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ASYMPTOTIC ANALYSIS ON STEADY-STATE RESPONSE OF AXIALLY ACCELERATING BEAM CONSTITUTED BY THE STANDARD LINEAR SOLID MODEL

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Abstract: The transverse bending vibrations of an axially accelerating viscoelastic beam are studied. A material of beam is constituted by the standard viscoelastic model. The method of multiple scales is applied to determine the steady-state response of beam. The numerical examples illustrate the asymptotic solution.

Key words: viscoelastic beam, axial acceleration, standard viscoelastic model, steady-state response of beam.

1, Introduction.

Due to their technological importance, the vibrations of axially moving beams have been investigated by many researchers. Transverse parametric vibration of axially accelerating elastic beams has been extensively analyzed since first study by Pasin [1]. Wickert and Mote reviewed the literature on axially moving materials [2]. Öz and Pakdemirli [3] employed the method of multiple scales to study dynamic stability of an axially accelerating beam with small bending stiffness. In addition to elastic beams, axially accelerating viscoelastic beams have recently been investigated. Chen and Yang [4] applied the method of multiple scales to obtain analytically the stability boundaries. Yang and Chen [5] presents vibration and stability of an axially moving beam constituted by the viscoelastic constitutive law of an integral type. Chen and Yang [6] presents analytically vibration and stability of an axially moving beam constrained by simple supports with rotational springs. Chen and Wang [7] presented the comparison between the analytical and the numerical results parametric resonances via the differential quadrature method. In Ref. [4 – 7], the Kelvin model containing the partial time derivative was used to describe the viscoelastic behavior of beam materials. As parametric vibration excited by the variation of the axial tension or axial velocity, large transverse motion of axially moving beams may occur under certain conditions. Geometrical nonlinearity produced due to axially stretching of beam, can't be neglected when large transverse displacement takes place. Chen and Yang investigated the steady-state response and their stability of two nonlinear models of axially moving viscoelastic beams [8]. Chen and Ding investigates steady-state periodical response for planar vibration of axially moving viscoelastic beams with two nonlinear models subjected external transverse loads [9]. Compared with Kelvin model, standard linear solid model is more typical and representative, mean while this model can degenerate to the Kevin or Maxwell model by varying alternative of the stiffness of beam. Wang and Chen investigated the linear stability of axially accelerating beams via asymptotic analysis [10]. In present investigation, the standard linear solid model viscoelastic material is developed for two nonlinear models of axially accelerating viscoelastic beams. The method of multiple scales is applied to solution of governing equation and steady-state response of nonlinear axially moving beams.

2. Equation of motion.

A uniform axially moving viscoelastic beam, with density ρ, cross-sectional area A, and initial tension P, travels at time-dependent axial velocity $\gamma(t)$ between two transversely motionless ends separated by distance l. Consider only the bending vibration of beam in a reference frame described by the transverse displacement $v(x, t)$, where t is the time and x is the axial coordinate. The equation of motion in the transverse direction can be derived from Newton' second law as

$$\rho A \frac{d^2 v}{dt^2} + M,_{xx} = \frac{\partial}{\partial x}\left[\left(P + A\sigma\right) v,_x\right], \tag{1}$$

where $(\cdot),_x$ denotes partial differentiation with respect to x. $\sigma(x, t)$ is the disturbing stress, and the bending moment $M(x, t)$ is defined by

$$M\left(x,t\right) = -\int_A z\sigma\left(x,\, z,\, t\right) dA\,, \tag{2}$$

where the z-x plane is regarded as the principal plane of bending, and $\sigma\,(x, z, t)$ is the normal stress. The standard linear solid model is adopted to describe the viscoelastic property of the material of beam. The stress-strain relationship of the model is expressed in a differential form as [11 – 14]

$$\left(E_1 + E_2\right)\sigma\left(x,\, z,\, t\right) + \eta\frac{d}{dt}\sigma\left(x,\, z,\, t\right) = E_1E_2\varepsilon\left(x,\, z,\, t\right) + E_1\eta\frac{d}{dt}\varepsilon\left(x,\, z,\, t\right), \tag{3}$$

where both E_1 and E_2 are stiffness constants, $\varepsilon(x, z, t)$ is the axial strain, and η is viscous damping. The standard linear solid model can be employed to describe the behavior of linear viscoelastic materials of solid type with limited creep deformation. It can reduce to Kelvin model ($E_1 \to \infty$ and $E_2 \neq 0$) or Maxwell model ($E_1 \neq 0$ and $E_2 = 0$). For small deflections, the strain-displacement relation is

$$\varepsilon\left(x,z,t\right) = -z\frac{\partial^2 v\left(x,t\right)}{\partial x^2}\,. \tag{4}$$

Lagrangian strain is employed as a finite measure to account for geometric nonlinearity due to small but finite stretching of beam. For one-dimensional problems, the disturbing stress $\sigma\,(x, t)$ in equation (1) is still described by the standard linear solid model

$$\left(E_1 + E_2\right)\sigma\left(x,\, t\right) + \eta\frac{d}{dt}\sigma\left(x,\, t\right) = E_1E_2\varepsilon_L\left(x,\, t\right) + E_1\eta\frac{d}{dt}\varepsilon_L\left(x,\, t\right), \tag{5}$$

where $\varepsilon L(x, t)$ is the Lagrangian strain,

$$\varepsilon_L = \frac{1}{2} v,_x^2\,. \tag{6}$$

Introduce the material time derivative by defining differential operator $d\,/\,dt$ as

$$\frac{d}{dt} \leftrightarrow \frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial x}\,. \tag{7}$$

Inserting Eq. (7) in (1), (3) and (5) yields

$$\rho A\left(v,_{tt} + \dot{\gamma}v,_x + 2\gamma v,_{xt} + \gamma^2 v,_{xx}\right) + M,_{xx} = \left[\left(P + A\sigma\right)v,_x\right],_x\,; \tag{8}$$

$$\left(E_1 + E_2\right)\sigma + \eta\left(\sigma,_t + \gamma\sigma,_x\right) = E_1\left[E_2\varepsilon + \eta\left(\varepsilon,_t + \gamma\varepsilon,_x\right)\right]; \tag{9}$$

$$\left(E_1+E_2\right)\sigma+\eta\left(\sigma,_t+\gamma\sigma,_x\right)=E_1\left[E_2\varepsilon_{\mathrm{L}}+\eta\left(\varepsilon_{\mathrm{L}},_t+\gamma\varepsilon_{\mathrm{L}},_x\right)\right]. \tag{10}$$

Eqs. (8) – (10) are the governing equation of an axially moving viscoelastic beam. If the spatial variation of the tension is rather small compared with the initial tension, the exact form of the disturbing tension $A\sigma$ can be replaced by its spatially averaged value $\frac{1}{l}\int_0^l A\sigma \mathrm{d}x$ [15, 16]. Then Eq. (8) leads to

$$\rho A\left(v,_{tt}+\dot{\gamma}v,_x+2\gamma v,_{xt}+\gamma^2 v,_{xx}\right)+M,_{xx}=\left(P+\frac{1}{l}\int_0^l A\sigma \mathrm{d}x\right)v,_{xx}. \tag{11}$$

Let the dimensionless variables be:

$$v\leftrightarrow\frac{v}{\sqrt{\varepsilon}l};\ x\leftrightarrow\frac{x}{l};\ t\leftrightarrow t\sqrt{\frac{P}{\rho Al^2}};c=\gamma\sqrt{\frac{\rho A}{P}};$$

$$\varsigma(x,t)=\frac{1}{Pl}\int_{\mathrm{A}}z\sigma(x,z,t)\mathrm{d}A;\ \zeta(x,t)=\frac{A\sigma}{\varepsilon P};\ \alpha=\frac{\eta}{\left(E_1+E_2\right)}\sqrt{\frac{P}{\rho Al^2}}; \tag{12}$$

$$E_a=\frac{IE_1E_2}{Pl^2\left(E_1+E_2\right)};\ E_b=\frac{IE_1}{Pl^2};\ E_c=\frac{E_1E_2A}{P\left(E_1+E_2\right)};\ E_d=\frac{E_1A}{P},$$

where I is the moment of inertial, book keeping device ε is a small dimensionless parameter accounting for the fact that the transverse displacement and the viscous damping is very small. Eqs. (8) – (10) can be respectively cast into the dimensionless form of governing equation

$$v,_{tt}+\dot{c}v,_x+2cv,_{xt}+\left(c^2-1\right)v,_{xx}-\varsigma,_{xx}=\varepsilon\left(\zeta v,_x\right),_x; \tag{13}$$

$$\varsigma+\varepsilon\alpha\varsigma,_t+\varepsilon\alpha c\varsigma,_x=-E_a v,_{xx}-\varepsilon\alpha E_b\left(v,_{xxt}+cv,_{xxx}\right); \tag{14}$$

$$\zeta+\varepsilon\alpha\zeta,_t+\varepsilon\alpha c\zeta,_x=\frac{1}{2}E_c v,_x^2+\frac{1}{2}\varepsilon\alpha E_d\left[\left(v,_x^2\right),_t+c\left(v,_x^2\right),_x\right]. \tag{15}$$

We suppose that the beam is constrained at both ends by simple supports with rotational springs whose stiffness constants are k respectively, the boundary conditions are expressed in dimensionless form as follows then:

$$v(0,t)=0;\ v,_{xx}(0,t)-kv,_x(0,\ t)=0;\ v(1,\ t)=0;\ v,_{xx}(1,t)+kv,_x(1,\ t)=0. \tag{16}$$

In the present investigation, the axial velocity is assumed to be a small simple harmonic variation about the constant mean speed

$$c(t)=c_0+\varepsilon c_1\sin\omega t, \tag{17}$$

where c_0 is the constant mean speed, and $\varepsilon c1$ and ω are respectively the disturbed amplitude and the excitation frequency, all in the dimensionless form. Here the bookkeeping device ε is used to indicate the fact that disturbed amplitude is small, with the same order as the dimensionless viscosity.

Substitution of Eqs. (14), (15) and (17) in Eq. (13) and neglect of higher order ε terms in the resulting equation yield the following nonlinear partial-differential equation

$$Mv,_{tt}+Gv,_t+Kv=\frac{3}{2}\varepsilon E_c v,_x^2 v,_{xx}-\varepsilon\left[c_1\omega v,_x\cos\omega t+2c_1\sin\omega t\left(v,_{xt}+c_0v,_{xx}\right)+\right.$$

$$+\alpha\left(E_b-E_a\right)\left(v,_{xxxxt}+c_0 v,_{xxxxx}\right)\Big]+O\left(\varepsilon^2\right), \tag{18}$$

where the mass, gyroscopic, and linear stiffness operators are, respectively, defined as follows:

$$M=I;\ G=2c_0\frac{\partial}{\partial x};\ \ K=\left({c_0}^2-1\right)\frac{\partial^2}{\partial x^2}+E_a\frac{\partial^4}{\partial x^4}. \tag{19}$$

Resting on Eq. (11), the nonlinear integro-partial-differential equation below can be cast into by means of the previous similar procedures.

$$\begin{gathered}Mv,_{tt}+Gv,_t+Kv=\frac{1}{2}\varepsilon E_c v,_{xx}\int_0^1 v,_x^2\,\mathrm{d}\,x-\varepsilon\Big[c_1\omega v,_x\cos\omega t+2c_1\sin\omega t\left(v,_{xt}+c_0 v,_{xx}\right)+\\ +\alpha\left(E_b-E_a\right)\left(v,_{xxxxt}+c_0 v,_{xxxxx}\right)\Big]+O\left(\varepsilon^2\right).\end{gathered} \tag{20}$$

3. Multi-scale procedure with numerical examples.

Both equations (18) and (20) are regarded as gyroscopic continuous system with weakly nonlinear and parametric disturbances. Using the method of multiple scales, defining a slow time scale $T=\varepsilon\, t$, and looking for an asymptotic solution of the form

$$v\left(x,t;\varepsilon\right)=v_0\left(x,t,T\right)+\varepsilon v_1\left(x,t,T\right)+O\left(\varepsilon^2\right). \tag{21}$$

Substitution of Eq. (21) in Eq. (18), equating coefficients of each power of ε to zero, we obtain

$$Mv_0,_{tt}+Gv_0,_t+Kv_0=0\,; \tag{22}$$

$$\begin{gathered}Mv_1,_{tt}+Gv_1,_t+Kv_1=-\gamma_1\omega\cos\omega t v_0,_x-2v_0,_{tT}-2\gamma_0 v_0,_{xT}-2\gamma_1\sin\omega t\left(v_0,_{xt}+\gamma_0 v_0,_{xx}\right)+\\ +\frac{3}{2}E_c\left(v_0,_x\right)^2 v_0,_{xx}+\alpha\left(E_a-E_b\right)\left(v_0,_{xxxxt}+\gamma_0 v_0,_{xxxxx}\right).\end{gathered} \tag{23}$$

We assume that the system is in summation parametric resonance and in order to express the nearness of the excitation frequency to sum of arbitrary two natural frequencies, and define a detuning parameter μ through the relation

$$\omega=\omega_n+\omega_m+\varepsilon\mu\,, \tag{24}$$

where ω_m and ω_n are respectively the mth and nth natural frequencies of the undisturbed gyroscopic continuous system (22) under boundary condition (16).

Wickert and Mote [2] have obtained the solution to Eq. (22)

$$v_0\left(x,\,t,\,T\right)=\phi_m\left(x\right)A_m\left(T\right)\mathrm{e}^{\mathrm{i}\omega_m t}+\phi_n\left(x\right)A_n\left(T\right)\mathrm{e}^{\mathrm{i}\omega_n t}+cc\,, \tag{25}$$

where cc stands for the complex conjugate of all preceding terms on the right hand of an equation. The expressions of modal functions $\phi m(x)$ and $\phi n(x)$ had given in Ref. [3].

Substituting Eqs. (24) and (25) into Eq. (23), we obtain

$$\begin{gathered}Mv_1,_{tt}+Gv_1,_t+Kv_1=\Big[-2\left(i\omega_m\phi_m+\gamma_0\phi_m'\right)\dot{A}_m+\frac{1}{2}e^{iT\mu}\gamma_1\overline{A}_n\overline{\phi}_n'\left(\omega_n-\omega_m\right)+ie^{iT\mu}\gamma_0\gamma_1\overline{A}_n\overline{\phi}_n''+\\ +\frac{3}{2}E_cA_m^2\overline{A}_m\left(2\phi_m'\overline{\phi}_m'\phi_m''+\phi_m'^2\overline{\phi}_m''\right)+3E_cA_m\left|A_n\right|^2\left(\phi_n'\overline{\phi}_n'\phi_m''+\phi_m'\overline{\phi}_n'\phi_n''+\phi_m'\phi_n'\overline{\phi}_n''\right)+\\ +\alpha A_m\left(E_a-E_b\right)\left(i\omega_m\phi_m^{(4)}+\gamma_0\phi_m^{(5)}\right)\Big]e^{i\omega_m t}+\Big[-2\left(i\omega_n\phi_n+\gamma_0\phi_n'\right)\dot{A}_n+i\gamma_0\gamma_1\overline{A}_m\overline{\phi}_m''e^{iT\mu}+\\ +\frac{3}{2}E_cA_n^2\overline{A}_n\left(2\phi_n'\overline{\phi}_n'\phi_n''+\phi_n'^2\overline{\phi}_n''\right)+3E_cA_n\left|A_m\right|^2\left(\phi_n'\overline{\phi}_m'\phi_m''+\phi_m'\overline{\phi}_m'\phi_n''+\phi_m'\phi_n'\overline{\phi}_m''\right)+\end{gathered} \tag{26}$$

$$+\frac{1}{2}e^{iT\mu}\gamma_1\bar{A}_m\bar{\phi}'_m\left(\omega_m-\omega_n\right)+\alpha A_n\left(E_a-E_b\right)\left(i\omega_n\phi_n^{(4)}+\gamma_0\phi_n^{(5)}\right)\Big]e^{i\omega_n t},$$

where the dot and the prime denote differentiation with respect to T and x respectively, and *NST* stands for the terms that will not bring secular terms into the solution. Eq. (26) has a bounded solution only if a solvability condition holds. The solvability condition demands the following orthogonal relationships [17,18]

$$\begin{aligned}&\Big\langle -2\dot{A}_m\left(i\omega_m\phi_m+\gamma_0\phi'_m\right)+\frac{1}{2}\bar{A}_n\gamma_1 e^{iT\mu}\left[\bar{\phi}'_n\left(\omega_n-\omega_m\right)+2i\gamma_0\bar{\phi}''_n\right]+\frac{3}{2}E_c A_m^2\bar{A}_m\left(2\phi'_m\bar{\phi}'_m\phi''_m+\phi'^2_m\bar{\phi}''_m\right)+\\&+3E_c A_m\left|A_n\right|^2\left(\phi'_n\bar{\phi}'_n\phi''_m+\phi'_m\bar{\phi}'_n\phi''_n+\phi'_m\phi'_n\bar{\phi}''_n\right)+\alpha A_m\left(E_a-E_b\right)\left(i\omega_m\phi_m^{(4)}+\gamma_0\phi_m^{(5)}\right),\phi_m\Big\rangle=0;\\&\Big\langle -2\dot{A}_n\left(i\omega_n\phi_n+\gamma_0\phi'_n\right)+\frac{1}{2}\bar{A}_m\gamma_1 e^{iT\mu}\left[\bar{\phi}'_m\left(\omega_m-\omega_n\right)+2i\gamma_0\bar{\phi}''_m\right]+\frac{3}{2}E_c A_n^2\bar{A}_n\left(2\phi'_n\bar{\phi}'_n\phi''_n+\phi'^2_n\bar{\phi}''_n\right)+\\&+3E_c A_n\left|A_m\right|^2\left(\phi'_n\bar{\phi}'_m\phi''_m+\phi'_m\bar{\phi}'_m\phi''_n+\phi'_m\phi'_n\bar{\phi}''_m\right)+\alpha A_n\left(E_a-E_b\right)\left(i\omega_n\phi_n^{(4)}+\gamma_0\phi_n^{(5)}\right),\phi_n\Big\rangle=0.\end{aligned}\tag{27}$$

Substituting the polar form

$$A_m=a_m e^{i\beta_m};\;\; A_n=a_n e^{i\beta_n}\tag{28}$$

into Eq. (27), where ak (T) and $\beta k(T)$ ($k = m$, n) are respectively the amplitude and phase of the steady-state response in the summation parametric resonance. By the means of separating real and imaginary parts of the resulting equations, we arrive at the relationship between $ak(T)$ and μ to determine steady-state response of nonlinear system.

Specify system parameters of an axially moving beam with E_a=0,64, c_0=2,0 and k = 2 under boundary condition (16). The first two natural frequencies of undisturbed system (22) are ω_1=8,1570 and ω_2=32,9441. Fig. 1 depicts the relationship between the amplitude and the detuning parameter in the summation parametric resonance, in which the solid and dashed lines stand for stable and unstable amplitudes, respectively. In the figures, c_1=0,2, E_c=400, α=0,0001 and E_1 =E_2=3. Eq. (12) gives $E_b = E_a (E_1 + E_2) / E_2$ = 1,28.

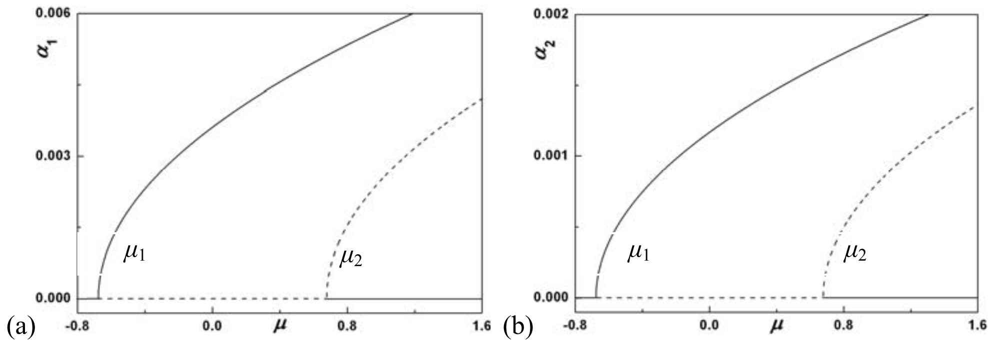
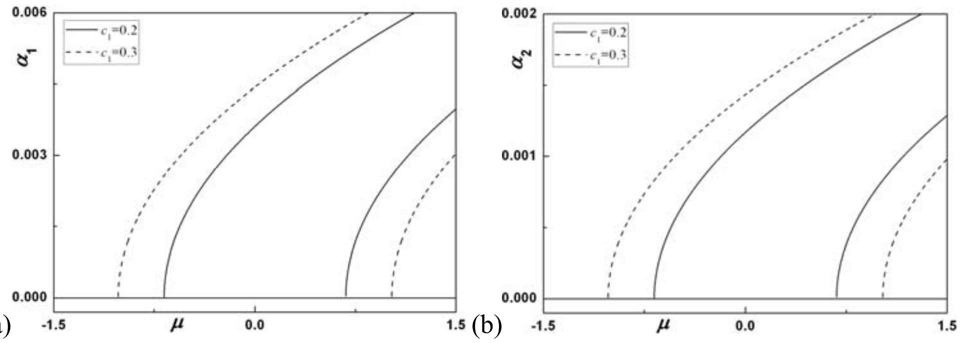


Fig. 1. The amplitude and the detuning parameter relationship in the summation resonance: (*a*) the first order and (*b*) the second order.

In the summation resonance, only the trivial solution exists and is stable for $\mu<\mu_1$. At $\mu=\mu_1$ the trivial solution losses its stability and a stable nontrivial solution occurs. At $\mu=\mu_2$ the unstable trivial solution becomes stable again, and an unstable nontrivial solution bifurcates.

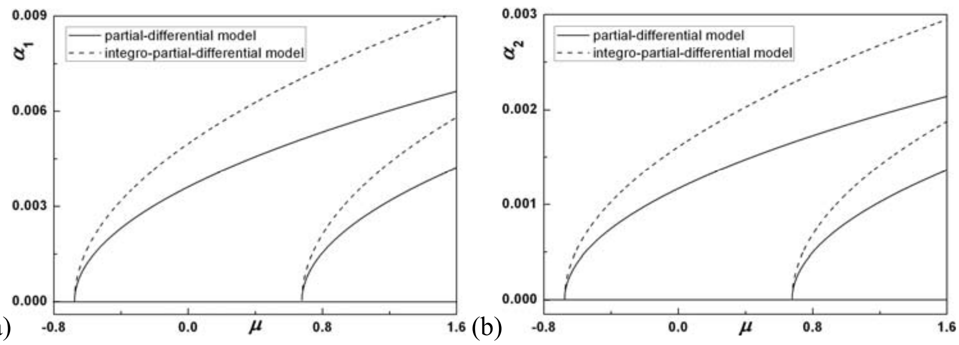
The instability interval in the first order is larger than in the second, which indicates that effect of the low order is more significant.

(a) 

(b)

Fig. 2. The effect of the axial velocity variation amplitude in the summation resonance: (*a*) the first order and (*b*) the second order.

Fig. 2 illustrates the effect of the axial velocity variation amplitude in the summation resonance, and the increasing velocity variation amplitude leads to the larger instability interval. Fig. 3 show the comparison of the two nonlinear models, and the nontrivial solution amplitude of partial-differential equation model is smaller, but both of them possess the same instability intervals and changing trend with related parameters.

(a) 

(b)

Fig. 3. Comparison of two nonlinear models in the summation resonance: (*a*) the first order and (*b*) the second order.

4. Conclusions.

With conclusions: (1) There exists an instability interval of the detuning on which straight equilibrium is unstable, and the instability interval of lower order resonance is more than the higher order one. The first nontrivial solution is always stable but the second one. (2) Increasing velocity variation amplitude lead to the larger instability interval. (3) By the means of the comparison of the two nonlinear models, both of them have the same tendencies and instability intervals with changing relevant parameters. Besides partial-differential equation model has smaller amplitude of nontrivial steady-state response.

Р Е З Ю М Е . Вивчено поперечні згинні коливання в'язкопружної балки яка допускає рухи вздовж осі. Матеріал балки припускається деформівним за стандартною триконстантною моделлю теорії в'язкопружності. Для визначення стаціонарної реакції балки застосовано метод багатьох масштабів. Числові приклади ілюструють асимптотичний розв'язок.

1. *Pasin F.* Ueber die Stabilität der Beigeschwingungen von in Laengsrichtung periodisch hin und herbewegten Stäben // Ingenieur-Archiv. – 1972. – **41**, N 6. – P. 387 – 393.

2. *Wickert J. A., Mote C.D.Jr.* Current Research on the Vibration and Stability of Axially Moving Materials // Shock and Vibration Digest. – 1988. – **20**, N 2. – P. 3 –13.
3. *Öz H.R., Pakdemirli M.* Vibrations of an axially moving beam with time dependent velocity // J. Sound and Vibration. – 1999. – **227,** N 2. – P. 239 –257.
4. *Chen L.Q., Yang X.D.* Stability in parametric resonances of an axially moving viscoelastic beam with time-dependent velocity // J. Sound and Vibration. – 2005. – **284**, N 3 – 5. – P.879 – 891.
5. *Yang X.D., Chen L.Q.* Stability in parametric resonance of axially accelerating beams constituted by Boltzmann's superposition principle // J. Sound and Vibration. – 2006. – **289,** N1 – 2. – P. 54 – 65.
6. *Chen L.Q., Yang X.D.* Vibration and stability of an axially moving viscoelastic beam with hybrid supports // Eur. J. Mech. A/Solid. – 2006. – **25**, N 6. – P. 996 – 1008.
7. *Chen L.Q., Wang B.* Stability of axially accelerating viscoelastic beams: asymptotic perturbation analysis and differential quadrature validation // Eur. J. Mech. A/Solid. – 2009. – **28**, N 4. – P. 786 – 791.
8. *Chen L.Q., Yang X.D.* Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models // Int. J. Solids Struct. – 2005. – **42,** N 1. – P. 37 – 50.
9. *Chen L.Q., Ding H.* Steady-State Transverse Response in Coupled Planar Vibration of Axially Moving Viscoelastic Beams // J. Vib. Acoust. – 2010. – **132**, N 1. – P. 256 – 280.
10. *Wang B., Chen L.Q.* Asymptotic stability analysis with numerical confirmation of an axially accelerating beam constituted by the standard linear solid model // J. Sound and Vibration. – 2009. – **328**, N 2. – P. 456 – 466.
11. *Hou Z., Zu J.W.* Non-linear free oscillations of moving viscoelastic belts // Mech. Mach. Theory. – 2002. – **37**, N 9. – P. 925 – 940.
12. *Fung R.F., Huang J.S., Chen Y.C., Yao C.M.* Nonlinear dynamic analysis of the viscoelastic string with a harmonically varying transport speed // Comput . Struct. – 1998. – **66**, N 6. – P. 777 – 784.
13. *Ha J.L., Chang J.R. Fung R.F.* Nonlinear dynamic behavior of a moving viscoelastic string undergoing three-dimensional vibration // Chaos, Solitons and Fractals. – 2007. – **33**, N 4. – P. 1117 – 1134.
14. *Chen L.Q., Chen H.* Asymptotic analysis on nonlinear vibration of axially accelerating viscoelastic strings with the standard linear solid model // J. Eng. Math. – 2010. – **67**, N 3. – P. 205 – 218.
15. *Chen L.Q., Ding H.* Steady-state responses of axially accelerating viscoelastic beams: approximate analysis and numerical confirmation // Sci. China Ser. G-Phys Mech. Astron. – 2008. – **51**, N 11. – P. 1707 – 1721.
16. *Ding H., Chen L.Q.* On two transverse nonlinear models of axially moving beams // Sci. China Ser. E-Tech Sci. – 2009. – **52**, N 3. – P. 743 – 751.
17. *Chen L.Q., Lim C.W., Hu Q.Q., Ding H.* Asymptotic analysis of a vibrating cantilever with a nonlinear boundary // J. Appl. Mech. – 2009. – **52,** N 9. – P. 1414 – 1422.
18. *Chen L.Q., Zu J.W.* Solvability condition in multi-scale analysis of gyroscopic continua // J. Sound and Vibration. – 2008. – **309**, N1 – 2. – P. 338 – 342.

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