$

is reached by $|\varphi_\xi|$ at some point $\eta \in \mathbb{R}^n$. If $|\eta - \xi| \ge |\xi|/\sqrt{q}$, then

$$M_\xi \le C\exp\{-\Delta|\xi|^\rho\};$$

otherwise there exists such a point $\zeta \in \mathbb{E}$ that $\zeta \in \left\{x \in \mathbb{R}^n : |x - \xi| \le |\xi|/\sqrt{q}\right\}$ and $|\eta - \zeta| \le |\xi|^{1-\rho}\varepsilon(|\xi|)$ for some function $\varepsilon(R)$ that monotonically decreases to 0 as R tends to ∞ and does not depend on ξ and ζ.

In the first case

$$|f(\xi)| \le C\exp\{-\Delta|\xi|^\rho\}$$

by statement 2 of Lemma 1.

To handle the second case, we need the well-known estimate (proved by S. Bernstein) for derivatives of functions of exponential type: Let $g : \mathbb{C}^n \to \mathbb{C}$ be an element of $H_n(1, \sigma)$. Then

$$\sup\left\{|\nabla g(x)| : x \in \mathbb{R}^n\right\} \le \widehat{\sigma}\sup\left\{|g(x)| : x \in \mathbb{R}^n\right\}$$

where

$$\nabla g(x) = \left(\frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_n}(x)\right).$$

According to this estimate, we have

$$M_\xi = |\varphi_\xi(\eta)| \le |\varphi_\xi(\zeta)| + \sup\{|\nabla\varphi_\xi(x)| : x \in \mathbb{R}^n\}|\eta-\zeta| \le$$
$$\le |f(\zeta)| + C\exp\{-\Delta|\xi|^\rho\} + q^{\rho+1}|\xi|^{\rho-1}M_\xi|\xi|^{1-\rho}\varepsilon(|\xi|).$$

If ξ is far enough from the origin — the more remote, the larger is q, this inequality means that

$$M_\xi \le 2|f(\zeta)| + C\exp\{-\Delta|\xi|^\rho\},$$

and, according to statement 2 of Lemma 1,

$$|f(\xi)| \le 8|f(\zeta)| + C\exp\{-\Delta|\xi|^\rho\} \le$$
$$\le 8\sup\left\{|f(y)|: y \in \mathbb{E}\cap\left\{x: |x-\xi| \le |\xi|/\sqrt{q}\right\} + C\exp\left\{-\Delta|\xi|^\rho\right\}\right\}.$$

For $\xi = t\omega$ it means that

$$\lim_{t\to\infty} \sup_{t\omega\in\mathbb{F}_\omega\cap\mathbb{K}(z^0,1/(2q))} \frac{\log|f(t\omega)|}{|t|^\rho} \le$$
$$\le \lim_{t\to\infty} \sup_{t\omega\in\mathbb{F}_\omega\cap\mathbb{K}(z^0,1/(2q))} |t|^{-\rho}\log\Big(8\sup\{|f(y)|: y\in\mathbb{E}, |y-t\omega| \le$$
$$\le t|\omega|/\sqrt{q}\} + C\exp\left\{-\Delta|t|^\rho|\omega|^\rho\right\}\Big).$$

V. Bernstein's theorem allows us to get rid of the restriction that $t\omega \in \mathbb{F}_\omega$ on the left. Besides, the expression on the right does not exceed

$$\alpha(q) \lim_{|\xi|\to\infty} \sup_{\xi\in\mathbb{E}\cap\mathbb{K}(z^0,2/\sqrt{q})} \frac{\log\left(8|f(\xi)| + C\exp\left\{-\Delta|\xi|^\rho\right\}\right)}{|\xi|^\rho}$$

where $\alpha(q) \to 1$ as $q \to \infty$. It means that

$$h_f(z^0) \le \lim_{q\to\infty} \lim_{|\xi|\to\infty} \sup_{|\xi|\in\mathbb{E}\cap\mathbb{K}(z^0,2/\sqrt{q})} \frac{\log\left(8|f(\xi)| + C\exp\left\{-\Delta|\xi|^\rho\right\}\right)}{|\xi|^\rho}.$$

If $h_{f,\mathbb{E}}(z^0) \ge -\sigma$, then $h_f(z^0) \le h_{f,\mathbb{E}}(z^0)$; otherwise $h_f(z^0) < -\sigma$, and $f(z) \equiv$ $\equiv 0$. Theorems 1 and 3 are proved.

***Proofs of Theorems* 2 *and* 4.** Once again, we don't lose much generality assuming that

$$z^0 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in \mathbb{R}^n.$$

Let q in Lemma 1 be so large that for some $\omega > 2/\sqrt{q}$ the intersection of $\mathbb{F}_\omega$ and any ray of the cone $\mathbb{K}(z^0, 1/(2q))$ is relatively logarithmically dense.

Let $\xi \in \mathbb{F}_\omega \cap \mathbb{K}(z^0, 1/(2q))$, and let $\varphi_\xi \in H(1, q^{\rho+1}|\xi|^{\rho-1})$ be the function defined in Lemma 1 for these f, $\Delta = 2\left(1+\sqrt{q}\right)^\rho\sigma$, q, and ξ. Assume that there exists such a point $\eta \in \mathbb{R}^n$ that

$$|\varphi_\xi(\eta)| = M_\xi = \sup\{|\varphi_\xi(x)| : x \in \mathbb{R}^n\}.$$

Let $\zeta \in \mathbb{E}$ be a point at the distance of less than ε from η. Applying just mentioned S. Bernstein's estimate of the derivative of entire function, we obtain

$$M_\xi = |\varphi_\xi(\eta)| \le |\varphi_\xi(\zeta)| + \sup\{|\nabla\varphi_\xi(x)| : x \in \mathbb{R}^n\}|\eta - \zeta| \le$$
$$\le |\varphi_\xi(\zeta)| + Cq^{\rho+1}|\xi|^{\rho-1}M_\xi\varepsilon|\xi|^{-\rho+1}.$$

If $\varepsilon > 0$ is sufficiently small, the second term on the right is smaller than $M_\xi/2$. Therefore,

$$M_\xi \le 2\sup\left\{\left|\varphi_\xi(x)\right| : x \in \mathbb{E} \cap \left\{y \in \mathbb{R}^n : |y - \xi| \le |\xi|/\sqrt{q}\right\}\right\}.$$

According to statement 2 of Lemma 1, we have

$$|f(\xi)| \le 4M_\xi + C\exp\{-\Delta|\xi|^\rho\} \le$$
$$\le 8\sup\left\{|f(x)| : x \in \mathbb{E} \cap \left\{y \in \mathbb{R}^n : |y - \xi| \le |\xi|/\sqrt{q}\right\}\right\} + C\exp\left\{-\Delta|\xi|^\rho\right\}.$$

By Bernstein's theorem it means that

$$h_f(z^0) \le \varlimsup_{|\xi|\to\infty} \sup_{\xi\in\mathbb{K}(z^0, 1/(2q))} \frac{\log|f(\xi)|}{|\xi|^\rho} \le$$
$$\le \varlimsup_{|\xi|\to\infty} \sup_{\xi\in\mathbb{K}(z^0, 1/(2q))} |\xi|^{-\rho}\log\Big(8\sup\{|f(y)| :$$
$$y \in \mathbb{E} \cap \left\{y \in \mathbb{R}^n : |x - \xi| \le |\xi|/\sqrt{q}\right\}\right\} + C\exp\left\{-\Delta|t|^\rho|\xi|^\rho\right\}.$$

If

$$\varlimsup_{|\xi|\to\infty} \sup_{\xi\in\mathbb{E}\cap\mathbb{K}(z^0, 1/(2q))} \frac{\log|f(\xi)|}{|\xi|^\rho} \le -2\left(1 + \frac{1}{\sqrt{q}}\right)^\rho \sigma, \tag{7}$$

then $h_f < -\sigma$, and therefore f vanishes identically. Theorem 4 is proved. For this case Theorem 2 is also proved. If (7) is false, the statement of Theorem 2 is still true. So, Theorem 2 is always true.

***Proof of Theorem* 6.** Let's prove the first statement of the theorem. As usual,

$$z^0 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in \mathbb{R}^n.$$

Assume that $f \in H(\rho)$ is of σ-type and bounded by 1 on $\mathbb{E}$. Let $\Delta = 2\sigma$, let $\xi \in l(z^0)$, and let $q = q(n, \rho, \sigma)$ in Lemma 1 be large enough to guarantee the inclusion

$$B\left(\xi, 2|\xi|/\sqrt{q}\right) = \left\{x \in \mathbb{R}^n : |x - \xi| \le 2|\xi|/\sqrt{q}\right\} \subset \mathbb{K}.$$

Let $\varphi_\xi \in H(1, q^{\rho+1}|\xi|^{\rho-1})$ be the function defined in Lemma 1 for these f, Δ, q, and ξ. The set

$$\mathbb{F}_\xi = \mathrm{Clos}\left(\left(\mathbb{R}^n \setminus B\left(\xi, 2|\xi|/\sqrt{q}\right)\right) \cup \left(B\left(\xi, 2|\xi|/\sqrt{q}\right) \cap \mathbb{F}\right)\right)$$

is relatively dense in $\mathbb{R}^n$ with density characteristics $L|\xi|^{1-\rho}$ and $\delta|\xi|^{n(1-\rho)}$. If $|\xi|$ is large enough, φ_ξ is bounded by 2 on $\mathbb{E}$. It is also bounded on $\mathbb{R}^n$ by some finite value that may a priori depend on ξ. However, it can be chosen independent of ξ. To show it, we need the following result proved by Schaeffer [16] for $n = 1$ and by Levin [17] for the general case (some Schaeffer – Levin-type results for plurisubharmonic and subharmonic functions one can find in [18 – 20]).

Theorem 9 (Schaeffer – Levin). *Let $\mathbb{F}$ be a relatively dense subset of $\mathbb{R}^n$*

with density characteristics L *and* δ. *Then the following inequality*

$$\sup\{|g(x)| : x \in \mathbb{R}^n\} \le \exp\{C\sigma L^{n+1}/\delta\} \sup\{|g(y)| : y \in \mathbb{F}\}$$

where $C < \infty$ *does not depend on* $g, \sigma,$ *and* $\mathbb{F}$ *holds for every* $g \in H_n(1, \sigma)$.

Let

$$M_\xi = \sup\{|\varphi_\xi(x)| : x \in \mathbb{R}^n\}$$

and

$$N_\xi = \sup\{|\varphi_\xi(y)| : y \in \mathbb{F}_\xi\}.$$

According to Levin – Schaeffer Theorem,

$$M_\xi \le \{Cq^{\rho+1}|\xi|^{\rho-1}L^{(n+1)}|\xi|^{(n+1)(1-\rho)}/\delta|\xi|^{n(1-\rho)}\}N_\xi \le CN_\xi$$

where the finite coefficient C on the right does not depend on f and ξ. Assume, for the sake of simplicity, that $|\varphi_\xi(x)|$ reaches the value N_ξ at some point $\eta \in \mathbb{F}_\xi$. Only the case of $|\eta - \xi| \le |\xi|/\sqrt{q}$ is of interest for us. In this case there exists such a point $\zeta \in \mathbb{E}$ that

$$|\zeta - \eta| \le |\xi|^{1-\rho}\varepsilon(|\xi|) \tag{8}$$

where $\varepsilon(R)$ monotonically decreases to 0 as $R \to \infty$. Applying S. Bernstein's estimate once again, we get that

$$M_\xi \le CN_\xi \le C(2 + q^{\rho+1}|\xi|^{\rho-1}M_\xi|\xi|^{1-\rho}\varepsilon(|\xi|)).$$

It means that $M_\xi \le \infty$ for all f of the class and $\xi \in l(\mathbf{1})$ provided that $|\xi|$ is large enough. (In fact, the condition that $\xi \in l(z^0)$ was never used in this argument, and since q may be arbitrarily large, f is bounded on any closed cone embedded in $\mathbb{K}$.) The first statement of Theorem 6 is proved.

To prove the second statement, we should almost literally repeat our preceding reasoning leading to (8). Now we have

$$|\zeta - \xi| < C\varepsilon|\xi|^{1-\rho}$$

instead of (8). It yields

$$M_\xi \le C + Cq^{\rho+1}|\xi|^{\rho-1}M_\xi|\xi|^{1-\rho}\varepsilon.$$

Here C stands for various constants that do not depend on ξ and f. So, if positive ε is small enough, all functions φ_ξ are uniformly bounded. Therefore, f is bounded on the ray $l(z^0)$. Theorem 6 is proved.

***Remark* 3.** As it was mentioned in Introduction, the estimates of entire functions that Theorem 6 guarantees are not uniform with respect to functions f even though there is such a common finite constant that all of them are bounded by it for all $\xi \in l(\mathbf{1})$ with $|\xi| \ge R_f$, $R_f < \infty$. The following example shows that it is impossible to get rid of dependence of R on f.

Let

$$g_m(\lambda) = \frac{1}{2}\left\{\frac{\sin\left(m\left(\sqrt{m\lambda} - m\right)\right)}{\sqrt{m}\left(\sqrt{m\lambda} - m\right)} + \frac{\sin\left(m\left(\sqrt{m\lambda} + m\right)\right)}{\sqrt{m}\left(\sqrt{m\lambda} + m\right)}\right\} \in H_1\left(\frac{1}{2}\right),$$

$$m = 1, 2, 3, \ldots .$$

For $\lambda \ge 0$ the inequalities $\sqrt{m}\left|\sqrt{m\lambda} - m\right| \ge 1$ and $\left|\lambda - (m + 1/m^2)\right| \ge 2/\sqrt{m}$ are equivalent. Let

$$\mathbb{F} = \mathbb{E} = \mathbb{R}^n \setminus \bigcup_{m=1}^{\infty} \Big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n :$$

$$\left| x_1 + \ldots + x_n - (4m + 1/16m^2) \right| \le 1/\sqrt{m} \Big\},$$

and let

$$f_m(z) = g_{4m}(z_1 + \ldots + z_n) \in H_n\left(\frac{1}{2}\right), \quad m = 1, 2, \ldots.$$

$\mathbb{E}$ itself is relatively dense in $(R_+)^n$ and besides an asymptotic net in it of any order less than $3/2$. All f_m are bounded by 1 on $\mathbb{E}$. However,

$$f_m\left(\frac{4m}{n}, \ldots, \frac{4m}{n}\right) = 2\sqrt{m} + o(1), \quad m \to \infty.$$

***Remark* 4.** Actually, we have proved more than was formulated in the first statement of Theorem 6: Each of the functions in question is bounded on any closed cone embedded in $\mathbb{K}$. The following example shows that we cannot change embedded cones to shifts of $\mathbb{K}$.

For the sake of brevity, assume that $n = 2$. Let

$$f(z_1, z_2) = \sum_{j=1}^{\infty} g_j(z_1) \frac{\sin(z_2 - m_j)}{z_2 - m_j}$$

where numbers $m_j \uparrow \infty$ as $m \to \infty$. If increasing is fast enough, then $f \in H_2(1, 1)$. Evidently, this function is bounded on

$$\mathbb{F} = \mathbb{E} = \mathbb{R}^2 \setminus \bigcup_{j=1}^{\infty} \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \left| x_1 + x_2 - (j + 1/j^2) \right| \le 2/\sqrt{j} \right\},$$

which is relatively dense in $\mathbb{K} = (R_+)^2$ and an asymptotic net in it of any order less than $3/2$ simultaneously. On the other hand, $f(j, m_j) \to \infty$ as $j \to \infty$. For each vector $\mathbf{a} = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$, all pairs of the sequence $\{(j, m_j)\}$, but a finite number belong to $\mathbb{K} + \mathbf{a}$.

4. Generalization. To extend our results on functions which are analytic in a cone, we need a Keldysh-type theorem of Russakovskii mentioned in Introduction. First let's introduce some notations.

Let ω and φ be plurisubharmonic functions in $\mathbb{C}^n$, the following "nonoscillating property":

$$(u)^{[1]}(z) \le -A(-u)^{[1]}(z) + B$$

where by $(u)^{[r]}(z)$ we denote $\sup\{u(\omega) : |\omega - z| < r\}$ and $A, B \ge 0$. Assume also that $\varphi(z) \ge 0$ and

$$\log(1 + |z|) = o(\varphi(z))$$

as $z \to \infty$. For $\varepsilon > 0$ let

$$\Omega_\varepsilon = \{ z \in \mathbb{C}^n : \omega(z) < -\varepsilon\varphi(z) \},$$

and let

$$\forall \varepsilon_1 > \varepsilon_2 : \inf\left\{\left|z^{(1)} - z^{(2)}\right| : z^{(1)} \in \Omega_{\varepsilon_1}, z^{(2)} \in \mathbb{C}^n \setminus \Omega_{\varepsilon_2}\right\} > 0,$$

which is a kind of smoothness condition on ω and φ.

Theorem 10 (Russakovskii [13]). *Let $f(z)$ be an analytic function in Ω_0 satisfying the estimate*

$$|f(z)| \le C_f \exp\{C_f \varphi(z)\}, \quad z \in \Omega_0.$$

Then for each $\varepsilon > 0$ and each $N \ge 1$ there exists such an entire function $g(z)$ that

$$|f(z) - g(z)| \le C\exp\{-N\varphi(z)\}, \quad z \in \Omega_\varepsilon,$$

$$|g(z)| \le C\exp\left\{C\max\{C_f, N\}\left(\frac{2}{\varepsilon}\omega^+ + \varphi\right)(z)\right\}$$

where C does not depend on N.

***Proof of Theorem* 7.** Without loss of generality, we can assume that

$$z^0 = \frac{\mathbf{1}}{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in \mathbb{R}^n.$$

If M is a constant which is large enough, then the connected component Ω_0, $\mathbf{1}/\sqrt{n} \in \Omega_0$, of the set

$$\left\{z \in \mathbb{C}^n : \max\left\{M\left|\operatorname{Im} z_j^\rho\right| - \operatorname{Re} z_j^\rho : j = 1, \ldots, n\right\} < 0\right\}$$

is a subset of G. Define

$$\omega(z) = \max\left\{M\left|\operatorname{Im} z_j^\rho\right| - \operatorname{Re} z_j^\rho : j = 1, \ldots, n\right\}$$

outside the union of the other components and 0 on them. Define also

$$\varphi(z) = \max\left\{1, \max\left\{\left|z_j\right|^\rho : j = 1, \ldots, n\right\}\right\}.$$

It is evident that the functions and domains defined in such a way

$$\Omega_\varepsilon = \{z \in \mathbb{C}^n : \omega(z) < -\varepsilon\varphi(z)\}$$

satisfy the conditions of Russakovskii theorem. According to it, for each $f \in H_G(\rho, \sigma)$, each $\varepsilon > 0$ and each finite N, $N \ge N_0(G, \rho, \sigma)$, there exists such $g \in H(\rho, \psi(N))$ that

$$|f(z) - g(z)| \le C\exp\left\{-\psi(N)|z|^\rho\right\}, \quad z \in \Omega_0. \tag{9}$$

Here $\psi(N) \to \infty$ as $N \to \infty$. If $h_{f,\mathbb{E}}(\mathbf{1}) > -\infty$, then g behaves on $\mathbb{E}$ essentially in the same way as f. Applying Theorem 1 to the function g, we see that

$$h_f(\mathbf{1}) = h_g(\mathbf{1}) = h_{g,\mathbb{E}}(\mathbf{1}) = h_{f,\mathbb{E}}(\mathbf{1}).$$

If $h_{f,\mathbb{E}}(\mathbf{1}) = \infty$, (9) yields inequality $h_{g,\mathbb{E}}(\mathbf{1}) < -\psi(N)$. Therefore, $h_g(\mathbf{1}) < -\psi(N)$, and applying (9) once again, we get that $h_f(\mathbf{1}) < -\psi(N)$. Since N is arbitrary large, we are done in this case too.

***Proof of Theorem* 8.** This proof may be derived from Theorem 3 and Russakovskii's theorem in exactly the same way as the proof of Theorem 6 was derived from the latter and Theorem 1. The only point that should be added to the previous argument is the following Phragmen – Lindelöf-type result: Since the opening of G is wide enough, the indicator of each nontrivial function $f \in H(\rho)$ should be bounded from below. Theorem 8 is proved.

1. *Lelong P.* Fonctions plurisousharmoniques et fonctions analytiques reeles // Ann. fast. Fourier. – 1961. – **11**. – P. 263 – 303.
2. *Ronkin L. I.* Introduction to the theory of entire functions of several variables. – Providence: Amer. Math. Soc., 1974.
3. *Bernstein V.* Sur les proprietes caracteristiques des indicatrices de croissance // C. R. – 1936. – **202**. – P. 108 – 110.
4. *Pfluger A.* Über das Anwachsen von Funktionen, die in einem Winkelraum reqular und von Exponentialtypus sind // Compos. math. – 1937. – **4**. – S. 367 – 372.
5. *Levinson N.* Gap and density theorems. – New York: Amer. Math. Soc., 1940.
6. *Boas R. P.* On the growth of analytic functions // Duke Math. J. – 1936. – **13**. – P. 433 – 448.
7. *Fuchs W. H. I.* On the growth of functions of mean type // Proc. Edinburgh Math. Soc. – 1954. – **9**. – P. 53 – 70.
8. *Levin B. Ja.* Distribution of zeros of entire functions. – Providence: Amer. Math. Soc., 1964.
9. *Malliavin P.* Sur le croissance radiale d'une fonction meromorphe // III. J. Math. – 1957. – **1**. – P. 256 – 296.
10. *Ronkin L. I.* On discrete uniqueness sets for entire functions of exponential type and several variables // Sib. Mat. Zh. – 1978. – **19**. – P. 142 – 152 [in Russian].
11. *Logvinenko V. N.* On entire functions of exponential type bounded or slowly growing along the real hyperplane // Sib. Mat. Zh. – 1988. – **29**. – P. 126 – 138 [in Russian].
12. *Keldysh M. V.* Sur l'approximation des fonctions holomorphes par les fonctions entiers // Dokl. Akad. Nauk SSSR. – 1946. – **47**. – P. 239 – 241.
13. *Russakovskii A.* Approximation by entire functions on unbounded domains in $\mathbb{C}^n$ // J. Approxim. Theory. – 1993. – **70**. – P. 353 – 358.
14. *Logvinenko V., Russakovskii A.* Quasi-sampling sets for functions analytic in a cone. – Preprint, 1995.
15. *Logvinenko V., Chystyakova N.* Polynomial approximation on varying sets // J. Approxim. Theory. – 1996. – **86**. – P. 144 – 178.
16. *Schaeffer A. C.* Entire functions and trigonometric polynomials // Duke Math J. – 1953. – **20**. – P. 77 – 78.
17. *Levin B. Ja.* Majorants in classes of subharmonic functions // Teoriya funktsii, funkts. Analiz i ikh Pril. – 1989. – **51**. – P. 3 – 17 [in Russian].
18. *Katznelson V. E.* Equivalent norms in spaces of entire functions // Mat. Sb. – 1973. – **92(134)**. – P. 34 – 54 [in Russian].
19. *Levin B. Ja., Logvinenko V. N.* On classes of functions subharmonic in $\mathbb{R}^n$ and bounded on some subsets // Zap. Nauchn. Sem. LOMI. – 1989. – **170**. – P. 157 – 175 [in Russian].
20. *Levin B. Ja., Logvinenko V. N., and Sodin M. L.* Subharmonic functions on finite degree bounded on subsets of the "Real Hyperplane" // Adv. Sov. Math. – 1992. – **11**. – P. 181 – 197.

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