5. Черкесов Г. Н. Анализ надёжности сложных систем при помощи вероятностной логики. *Основные вопросы теории и практики надёжности*. Сб. тр. семинара научного совета по проблемам надёжности отделения механики и процессов управления АН СССР. М.: Сов. радио, 1980. 328 с.
6. Половко А. М., Гуров С. В. Основы теории надежности. С. Пб.: БХВ-Петербург, 2006. 704 с.
7. Вентцель Е. С. Исследование операций. М.: Сов. радио, 1972. 550 с.
8. Зевин Л. И., Инкулис В. В., Зевин С. Л. Методика расчета и анализа структурной надежности блоков атомных станций. *Проблемы машиностроения*. 2002. Т. 5. № 2. С. 34–37.
9. Зевин С. Л. Распознавание структурных схем в задачах моделирования надежности энергоустановок. *Вестн. Нац. техн. ун-та "ХПИ". Серия: Электроэнергетика и преобразовательная техника.* 2002. Вып. 9. Т. 3. С. 33–38.
10. Обоскалов В. П. Проблемы расчета структурной надежности систем электроснабжения с использованием метода вероятностного эквивалентирования. *Электричество*. 2015. № 12. С. 4–12.
11. Бурмутаев А. Е. Сложность моделирования интервальных оценок показателей структурной надежности электротехнических комплексов методом Монте-Карло. *Вектор науки Тольяттин. ун-та.* 2011. № 3 (17). С. 72–75.

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MINIMIZATION OF THE STRESSED STATE OF A STRINGER PLATE WITH A HOLE AND RECTILINEAR CRACKS

Minavar V. Mir-Salim-zade
minavar.mirsalimzade@imm.az
ORCID: 0000-0003-4237-0352

Institute of Mathematics and Mechanics of the NAS of Azerbaijan,
9, Vahabzade str., Baku,
AZ1141, Azerbaijan

As is known, thin plates with holes are one of the most common structural elements. To increase their reliability and service life, it is of interest to find such a hole contour that ensures the minimum circumferential stress thereon, and also prevents the growth of possible cracks in the plate. This article deals with the problem of minimizing the stress state on the contour of a hole in an unbounded isotropic stringer plate weakened by two rectilinear cracks. Crack faces are considered to be free of stress. Determined is the optimal hole contour, at which no crack growth occurs, and the maximum circumferential stress thereon is minimal. The minimax criterion is used. The parameter characterizing the stress state in the vicinity of crack tips, according to the Irwin-Oroan theory of quasi-brittle fracture, is the stress intensity factor. The plate undergoes uniform stretching at infinity along the stringers. It is believed that the plate and the stringers are made of various elastic materials. The action of the stringers is replaced by the unknown equivalent concentrated forces applied at the points of their attachment to the plate. To determine these forces, Hooke's law is used. Applying the small parameter method, the theory of analytic functions and the method of direct solution to singular equations, we constructed a closed system of algebraic equations. This system depends on the mechanical and geometrical parameters of the plate and stringers, ensures the on-hole contour stress state minimization and equality of stress intensity factors to zero in the vicinity of crack tips. The minimization problem is reduced to a linear programming problem. The simplex method is applied.

Keywords*: stringer plate, stress minimization, cracks, optimal hole contour, minimax criterion.*

Introduction

One of the most common structural elements is thin plates. Frequently, such plates have technological holes. Since the holes are stress concentrators and can lead to premature failure, the problem of minimizing the stress state on the hole contour is of great interest [1–15]. Article [1], based on the finite element method (FEM), develops an iterative method to optimize the hole contour to simultaneously minimize the tangential stresses in several areas around the hole boundary. It shows that such an optimal hole contour can significantly reduce peak stress in all the areas around the hole boundary, compared to typical non-optimal circular holes. Article [2] describes a piecewise-smooth optimal contour that minimizes local stresses under remote shear for a single, stress-free hole in an elastic plate, with the methods of conformal mapping and genetic algorithm used. It shows numerically that the hole contour found provides a shear stress by 30% lower than the stress concentration factor for commonly used circular holes. Article [6], considers three holes for the case when two identical side holes differ in shape and area from the central one. Compared with one hole, interacting optimal holes create 15–19% less energy, depending on the distance between them and the shape of the central hole. Article [3] gives a solution to the inverse elastic problem of determining the optimal shape of the hole contour for an isotropic medium with a system of foreign transverse straight inclusions (an unbounded plate reinforced by a regular stringer system). The criterion for determining the optimal shape is the condition that there is no concentration of stresses on the surface of the hole or the requirement for the plastic region to form at once over the entire surface of the hole. Article [5] obtains a solution for the inverse elastic-plastic problem of determining the optimal shape of hole contours for a riveted perforated plate. Article [4], investigates the mixed problem of the theory of elasticity for a rectangle weakened by equal-strength holes. It assumes that the tangential stresses at the outer boundary of the rectangle are zero, and the normal displacements are constant, with the tangential stresses on the contours of the holes being zero, and the normal stresses being constant. It also determines the boundaries of the holes for which the tangential normal stress is constant. Article [7] carries out the optimization of the shape of staggered holes, free from stresses. Article [8] provides common basic equal-strength forms and structural elements, gives equal-strength forms of an aircraft swept-back wing, and finds some new equal-strength forms for elastic bodies with any number of infinite branches pulled out of the body under the conditions of plane deformation and plane stress state. Article [9] considers the problem of finding the equal-strength hole form at the crack tip and its effect on crack growth. The obtained solution to the problem of optimal design allows selecting the optimal geometric parameters of the body, ensuring effective crack retardation. Article [10] presents an evolutionary method for optimizing structures, based on the displacement of control points along the boundary of elements, and determines the optimal element form, i.e. one for which the stress concentration factor is reduced. Article [11] conducts the minimization of the stress concentration around the hole edge in an orthotropic plate, and investigates the optimal holes and stress distribution at different loads, Young's modulus and fiber direction in the plate. Article [12] carries out a semi-analytical study of periodic and doubly periodic non-standardly located equal-strength holes in an infinite plate with a given volume load, with the main attention paid to periodic structures with some rotational symmetry. Article [13] conducts a theoretical analysis to determine the mine working form ensuring maximum strength. Article [14] proposes a criterion and method for solving the inverse problem of preventing the destruction of an isotropic elastic plate weakened by a hole and an arbitrary system of cracks under the action of a given system of external loads, with the principle of equal strength and minimization of stress intensity factors implemented. Article [15] determines the optimal hole shape in an isotropic elastic plate weakened by an arbitrary system of cracks on the basis of the minimax criterion.

To increase the service life and reliability of a structure, it is important to consider the possibility of the presence of cracks, i.e. determine the hole contour, at which no crack growth will occur [9, 13–15].

The problem under consideration is to find such a hole contour in the stringer plate, weakened by two rectilinear cracks, at which no crack growth will occur, and the maximum circumferential stress on the contour will be minimal.

Formulation of the Problem

Consider an unbounded thin plate reinforced by a regular stringer system (Fig. 1). The plate and stringers are isotropic and made of various elastic materials. At infinity, the reinforced plate is subjected to uniform tension along the stringers with the stress $\sigma_y^\infty = \sigma_0$. The plate with the thickness h is weakened by a hole and two rectilinear cracks.

The following assumptions are taken: during deformation, the thickness of the stringers is unchanged, and the stress state is uniaxial; the stringers are not subjected to bending and work solely in tension; in the plate, the flat stress state is realized; the truss-type stringer system and the weakening of the stringers due to the setting of attachment points is not taken into account; the attachment points are the same, their radius a_0 (point adhesion area) is small, compared to their $2L$ step and other characteristic dimensions; the plate and stringers interact with each other in the same plane and only at the attachment points.

It is believed that the attachments of the stringers are arranged in a discrete manner with a constant step y_0 along the entire length of the stringer, symmetrically relative to the plate surface. The action of the attachment points is modeled by the action of the concentrated forces applied at the points corresponding to the centers of the attachment points: $z = \pm(2m+1)L \pm iky_0$ ($m=0, 1, 2,\ldots$; $k=1, 2,\ldots$). The action of the stringers is replaced by the unknown equivalent concentrated forces applied at the points of their attachment to the plate. The magnitude of the concentrated forces are determined in the course of solving the problem.

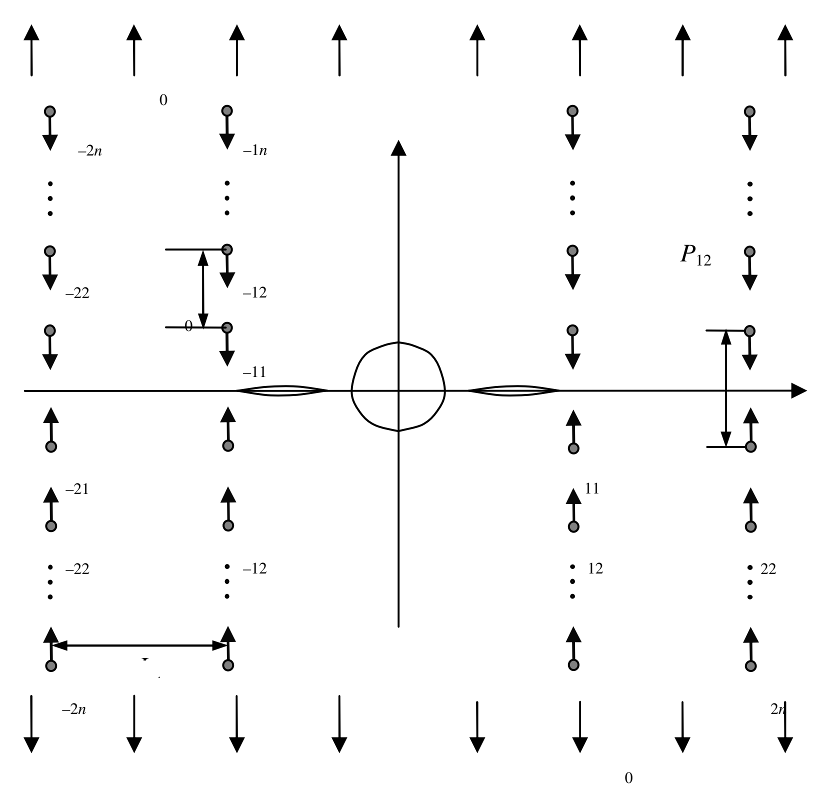

Fig. 1. The design scheme of the problem

On the unknown hole contour L_0, the boundary conditions have the form

$$\sigma_n = 0, \qquad \tau_{nt} = 0;$$

on the crack faces,

$$\sigma_y = 0, \qquad \tau_{xy} = 0 \qquad a \le |x| \le b.$$

Here n and t are the tangent and normal to the hole contour.

It is required that a hole shape be found, with no crack growth occurring, and the tangential normal stress σ_t, acting on the contour, being minimal. According to the Irwin-Oroan theory of quasi-brittle fracture, the stress intensity factor is taken as the parameter characterizing the stress state in the vicinity of crack tips. Thus, it is necessary that there be fulfilled the conditions for minimizing the maximum circumferential stress σ_t on the hole contour and the equality of stress intensity factors to zero in the vicinity of crack tips.

Therefore, we require that there be met the conditions

$$\min_{\eta \in C} \max_{\theta \in [0,2\pi]} \sigma_t(\theta, \eta) \tag{1}$$

$$K_{\mathrm{I}}^a = 0, \qquad K_{\mathrm{I}}^b = 0. \tag{2}$$

Here C is the set of constraints to be defined; η is the design parameters; K_{I}^a, K_{I}^b are the stress intensity factors in the vicinity of crack tips. Since in the problem under consideration the cracks are arranged symmetrically, $K_{\mathrm{I}}^a = K_{\mathrm{I}}^{-a}$, $K_{\mathrm{I}}^b = K_{\mathrm{I}}^{-b}$.

The problem set is to determine such a hole contour, at which the maximum circumferential stress σ_t is minimal, and the stress intensity factors in the vicinity of crack tips are zero, as well as to determine the magnitudes of the concentrated forces P_{mn} and the stress-strain state of the reinforced plate.

Solution to the Boundary Problem

We will seek an unknown hole contour L_0 in the class of contours close to circular. Imagine an unknown contour L_0 in the form

$$r = \rho(\theta) = R + \varepsilon H(\theta),$$

where $\varepsilon = R_{\max}/R$ is the small parameter; $R_{\max}$ is the maximum height of the roughness of the hole contour profile L_0 of the cirumference $r=R$; the function $H(\theta)$ is to be determined in the process of solving the inverse problem.

Without reducing the generality of the problem under consideration, we assume that the desired function $H(\theta)$ is symmetric about the coordinate axes and can be represented as a Fourier series $H(\theta) = \sum_{k=1}^{\infty} d_{2k} \cos 2k\theta$.

The required functions (stresses, displacements, concentrated forces P_{mn} and stress intensity factors) will be sought in the form of expansions in the small parameter ε

$$\sigma_n = \sigma_n^{(0)} + \varepsilon\sigma_n^{(1)} + \ldots, \qquad \sigma_t = \sigma_t^{(0)} + \varepsilon\sigma_t^{(1)} + \ldots, \qquad \tau_{nt} = \tau_{nt}^{(0)} + \varepsilon\tau_{nt}^{(1)} + \ldots,$$

$$u_n = u_n^{(0)} + \varepsilon u_n^{(1)} + \ldots, \qquad v_n = v_n^{(0)} + \varepsilon v_n^{(1)} + \ldots,$$

$$P_{mn} = P_{mn}^{(0)} + \varepsilon P_{mn}^{(1)} + \ldots,$$

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{(0)} + \varepsilon K_{\mathrm{I}}^{(1)} + \ldots$$

in which we neglect, for simplicity, the terms containing ε degrees higher than the first one.

Each of the approximations satisfies the system of differential equations of the plane problem of the theory of elasticity.

The values of the stress tensor components at $r=\rho(\theta)$ are obtained by decomposing expressions for the stresses in the vicinity of $r = R$. Using the well-known formulas [16] for the stress components σ_n and τ_{nt}, the boundary conditions of the problem are written as follows:

– for a zeroth-order approximation:

$$\text{on the contour, } r=R \qquad \sigma_r^{(0)} = 0, \qquad \tau_{r\theta}^{(0)} = 0; \tag{3}$$

$$\text{on the crack faces, } \sigma_x^{(0)} = 0, \qquad \tau_{xy}^{(0)} = 0 \qquad a \le |x| \le b; \tag{4}$$

– for a first-order approximation:

$$\text{on the circuit, } r=R \qquad \sigma_r^{(1)} = N, \qquad \tau_{r\theta}^{(1)} = T;$$

$$\text{on the crack faces, } \sigma_x^{(1)} = 0, \qquad \tau_{xy}^{(1)} = 0 \qquad a \le |x| \le b.$$

Here, $N = -H(\theta)\dfrac{\partial\sigma_r^{(0)}}{\partial r} + 2\dfrac{\tau_{r\theta}^{(0)}}{R}\dfrac{\partial H(\theta)}{\partial\theta}$, $T = -H(\theta)\dfrac{\partial\tau_{r\theta}^{(0)}}{\partial r} + \dfrac{\sigma_\theta^{(0)} - \sigma_r^{(0)}}{R}\dfrac{\partial H(\theta)}{\partial\theta}$.

Based on the Kolosov-Muskhelishvili formulas [16] and boundary conditions (3)–(4), both on the hole contour and on the crack faces, the problem in the zeroth-order approximation is reduced to the definition of two analytic functions $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ from the boundary condition

$$\Phi^{(0)}(\tau) + \overline{\Phi^{(0)}(\tau)} - e^{2i\theta}\left[\bar{\tau}\Phi^{(0)\prime}(\tau) + \Psi^{(0)}(\tau)\right] = 0 \qquad \text{at } \tau = Re^{i\theta}, \tag{5}$$

$$\Phi^{(0)}(x) + \overline{\Phi^{(0)}(x)} + x\overline{\Phi^{(0)\prime}(x)} + \overline{\Psi^{(0)}(x)} = 0 \qquad a \le |x| \le b. \tag{6}$$

The solution to the boundary value problem (5)–(6) is sought in the form (k=0)

$$\Phi^{(k)}(z) = \Phi_0^{(k)}(z) + \Phi_1^{(k)}(z) + \Phi_2^{(k)}(z), \tag{7}$$

$$\Psi^{(k)}(z) = \Psi_0^{(k)}(z) + \Psi_1^{(k)}(z) + \Psi_2^{(k)}(z).$$

The potentials $\Phi_0^{(0)}(z)$, $\Psi_0^{(0)}(z)$ describe the stress and strain field in a solid plate under the action of a system of the concentrated forces $P_{mn}^{(0)}$ and stress σ_0, and are defined by the following formulas:

$$\Phi_0^{(0)}(z)=\frac{1}{4}\sigma_0-\frac{i}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{1}{z-mL+iny_0}-\frac{1}{z-mL-iny_0}\right], \tag{8}$$

$$\Psi_0^{(0)}(z)=\frac{1}{2}\sigma_0-\frac{i\kappa}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{1}{z-mL+iny_0}-\frac{1}{z-mL-iny_0}\right]+$$

$$+\frac{i}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{mL-iny_0}{(z-mL+iny_0)^2}-\frac{mL+iny_0}{(z-mL-iny_0)^2}\right].$$

Here κ=(3–ν)/(1+ν); ν is Poisson's ratio of the plate material; the prime symbol of the summation symbol indicates that during summation the index m=n=0 is excluded.

We seek the functions $\Phi_1^{(0)}(z)$ and $\Psi_1^{(0)}(z)$ in the form

$$\Phi_1^{(0)}(z)=\frac{1}{2\pi}\int_{L_1}\frac{g^{(0)}(t)}{t-z}dt, \quad \Psi_1^{(0)}(z)=\frac{1}{2\pi}\int_{L_1}\left[\frac{1}{t-z}-\frac{t}{(t-z)^2}\right]g^{(0)}(t)dt, \tag{9}$$

where L_1=[a, b]+[–a, –b]; $g^{(0)}(x)=\frac{2\mu}{1+\kappa}\frac{d}{dx}\left[v^+(x,0)-v^-(x,0)\right]$; μ is the shear modulus of the plate material.

The unknown function $g^{(0)}(x)$ and the potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$ must be determined from the boundary conditions (5) – (6). Imagine the boundary condition (6) in the form

$$\Phi_2^{(0)}(\tau)+\overline{\Phi_2^{(0)}(\tau)}-e^{2i\theta}\left[\bar{\tau}\Phi_2^{(0)\prime}(\tau)+\Psi_2^{(0)}(\tau)\right]=-\Phi_*^{(0)}(\tau)-\overline{\Phi_*^{(0)}(\tau)}+e^{2i\theta}\left[\bar{\tau}\Phi_*^{(0)\prime}(\tau)+\Psi_*^{(0)}(\tau)\right] \tag{10}$$

where $\Phi_*^{(0)}(\tau)=\Phi_0^{(0)}(\tau)+\Phi_1^{(0)}(\tau)$, $\Psi_*^{(0)}(\tau)=\Psi_0^{(0)}(\tau)+\Psi_1^{(0)}(\tau)$.

To solve the boundary value problem (10) (definitions of the potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$), we use N. I. Muskhelishvili's solution [16]. As a result, we have

$$\Phi_2^{(0)}(z)=\frac{\sigma_0}{2z^2}+\frac{1}{2\pi}\int_{L_1}\left[\frac{1-t^2}{t(1-tz)}+\frac{z-t}{(1-tz)^2}\right]g^{(0)}(t)dt-\frac{i}{2\pi h(1+\kappa)}\times$$

$$\times\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{(mL-iny_0)(mL+iny_0)-1}{(mL-iny_0)[z(mL-iny_0)-1]^2}-\frac{(mL+iny_0)(mL-iny_0)-1}{(mL+iny_0)[z(mL+iny_0)-1]^2}\right\}+;$$

$$+\frac{i\kappa}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{1}{z[z(mL-iny_0)-1]}-\frac{1}{z[z(mL+iny_0)-1]}\right\}; \tag{11}$$

$$\Psi_2^{(0)}(z)=\frac{\sigma_0}{2z^2}+\frac{\Phi_2^{(0)}(z)}{z^2}-\frac{\Phi_2^{(0)\prime}(z)}{z^2}+\frac{1}{2\pi z}\int_{L_1}\left[\frac{2}{tz}-\frac{t}{z(1-tz)}+\frac{t^2z-z-t}{z(1-tz)^2}-\frac{2t(z-t)}{(1-tz)^3}\right]g^{(0)}(t)dt+$$

$$+\frac{i}{2\pi h(1+\kappa)z}\times\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{1}{z(mL-iny_0)-1}-\frac{1}{z(mL+iny_0)-1}+\frac{1}{z(mL-iny_0)}-\frac{1}{(zmL+iny_0)}\right\}.$$

In formulas (11), all the linear dimensions are related to the radius of the circular hole R.

Requiring that functions (7) with k=0 satisfy the boundary condition (6) on the crack faces, we obtain a singular integral equation for $g^{(0)}(x)$

$$\frac{1}{\pi}\int_{L_1}\frac{g^{(0)}(t)}{t-x}dt+\frac{1}{\pi}\int_{L_1}K(t,x)g^{(0)}(t)dt=F(x), \tag{12}$$

$$K(t,x)=\frac{x-t}{xt(1-tx)^2}+\frac{1}{x^2t}+\frac{1}{2}\left[\frac{2t(x-t)(x^2-1)}{x(1-tx)^3}+\frac{2x^3-x-2t+2t^2x-x^3t^2}{x^2(1-tx)^2}\right],$$

$$F(x)=f_0^{(0)}(x)+f_1^{(0)}(x),$$

$$f_0^{(0)}(x)=-\sigma_0+\frac{\kappa+2}{\pi h(1+\kappa)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{-mn}^{(0)}ny_0\left[\frac{1}{(x-mL)^2+n^2y_0^2}\right]+\sum_{m,n=1}^{\infty}P_{mn}^{(0)}ny_0\left[\frac{1}{(x+mL)^2+n^2y_0^2}\right]\right\}-$$

$$-\frac{1}{\pi h(1+\kappa)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{mn}^{(0)}ny_0\frac{(x-mL)^2-n^2y_0^2-(x^2-m^2L^2)}{\left[(x-mL)^2+n^2y_0^2\right]^2}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{-mn}^{(0)}ny_0\frac{(x+mL)^2-n^2y_0^2-(x^2-m^2L^2)}{\left[(x+mL)^2+n^2y_0^2\right]^2}\right\},$$

$$f_1^{(0)}(x)=-\frac{1}{2\pi(1+\kappa)h}\sum_{m,n=1}^{\infty}P_{-mn}^{(0)}n\left\{\left(2+\frac{1}{x^2}\right)\times\right.$$

$$\times\left\langle\frac{2(m^2L^2+n^2y_0^2-1)\left[x^2(3m^2L^2-n^2y_0^2)+4xmL+1\right]}{(m^2L^2+n^2y_0^2)\left[(xmL+1)^2+x^2n^2y_0^2\right]^2}+\frac{2\kappa}{(mxL+1)^2+x^2n^2y_0^2}\right\rangle+$$

$$+\left(\kappa-\frac{1}{\kappa}\right)\left\langle-4(m^2L^2+n^2y_0^2-1)\frac{\left[x^3(3m^2L^2-n^2y_0^2)+6x^2mL+3x\right]}{\left[(xmL+1)^2+x^2n^2y_0^2\right]^3}-\right.$$

$$\left.\left.-4\kappa\frac{mL+x(m^2L^2+n^2y_0^2)}{\left[(mxL+1)^2+x^2n^2y_0^2\right]^2}\right\rangle-2\left[\frac{1}{(mxL+1)^2+x^2n^2y_0^2}+\frac{1}{m^2L^2+n^2y_0^2}\right]\right\}-$$

$$-\frac{1}{2\pi(1+\kappa)h}\sum_{m,n=1}^{\infty}P_{mn}^{(0)}\left(2+\frac{1}{x^2}\right)\left\langle\frac{2(m^2L^2+n^2y_0^2-1)\left[x^2(3m^2L^2-n^2y_0^2)-4xmL+1\right]}{(m^2L^2+n^2y_0^2)\left[(xmL-1)^2+x^2n^2y_0^2\right]^2}+\right.$$

$$\left.+\frac{2\kappa_0}{(mxL-1)^2+x^2n^2y_0^2}\right\rangle+\left(\kappa-\frac{1}{\kappa}\right)\times\left\langle-4(m^2L^2+n^2y_0^2-1)\frac{x^3(3m^2L^2-n^2y_0^2)-6x^2mL+3x}{\left[(xmL-1)^2+x^2n^2y_0^2\right]^3}+\right.$$

$$\left.+4\kappa\frac{mL-x(m^2L^2+n^2y_0^2)}{\left[(mxL-1)^2+x^2n^2y_0^2\right]^2}\right\rangle-2\left[\frac{1}{(mxL-1)^2+x^2n^2y_0^2}+\frac{1}{m^2L^2+n^2y_0^2}\right]\Bigg\}-\frac{\sigma_0}{2x^2}-\frac{3}{2}\frac{\sigma_0}{x^4}.$$

To construct a solution to the singular integral equation (12), we use the method of direct solution of singular equations [17, 18]. Proceeding to the dimensionless variables, we represent the solution in the form

$$g^{(0)}(\eta)=\frac{g_0^{(0)}(\eta)}{\sqrt{1-\eta^2}},$$

where $g_0^{(0)}(\eta)$ is a bounded function, continuous on the segment [−1, 1]; it is replaced by the Lagrange interpolation polynomial constructed through using the Chebyshev nodes.

Using the algebraization procedure [17, 18], the singular integral equation (12), with an additional condition that ensures the uniqueness of displacements during the path-tracing of crack contours

$$\int_a^b g_0^{(0)}(t)dt=0,\qquad \int_{-a}^{-b} g_0^{(0)}(t)dt=0,$$

reduces to the system of M linear algebraic equations to define the M unknowns (m=1, 2, ..., M)

$$\begin{cases} \sum_{k=1}^{M} A_{mk} g_k^{(0)} = f_0^{(0)}(\eta_m) + f_1^{(0)}(\eta_m) \\ \sum_{k=1}^{M} g_k^{(0)}(\eta_m) = 0 \end{cases} \tag{13}$$

where m=1, 2, ..., M-1; $A_{mk} = \frac{1}{M}\left[\frac{1}{\sin\theta_m}\operatorname{ctg}\frac{\theta_m + (-1)^{|m-k|}\theta_k}{2} + K_0(\eta_m, \tau_k)\right]$; $g_k^{(0)} = g^{(0)}(\tau_k)$; $\eta_m = \cos\theta_m$;

$\theta_m = \frac{2m-1}{2M}\pi$; $\eta_m = \cos\theta_m$; $\tau_k = \eta_k$.

To determine the unknown concentrated forces $P_{mn}^{(0)}$, we use Hooke's law, according to which the magnitude of the concentrated force $P_{mn}^{(0)}$ acting on each point of attachment from the side of the stringer will be

$$P_{mn}^{(0)} = \frac{E_s A_s}{2 y_0 n}\Delta v_{mn}^{(0)} \qquad (m, n = 1,2,\ldots.), \tag{14}$$

where E_s is the Young's modulus of the stringer material; A_s is the cross section of the stringer; $2y_0n$ is the distance between the attachment points; $\Delta v_{mn}^{(0)}$ is the mutual displacement of the attachment points considered, with the displacement equal to the elongation of the corresponding stringer section.

Let us take the natural assumption in [19] that the displacement compatibility condition is satisfied, i.e. the mutual elastic displacement of the points $mL+i(ny_0-a_0)$ and $mL-i(ny_0-a_0)$ in the elasticity theory problem under consideration is equal to the mutual displacement of the attachment points $\Delta v_{mn}^{(0)}$. Using the Kolosov-Muskhelishvili formulas [16] and relations (7)–(9), (11), we find the mutual displacement of the indicated points $\Delta v_{mn}^{(0)}$. In view of some cumbersomeness, these expressions are not given. Solving systems (13) and (14), we determine the magnitudes of the concentrated forces $P_{mn}^{(0)}$, the approximate values at the nodal points $g^{(0)}(\tau_m)$, and thus, the complex potentials of the zeroth-order approximation.

For the stress intensity factors in the vicinity of the crack tip at $x=a$ in the zeroth-order approximation, we have

$$K_{\mathrm{I}}^{(0)} = \sqrt{\pi(b-a)}\sum_{m=1}^{M}(-1)^{m+M} g^{(0)}(t_m)\operatorname{tg}\frac{2m-1}{4M}\pi,$$

in the vicinity of the crack tip $x=b$,

$$K_{\mathrm{I}}^{(0)} = \sqrt{\pi(b-a)}\sum_{m=1}^{M}(-1)^{m} g^{(0)}(t_m)\operatorname{ctg}\frac{2m-1}{4M}\pi.$$

According to the Kolosov-Muskhelishvili formulas and relations (7), the stress components in the reinforced plate are found in the zeroth-order approximation. Knowing the stress state in the zeroth-order approximation, we find the functions N and T.

After finding the solution to the zeroth-order approximation, we proceed to solving the problem in the first-order approximation. The boundary conditions of the problem for the first-order approximation are written as

$$\Phi^{(1)}(\tau) + \overline{\Phi^{(1)}(\tau)} - e^{2i\theta}\left[\overline{\tau}\Phi^{(1)\prime}(\tau) + \Psi^{(1)}(\tau)\right] = N - iT, \tag{15}$$

$$\Phi^{(1)}(x) + \overline{\Phi^{(1)}(x)} + x\overline{\Phi^{(1)\prime}(x)} + \overline{\Psi^{(1)}(x)} = 0 \qquad a \le |x| \le b. \tag{16}$$

We seek the solution to the boundary value problem (15) similarly to the zeroth-order approximation in the form (7) for k=1, where the potentials $\Phi_0^{(1)}(z)$ and $\Psi_0^{(1)}(z)$ describe the stress and strain field under the action of a system of the concentrated forces $P_{mn}^{(1)}$, and are determined by formulas similar to (8), in which σ_0 should be set equal to zero and $P_{mn}^{(0)}$ should be replaced with $P_{mn}^{(1)}$.

We seek the potentials $\Phi_1^{(1)}(z)$ and $\Psi_1^{(1)}(z)$ in the form similar to (9), where the function $g^{(0)}(x)$ should be replaced with $g^{(1)}(x)$.

We find the functions $\Phi_2^{(1)}(z)$ and $\Psi_2^{(1)}(z)$ from the boundary condition (15), again using N. I. Muskhelishvili's method

$$\Phi_2^{(1)}(z)=\Phi_*^{(1)}(z)+\sum_{k=0}^{\infty}a_{2k}z^{-2k}, \qquad \Psi_2^{(1)}(z)=\Psi_*^{(1)}(z)+\sum_{k=0}^{\infty}b_{2k}z^{-2k}.$$

Here, $\Phi_*^{(1)}(z)$, $\Psi_*^{(1)}(z)$ are determined by formulas similar to (11), where σ_0 should be put equal to zero, $P_{mn}^{(0)}$ should be replaced with $P_{mn}^{(1)}$, and $g^{(0)}(x)$ should be replaced with $g^{(1)}(x)$. The coefficients a_{2k} and b_{2k} are found by the formulas

$$a^{2n}=C_{2n}R^{2n} \quad (n=1, 2,\ldots), \qquad a_0=0, \qquad b_{2n}=(2n-1)R^2a_{2n-2}-R^{2n}a_{-2n+2} \quad (n\geq 2),$$

$$b_0=0, \qquad b_2=-C_0R^2, \qquad N-iT=\sum_{k=-\infty}^{\infty}C_{2k}e^{-2ki\theta}.$$

For the concentrated $P_{mn}^{(1)}$ we have

$$P_{mn}^{(1)}=\frac{E_sA_s}{2y_0n}\Delta v_{mn}^{(1)},$$

where the mutual displacement $\Delta v_{mn}^{(1)}$ is determined similarly to the zeroth-order approximation.

Requiring that functions (7) with k=1 satisfy the boundary condition (16) on the crack faces in the first-order approximation, we obtain, after some transformations, a singular integral equation with respect to the function $g^{(1)}(x)$:

$$\frac{1}{\pi}\int_{L_1}\frac{g^{(1)}(t)}{t-x}dt+\frac{1}{\pi}\int_{L_1}K(t,x)g^{(1)}(t)dt=F^{(1)}(x). \tag{17}$$

As in the zeroth-order approximation, using the algebraization procedure [17, 18], we reduce the singular integral equation (17), with an additional condition that ensures uniqueness of displacements during the path-tracing of crack contours in the first-order approximation

$$\int_a^b g_0^{(1)}(t)dt=0, \qquad \int_{-a}^{-b} g_0^{(1)}(t)dt=0,$$

to the system M linear algebraic equations to define the M unknowns $g^{(1)}(\tau_m)$ (m=1, 2,…, M)

$$\begin{cases}\sum_{k=1}^{M}A_{mk}g_k^{(1)}=f_0^{(1)}(\eta_m)+f_1^{(1)}(\eta_m)\\ \sum_{k=1}^{M}g_k^{(1)}(\eta_m)=0\end{cases},$$

where m=1, 2,…, $M-1$; $g_k^{(1)}=g^{(1)}(\tau_k)$.

In the first-order approximation, for the stress intensity factors in the vicinity of the crack tip at x=a we have

$$K_{\mathrm{I}}^{(1)}=\sqrt{\pi(b-a)}\sum_{m=1}^{M}(-1)^{m+M}g^{(1)}(t_m)\,\mathrm{tg}\frac{2m-1}{4M}\pi,$$

in the vicinity of the crack tip x=b we have

$$K_{\mathrm{I}}^{(1)}=\sqrt{\pi(b-a)}\sum_{m=1}^{M}(-1)^{m}g^{(1)}(t_m)\,\mathrm{ctg}\frac{2m-1}{4M}\pi.$$

The resulting systems of equations of the first-order approximation are not yet closed, since the right-hand sides of these systems include the coefficients d_{2k} of the expansion of the function $H(\theta)$ in the Fourier series.

To construct the missing equations, we use the boundary condition (1) with additional constraints (2). Using the solution obtained, we find σ_t in the surface layer of the contour $L_0(r=\rho(\theta))$ to the nearest first-order values with respect to the small parameter ε

$$\sigma_t = \sigma_t^{(0)}(\theta)\Big|_{r=R} + \varepsilon\left[H(\theta)\frac{\sigma_t^{(0)}(\theta)}{\partial r} + \sigma_t^{(1)}(\theta)\right]\Bigg|_{r=R}$$

We find the maximum value of the function $\sigma_t(\theta,d_{2k})$ on the contour L_0,

$$\max \sigma_t(\theta_*, d_{2k}),$$

where θ_* is the solution to the equation

$$\frac{d\sigma_t(\theta)}{d\theta} = 0.$$

To construct the missing equations that allow us to determine the coefficients d_{2k},, we require that the maximum circumferential stress σ_t on the hole contour (1) be minimized under the constraints

$$\sum_{m=1}^{M}(-1)^m\left[g^{(0)}(t_m)+\varepsilon g^{(1)}(t_m)\right]\mathrm{tg}\frac{2m-1}{4M}\pi = 0, \quad \sum_{m=1}^{M}(-1)^m\left[g^{(0)}(t_m)+\varepsilon g^{(1)}(t_m)\right]\mathrm{ctg}\frac{2m-1}{4M}\pi = 0,$$

$$\max \sigma_t \le [\sigma].$$

Here [σ] is the permissible circumferential stress determined experimentally.

It is necessary to make use of the function $H(\theta)$ in such a way as to ensure the minimization of the maximum stress σ_t (minimax criterion). It is necessary to find such values of the coefficients d_{2k} that satisfy the obtained system of equations and minimize the linear function $\max\sigma_t$ (the objective function).

Since the stresses $\sigma_t(\theta,d_{2k})$ (control performance index) and $\max\sigma_t$ linearly depend on the required coefficients d_{2k}, the problem under consideration reduces to a linear programming problem. In the problem set, the simplex algorithm method turned out to be the most effective.

The calculations were carried out for the following values of free parameters: a_0/L was equal to 0.01; y_0/L was equal to 0.25. It was believed that the stringers were made of composite Al-steel, the plate, of B95alloy; E was equal to $7.1\cdot10^4$ MPa; E_s was equal to $11.5\cdot10^4$ MPa. The number of the stringers and attachment points was assumed to be 14, the value of M was equal to 72. The value of M could be different, but not less than 20, since that was the minimum value for good convergence of the numerical solution to singular integral equations [17, 18]. For simplicity, we took: $A_s/y_0h=1$. The results of the calculations of the expansion coefficients of the unknown function $H(\theta)$ are given below.

Fourier coefficients for the optimal contour

d_2	d_4	d_6	d_8	d_{10}	d_{12}	d_{14}
0.1079	0.0869	0.0558	0.0368	0.0231	0.0014	0.0005

Conclusions

Thus, the problem of minimizing the stress state on the contour of the hole in the stringer plate with two rectilinear cracks is solved. A closed system of algebraic equations is constructed. This system allows us to find the optimal shape of the hole contour for the stringer plate weakened by two rectilinear cracks, depending on the geometric and mechanical characteristics of the plate and stringers.

The study presented should be continued for other optimization criteria (equal strength, etc.) and other types of plates that are widely used, for example, perforated ones weakened by several holes, etc.

References

1. Waldman, W., & Heller, M. (2006). Shape optimisation of holes for multi-peak stress minimisation. *Australian Journal of Mechanical Engineering*, vol. 3, iss. 1, pp. 61–71. https://doi.org/10.1080/14484846.2006.11464495
2. Vigdergauz, S. (2006). The stress-minimizing hole in an elastic plate under remote shear. *Journal of Mechanics of Materials and Structures*, vol. 1, no. 2, pp. 387–406. https://doi.org/10.2140/jomms.2006.1.387

3. Mir-Salim-zada, M. V. (2007). *Opredeleniye formy ravnoprochnogo otverstiya v izotropnoy srede, usilennoy regulyarnoy sistemoy stringerov* [Determination of equistrong hole shape in isotropic medium reinforced by regular system of stringers]. *Materialy, tehnologii, instrumenty –Materials, Technology and Instruments*, no. 12(4), pp. 10–14 (in Russian).
4. Bantsuri, R., & Mzhavanadze, Sh. (2007). The mixed problem of the theory of elasticity for a rectangle weakened by unknown equi-strong holes. *Proceedings of A. Razmadze Mathematical Institute*, vol. 145, pp. 23–34.
5. Mir-Salim-zada, M. V. (2007). *Obratnaya uprugoplasticheskaya zadacha dlya klepanoy perforirovannoy plastiny* [Inverse elastoplastic problem for riveted perforated plate]. *Sbornik statey "Sovremennye problemy prochnosti, plastichnosti i ustoychivosti" – Collected papers "Modern problems of strength, plasticity and stability"*. Tver: TGTU. pp. 238– 46 (in Russian).
6. Vigdergauz, S. (2010). Energy-minimizing openings around a fixed hole in an elastic plate. *Journal of Mechanics of Materials and Structures*, vol. 5, no. 4, pp. 661–677. https://doi.org/10.2140/jomms.2010.5.661
7. Vigdergauz, S. (2012). Stress-smoothing holes in an elastic plate: From the square lattice to the checkerboard. *Mathematics and Mechanics of Solids*, vol. 17, iss. 3, pp. 289–299. https://doi.org/10.1177/1081286511411571
8. Cherepanov, G. P. (2015). Optimum shapes of elastic bodies: equistrong wings of aircraft and equistrong underground tunnels. *Physical Mesomechanics*, vol. 18, iss. 4, pp. 391–401. https://doi.org/10.1134/S1029959915040116
9. Kalantarly, N. M. (2017). *Ravnoprochnaya forma otverstiya dlya tormozheniya rosta treshchiny prodolnogo sdviga* [Equal strength hole to inhibit longitudinal shear crack growth]. *Problemy Mashinostroyeniya – Journal of Mechanical Engineering*, vol. 20, no. 4, pp. 31–37 (in Russian). https://doi.org/10.15407/pmach2017.04.031
10. Samadi, N, Abolbashari, M. H., & Ghaffarianjam H. R. (2017). An effective approach for optimal hole shape with evolutionary structural optimization [Retrieved from https://search.informit.com.au/documentSummary;dn=389813149728265;res=IELENG]. In the 9th Australasian Congress on Applied Mechanics (ACAM9). Sydney: Engineers Australia, [1]–[8].
11. Wang, S. J., Lu, A. Z., Zhang, X. L., & Zhang, N. (2018). Shape optimization of the hole in an orthotropic plate. *Mechanics Based Design of Structures and Machines*, vol. 46, iss. 1, pp. 23–37. https://doi.org/10.1080/15397734.2016.126103623
12. Vigdergauz, S. (2018). Simply and doubly periodic arrangements of the equi-stress holes in a perforated elastic plane: The single-layer potential approach. *Mathematics and Mechanics of Solids*, vol. 23, iss. 5, pp. 805–819. https://doi.org/10.1177/1081286517691807.
13. Mirsalimov, V. M. (2019). Maksimalnaya prochnost vyrabotki v gornom massive, oslablennom treshchinoy [Maximum strength of a working in a solid rock weakened by a crack]. *Fiziko-tekhnicheskiye problemy razrabotki poleznykh iskopayemykh – Journal of Mining Science*, vol. 55, iss. 1, pp. 12–21. https://doi.org/10.15372/FTPRPI20190102
14. Mirsalimov, V. M. (2019). Inverse problem of elasticity for a plate weakened by hole and cracks. *Mathematical Problems in Engineering*, vol. 2019, Article ID 4931489, 11 pages. https://doi.org/10.1155/2019/4931489
15. Mirsalimov, V. M. (2019). Minimizing the stressed state of a plate with a hole and cracks. *Engineering Optimization*. https://doi.org/10.1080/0305215X.2019.1584619
16. Muskhelishvili, N. I. (1977). Some basic problem of mathematical theory of elasticity. Dordrecht: Springer, 732 p. https://doi.org/10.1007/978-94-017-3034-1
17. Kalandija, A. I. (1973). *Matematicheskiye metody dvumernoy uprugosti* [Mathematical methods of two-dimensional elasticity]. Moscow: Nauka, 304 p. (in Russian).
18. Panasyuk, V. V., Savruk, M. P., & Datsyshin, A. P. (1976). *Raspredeleniye napryazheniy okolo treshchin v plastinakh i obolochkakh* [Stress distribution around cracks in plates and shells]. Kiev: Naukova Dumka, 443 p. (in Russian).
19. Mirsalimov, V. M. (1986). Some problems of structural arrest of cracks. *Physicochemical Mechanics of Materials*, vol. 22, iss. 1, pp. 81–85. https://doi.org/10.1007/BF00720871

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Мінімізація напруженого стану стрингерної пластини з отвором й прямолінійними тріщинами

Мір-Салім-заде М. В.

Інститут математики і механіки НАН Азербайджану,
Азербайджан, AZ1141, Баку, вул. Б. Вахабзаде, 9

Як відомо, тонкі пластини з отворами є одним з широко поширених елементів конструкцій. Для підвищення надійності і терміну служби становить інтерес знаходження такого контуру отвору, який забезпечує мінімальне окружне напруження на контурі отвору, а також перешкоджає росту можливих тріщин у пластині. У цій статті *розглядається задача мінімізації напруженого стану на контурі отвору в необмеженій ізотропній стрингерній пластині, ослабленій двома прямолінійними тріщинами. Береги тріщин вважаються вільними від навантажень. Визначається оптимальна форма отвору, така, що зростання тріщин не відбувається, а максимальне окружне напруження на контурі мінімальне. Використовується мінімаксний критерій. За параметр, що характеризує напружений стан в околі вершин тріщин, відповідно до теорії квазікрихкого руйнування Ірвіна-Орована приймається коефіцієнт інтенсивності напружень. Пластина піддається на нескінченності однорідному розтягуванню уздовж стрингерів. Вважається, що пластина і стрингери виконані з різних пружних матеріалів. Дія стрингерів замінюється невідомими еквівалентними зосередженими силами, прикладеними в точках їхнього з'єднання з пластиною. Для їх визначення використовується закон Гука. Застосувавши метод малого параметра, теорію аналітичних функцій і метод прямого розв'язання сингулярних рівнянь, була побудована замкнута система алгебраїчних рівнянь, що забезпечує в залежності від механічних і геометричних параметрів пластини та стрингерів мінімізацію напруженого стану на контурі отвору і рівність нулю коефіцієнтів інтенсивності напружень в околі вершин тріщин. Поставлена задача мінімізації зводиться до задачі лінійного програмування. Застосовано метод симплексного алгоритму.*

Ключові слова: *стрингерна пластина, мінімізація напруженого стану, тріщини, оптимальна форма отвору, мінімаксний критерій.*

Література

1. Waldman W., Heller M. Shape optimisation of holes for multi-peak stress minimization. *Australian J. Mech. Eng*. 2006. Vol. 3. Iss. 1. P. 61–71. https://doi.org/10.1080/14484846.2006.11464495
2. Vigdergauz S. The stress-minimizing hole in an elastic plate under remote shear. *J. of Mech. Materials and Structures*. 2006. Vol. 1. No. 2. P. 387–406. https://doi.org/10.2140/jomms.2006.1.387
3. Мир-Салим-заде М. В. Обратная упругопластическая задача для клепаной перфорированной пластины. Совр. проблемы прочности, пластичности и устойчивости: сб. статей. Тверь: Тверск. ун-т, 2007. С. 238–246.
4. Bantsuri R., Mzhavanadze Sh. The mixed problem of the theory of elasticity for a rectangle weakened by unknown equi-strong holes. *Proc. of A. Razmadze Math. Institute*. 2007. Vol. 145. P. 23–34.
5. Мир-Салим-заде М. В. Определение формы равнопрочного отверстия в изотропной среде, усиленной регулярной системой стрингеров. *Материалы, технологии, инструменты*. 2007. Т. 12. №4. С. 10–14.
6. Vigdergauz S. Energy-minimizing openings around a fixed hole in an elastic plate. *J. of Mech. Materials and Structures*. 2010. Vol. 5. No. 4. P. 661–677. https://doi.org/10.2140/jomms.2010.5.661
7. Vigdergauz S. Stress-smoothing holes in an elastic plate: From the square lattice to the checkerboard. *Math. and Mech. Solids*. 2012. Vol. 17. Iss. 3. P. 289–299. https://doi.org/10.1177/1081286511411571
8. Cherepanov G. P. Optimum shapes of elastic bodies: equistrong wings of aircrafts and equistrong underground tunnels. *Phys. Mesomechanics*. 2015. Vol. 18. Iss. 4. P. 391–401. https://doi.org/10.1134/S1029959915040116.
9. Калантарлы Н. М. Равнопрочная форма отверстия для торможения роста трещины продольного сдвига. *Проблемы машиностроения*. 2017. Т. 20. №. 4. С. 31–37. https://doi.org/10.15407/pmach2017.04.031
10. Samadi N, Abolbashari M. H., Ghaffarianjam H. R. An effective approach for optimal hole shape with evolutionary structural optimization [online]. 9th Australasian Congress on Appl. Mechanics (ACAM9). Sydney: Engineers Australia, 2017. P. 1–8.
11. Wang S. J., Lu A. Z., Zhang X. L., Zhang N. Shape optimization of the hole in an orthotropic plate. *Mechanics Based Design of Structures and Machines*. 2018. Vol. 46. Iss. 1. P. 23–37. https://doi.org/10.1080/15397734.2016.126103623
12. Vigdergauz S. Simply and doubly periodic arrangements of the equi-stress holes in a perforated elastic plane: The single-layer potential approach. *Math. and Mech. Solids*. 2018. Vol. 23. Iss. 5. P. 805–819. https://doi.org/10.1177/1081286517691807.
13. Мирсалимов В. М. Максимальная прочность выработки в горном массиве, ослабленном трещиной. *Физико-техн. проблемы разработки полезных ископаемых*. 2019. Т. 55. №1. С. 12–21. https://doi.org/10.15372/FTPRPI20190102
14. Mirsalimov V. M. Inverse problem of elasticity for a plate weakened by hole and cracks. *Math. Problems in Eng*. Vol. 2019. Article ID 4931489, 11 pages. https://doi.org/10.1155/2019/4931489
15. Mirsalimov V. M. Minimizing the stressed state of a plate with a hole and cracks. *Engineering Optimization*. 2019. https://doi.org/10.1080/0305215X.2019.1584619.
16. Мусхелишвили Н. И. Некоторые основные задачи математической теории упругости. М.: Наука, 1966. 707 с.
17. Каландия А. И. Математические методы двумерной упругости. М.: Наука. 1973. 304 с.
18. Панасюк В. В., Саврук М. П., Дацышин А. П. Распределение напряжений около трещин в пластинах и оболочках. Киев: Наук. думка, 1976. 443 с.
19. Мирсалимов В. М. Некоторые задачи конструкционного торможения трещины. *Физико-хим. механика материалов*. 1986. Т. 22. № 1. С. 84–88.