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TWO-LEVEL ALGORITHMS FOR RANNACHER-TUREK FEM

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In this paper a multiplicative two-level preconditioning algorithm for second order elliptic boundary value problems is considered, where the discretization is done using Rannacher-Turek non-conforming rotated bilinear finite elements on quadrilaterals. An important point to make is that in this case the finite element spaces corresponding to two successive levels of mesh refinement are not nested in general. To handle this, a proper two-level basis is required to enable us to fit the general framework for the construction of two-level preconditioners originally introduced for conforming finite elements. The proposed variant of hierarchical two-level splitting is first defined in a rather general setting. Then, the involved parameters are studied and optimized. The major contribution of the paper is the derived uniform estimates of the constant in the strengthened CBS inequality which allow the efficient multilevel extension of the related two-level preconditioners.

Introduction

In this paper we consider the elliptic boundary value problem

$$\begin{aligned} Lu \equiv -\nabla \cdot \big(\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})\big) &= f(\mathbf{x}) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma_D, \\ (\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x}))\cdot n &= 0 \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{1}$$

where Ω is a convex polygonal domain in, $\mathbf{R}^2$ $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, the coefficient matrix $a(x)$ is symmetric positive definite and uniformly bounded in Ω, $\mathbf{n}$ is the outward unit vector normal to the boundary $\Gamma = \partial\Omega$, and $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We assume that the elements of the diffusion coefficient matrix $\mathbf{a}(\mathbf{x})$ are a piece-wise smooth functions on $\overline{\Omega}$. The weak formulation of the above problem reads as follows: given $f \in L^2(\Omega)$ find $u \in V \equiv H_D^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$A(u,v) = (f,v) \quad \forall v \in \mathrm{H}_\mathrm{D}^1(\Omega \quad \text{where} \quad A(u,v) = \int_\Omega \mathbf{a}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}. \tag{2}$$

We assume that the domain Ω is discretized by the partition T_h which is obtained by a proper refinement of a given coarser partition T_H. We assume also that T_H is aligned with the discontinuities of the coefficient $\mathbf{a}(\mathbf{x})$ so that over each element $E \in T_H$ the coefficients $\mathbf{a}(\mathbf{x})$ of are smooth functions. The variational problem (2) is then discretized using the finite element method, i.e., the continuous space V is replaced by a finite dimensional subspace V_h. Then the finite element formulation is: find $u_h \in V_h$, satisfying

$$A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h \quad \text{where} \quad A_h(u_h, v_h) = \sum_{e \in T_h} \int_e \mathbf{a}(e)\nabla u_h \cdot \nabla v_h d\mathbf{x}. \tag{3}$$

Here $\mathbf{a}(e)$ is a piece-wise constant symmetric positive definite matrix, defined by the integral averaged values of $\mathbf{a}(\mathbf{x})$ over each element from the coarser triangulation T_h. We note that in this way strong coefficient jumps across the boundaries between adjacent finite elements from T_h are allowed. The resulting discrete problem to be solved is then a linear system of equations $A_h \mathbf{u}_h = \mathbf{f}_h$, with A and $\mathbf{f}_h$ being the corresponding global stiffness matrix and global right hand side, and h being the discretization (meshsize) parameter for the underlying partition T_h of Ω.

The two-level setting. We are concerned with the construction of a two-level preconditioner M for A_h, such that the spectral condition number $\kappa(M^{-1}A_h)$ of the preconditioned matrix $M^{-1}A_h$ is uniformly bounded with respect to the meshsize parameter h, and the possible coefficient jumps. The classical theory for constructing optimal order two-level preconditioners was first developed in [2, 3], see also [1]. The general framework requires to define two nested finite element spaces $V_H \subset V_h$, that correspond to two consecutive (regular) mesh refinements. The well studied case of conforming linear finite elements is the starting point in the theory of two-level and multi-level methods. Let T_h and T_H be two successive mesh refinements of the domain Ω, which correspond to V_H and V_h. Let $\{\varphi_H^{(k)}, k = 1,2,\cdots, N_H\}$ and $\{\varphi_h^{(k)}, k = 1,2,\cdots, N_h\}$ be the standard finite element nodal basis functions. We split the meshpoints $\mathbf{N}_h$ from T_h into two groups: the first group contains the nodes $\mathbf{N}_H$ from T_H and the second one consists of the rest, where the latter are the newly added node-points $\mathbf{N}_{h\setminus H}$ from $T_h \setminus T_H$. Next we define the so-called hierarchical basis functions

$$\{\tilde{\varphi}_h^{(k)}, k = 1,2,\cdots, N_h\} = \{\tilde{\varphi}_H^{(l)} \text{ on } T_H\} \cup \{\tilde{\varphi}_h^{(m)} \text{ on } T_h \setminus T_H\}. \tag{4}$$

Let then $\tilde{A}_h$ be the corresponding hierarchical stiffness matrix. Under the splitting (4) both matrices A_h and $\tilde{A}_h$ admit in a natural way a two-by-two block structure

$$A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \}N_{h\setminus H} \\ \}\ N_H \end{matrix}, \quad \tilde{A}_h = \begin{bmatrix} A_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & A_H \end{bmatrix} \begin{matrix} \}N_{h\setminus H} \\ \}\ N_H \end{matrix}. \tag{5}$$

As is well-known, there exists a transformation matrix $J = \begin{bmatrix} I_1 & 0 \\ J_{21} & I_2 \end{bmatrix}$, which relates the nodal point vectors for the standard and the hierarchical basis functions as follows,

$$\tilde{\mathbf{v}} = \begin{bmatrix} \tilde{\mathbf{v}}_1 \\ \tilde{\mathbf{v}}_2 \end{bmatrix} = J \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}, \quad \begin{matrix} \tilde{\mathbf{v}}_1 = \mathbf{v}_1 \\ \tilde{\mathbf{v}}_2 = J_{21}\mathbf{v}_1 + \mathbf{v}_2 \end{matrix}$$

Remark 1.1 Clearly, the hierarchical stiffness matrix $\tilde{A}_h$ is more dense than A_h and therefore its action on a vector is computationally more expensive. The transformation matrix J, however, enables us in practical implementations to work with A_h, since $\tilde{A}_h = JA_hJ^T$.

Two-level preconditioners and the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality. Consider a general matrix A, which is assumed to be symmetric positive definite and partitioned as in (5). The quality of this partitioning is characterized by the corresponding CBS inequality constant:

$$\gamma = \sup_{\mathbf{v}_1 \in \mathbf{R}^{n_1 - n_2},\ \mathbf{v}_2 \in \mathbf{R}^{n_2}} \frac{\mathbf{v}_1^T A_{12} \mathbf{v}_2}{\left(\mathbf{v}_1^T A_{11} \mathbf{v}_1\right)^{1/2} \left(\mathbf{v}_2^T A_{22} \mathbf{v}_2\right)^{1/2}}, \tag{6}$$

where $n_1 = N_h$ and $n_2 = N_H$. Let us assume also that

$$A_{11} \le C_{11} \le (1+\delta_1)A_{11} \text{ and } A_{22} \le C_{22} \le (1+\delta_2)A_{22}. \tag{7}$$

The inequalities (7) are in a positive semidefinite sense where C_{11} and C_{22} are symmetric and positive definite matrices for some positive constants δ_i, $i = 1,2$. The multiplicative preconditioner is then of the form

$$M_F = \begin{bmatrix} C_{11} & 0 \\ A_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I_1 & C_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}. \tag{8}$$

Then

$$\kappa(M_F^{-1}A) \le \frac{1}{1-\gamma^2}\left\{1+0.5\left[\delta_1+\delta_2+\sqrt{(\delta_1-\delta_2)^2+4\delta_1\delta_2\gamma^2}\right]\right\} \tag{9}$$

When $C_{11} = A_{11}$ and $C_{22} = A_{22}$, then estimate (9) reduces to $\kappa(M_F^{-1}A) \le 1/(1-\gamma^2)$. Detailed proof of (9) is found, for instance, in [1]. In the hierarchical bases context V_1 and V_2 are subspaces of the finite element space V_h spanned, respectively, by the basis functions at the new nodes $\mathbf{N}_{h\backslash H}$ and by the basis functions at the old nodes $\mathbf{N}_H$. For the strengthened CBS inequality constant, there holds that

$$\gamma = \cos(V_1, V_2) = \sup_{u\in V_1,\ v\in V_2} \frac{A_h(u,v)}{\sqrt{A_h(u,u)A_h(v,v)}} \tag{10}$$

where $A_h(\cdot,\cdot)$ is the bilinear form which appears in the variational formulation of the original problem. When $V_1 \cap V_2 = \{0\}$, the constant γ is strictly less than one. As shown in [2], it can be estimated locally over each finite element (macro-element) $E \in T_H$, which means that $\gamma = \max_E \gamma_E$, where

$$\gamma_E = \sup_{u\in V_1(E),\ v\in V_2(E)} \frac{A_E(u,v)}{\sqrt{A_E(u,u)A_E(v,v)}}, \quad v \ne const.$$

The spaces $V_k(E)$ above contain the functions from V_k restricted to E and $A_E(u,v)$ corresponds to $A_h(u,v)$ restricted over the element E of T_H (see also [5]). Using the local estimates, it is possible to obtain uniform estimates for γ. In the case of linear conforming finite elements, it is known that γ does not depend on h and on any discontinuities of the coefficients of the bilinear form $A_h(\cdot,\cdot)$, as long as they do not occur within any element of the coarse triangulation used. The h-independence means that if we have a hierarchy of refinements of the domain which preserve the properties of the initial triangulation (refinement by congruent triangles, for example), then γ is independent of the level of the refinement as well. For certain implementations, it is shown that γ is independent of anisotropy. Hence, as long as the rate of convergence is bounded by some function of γ, it is independent of various problem and discretization parameters, such as the ones mentioned above.

We stress here, that the above technique is originally developed and straightforwardly applicable for conforming finite elements and nested finite element spaces, i.e., when $V_H \subset V_h$.

1 Rannacher-Turek finite elements

Nonconforming finite elements based on *rotated* multilinear shape functions were introduced by Rannacher and Turek [6] as a class of simple elements for the Stokes problem. More generally, the recent activities in the development of efficient solution methods for non-conforming finite element systems are inspired by their attractive properties as a stable discretization tool for illconditioned problems.

The unit square $[-1,1]^2$ is used as a reference element $\hat{e}$ to define the isoparametric rotated bilinear element $e \in T_h$. Let $\psi_e : \hat{e} \to e$ be the corresponding bilinear one-to-one transformation, and let the nodal basis functions be determined by the relation

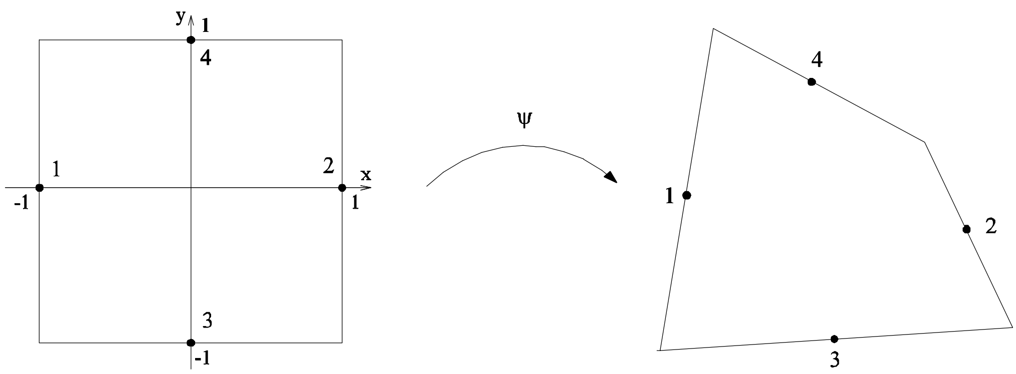


Fig. 1. Rotated bilinear finite element.

$$\{\varphi_i\}_{i=1}^4 = \{\hat{\varphi}_i \circ \psi^{-1}\}_{i=1}^4, \qquad \{\hat{\varphi}_i\} \in span\{1, x, y, x^2 - y^2\}.$$

For the variant MP (mid point), $\{\hat{\varphi}_i\}_{i=1}^4$ are found by the point-wise interpolation condition

$$\hat{\varphi}_i(b_\Gamma^j) = \delta_{ij},$$

where $b_\Gamma^j,\ j = 1,4$ are the midpoints of the edges of the quadrilateral $\hat{e}$. Then,

$$\hat{\varphi}_1(x,y) = \frac{1}{4}(1 - 2x + (x^2 - y^2)), \qquad \hat{\varphi}_2(x,y) = \frac{1}{4}(1 + 2x + (x^2 - y^2)),$$

$$\hat{\varphi}_3(x,y) = \frac{1}{4}(1 - 2y - (x^2 - y^2)), \qquad \hat{\varphi}_4(x,y) = \frac{1}{4}(1 + 2y - (x^2 - y^2)).$$

The variant MV (mid value) corresponds to integral midvalue interpolation conditions. Let $\Gamma_{\hat{e}} = \bigcup_{j=1}^4 \Gamma_{\hat{e}}^j$. Then $\{\hat{\varphi}_i\}_{i=1}^4$ are determined by the equality

$$|\Gamma_{\hat{e}}^j|^{-1} \int_{\Gamma_{\hat{e}}^j} \hat{\varphi}_i \, d\Gamma_{\hat{e}}^j = \delta_{ij},$$

which leads to

$$\hat{\varphi}_1(x,y) = \frac{1}{8}(2 - 4x + 3(x^2 - y^2)), \qquad \hat{\varphi}_2(x,y) = \frac{1}{8}(2 + 4x + 3(x^2 - y^2)),$$

$$\hat{\varphi}_3(x,y) = \frac{1}{8}(2 - 4y - 3(x^2 - y^2)), \qquad \hat{\varphi}_4(x,y) = \frac{1}{8}(2 + 4y - 3(x^2 - y^2)).$$

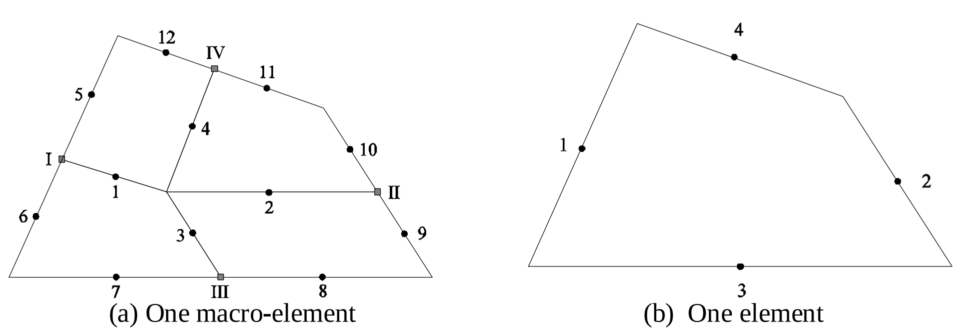
2 Hierarchical two-level splitting by differences and aggregates (DA)

Let us consider two consecutive discretizations T_H and T_h. Figure 2 illustrates a macro-element obtained after one regular mesh-refinement step. We see that in this case V_H and V_h are not nested.

The DA splitting is easily described for one macro-element. If $\varphi_1, \ldots, \varphi_{12}$ are the standard nodal basis functions for the macro-element, then we define

$$\begin{aligned}
V(E) &= span\{\varphi_1, \ldots, \varphi_{12}\} = V_1(E) \oplus V_2(E), \\
V_1(E) &= span\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 - \varphi_6, \varphi_9 - \varphi_{10}, \varphi_7 - \varphi_8, \varphi_{11} - \varphi_{12}\} \\
V_2(E) &= span\{\varphi_5 + \varphi_6 + \sum_{j=1,4} \alpha_{1j}\varphi_j, \varphi_9 + \phi_{10} + \sum_{j=1,4} \alpha_{2j}\varphi_j, \\
&\qquad \varphi_7 + \varphi_8 + \sum_{j=1,4} \alpha_{3j}\varphi_j, \varphi_{11} + \varphi_{12} + \sum_{j=1,4} \alpha_{4j}\varphi_j\}.
\end{aligned}$$

Using the related transformation matrix J_E,



(a) One macro-element (b) One element

Figure 2: Uniform refinement on a general mesh.

$$J_E = \frac{1}{2}\begin{bmatrix} 2 &&&&&&&&&&& \\ & 2 &&&&&&&&&& \\ && 2 &&&&&&&&& \\ &&& 2 &&&&&&&& \\ &&&& 1 & -1 &&&&&& \\ &&&&&&&& 1 & -1 && \\ &&&&&& 1 & -1 &&&& \\ &&&&&&&&&& 1 & -1 \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 1 & 1 &&&&&& \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} &&&&& 1 & 1 && \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} &&& 1 & 1 &&&& \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} &&&&&&& 1 & 1 \end{bmatrix}, \tag{11}$$

the vector of the macro-element basis functions $\varphi_E = \{\varphi_i\}_{i=1}^{12}$ is transformed to a new hierarchical basis $\tilde{\varphi}_E = \{\tilde{\varphi}_i\}_{i=1}^{12} = J_E \varphi_E$. Accordingly, J_E transforms the macro-element stiffness matrix into a hierarchical form

$$\tilde{A}_E = J_E A_E J_E^T = \begin{bmatrix} \tilde{A}_{E,11} & \tilde{A}_{E,12} \\ \tilde{A}_{E,21} & \tilde{A}_{E,22} \end{bmatrix} \begin{matrix} \tilde{\varphi}_i \in V_1(E) \\ \tilde{\varphi}_i \in V_2(E) \end{matrix}. \tag{12}$$

Following the local definitions, for the whole finite element space V_h with the standard nodal finite element basis $\varphi = \{\varphi_h^{(i)}\}_{i=1}^{N_h}$ we can similarly construct the new hierarchical basis $\tilde{\varphi} = \{\tilde{\varphi}_h^{(i)}\}_{i=1}^{N_h}$ and the corresponding splitting

$$V_h = V_1 \oplus V_2. \tag{13}$$

The transformation J such that $\tilde{\varphi} = J\varphi,$ can be used for transformation of the stiffness matrix A_h to hierarchical form $\tilde{A}_h = J A_h J^T,$ which allows preconditioning by the two-level preconditioners based on the splitting (13). Now, we are in a position to analyze the constant $\gamma = \cos(V_1, V_2)$ for the splitting (13). Again, as in the previous section, we would like to perform this analysis locally, by considering the corresponding problems on macro-elements. For this purpose we need to have satisfied the condition

(I) $\ker(\tilde{A}_{E,22}) = \ker(A_e)$, which is equivalent to $\sum_{i=1}^{4} \alpha_{ij} = 1,\ j \in \{1,2,3,4\}$.

There are obviously various DA splittings satisfying the condition (i). When the two-level algorithm is recursively generalized to the multilevel case, it is useful if

(II) $\tilde{A}_{E,22}$ is proportional to A_e.

Such a property holds in a very general setting for the DA splitting of the Crouzeix-Raviart finite element space, see [4]. Unfortunately, it seems to be rather complicated to find a parameter matrix $[\alpha_{ij}]$, which satisfies the condition (ii) in the general case of Rannacher-Turek finite elements.

3. Uniform estimates of the CBS constant

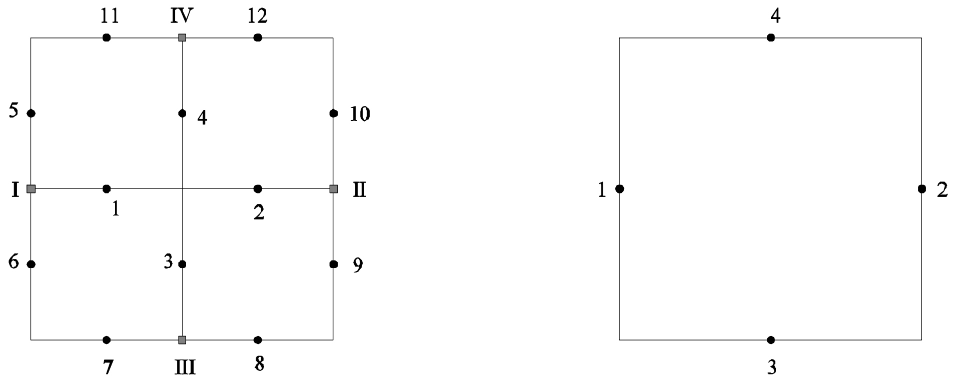
We study in this section the isotropic model problem where all elements $e \in T_H$ are squares, and the uniform refinement is as shown in Figure 3. Both variants MP and MV of rotated bilinear finite elements, are considered.

Due to the symmetry of the problem, the down-left block of the transformation matrix J_E can be simplified to the form

$$\begin{bmatrix} b & c & a & a \\ c & b & a & a \\ a & a & b & c \\ a & a & c & b \end{bmatrix} \tag{14}$$

The condition (i) is equivalent to $2a+b+c=1$. Let us write the condition (ii) in the form $\tilde{A}_{E,22} = pA_e$. Then, (ii) is reduced to a system of two nonlinear equations for, say, (b,c), with a parameter p. It appears, that the system for (b,c) has a solution if $p \in [p_0, \infty)$. In such a case, we can optimize the parameter p, so that the related CBS constant is minimal.

Variant MP:



(a) One macro-element (b) One element

Figure 3: Uniform refinement on a square mesh.

Lemma 4.1 *There exists a two-level splitting satisfying the condition* (ii)*, if and only if,* $p \geq \frac{3}{7}$. *Then, the solutions for* (b,c) *are invariant with respect to the local CBS constant* $\gamma_E^2 = 1 - \frac{1}{4p}$*, and for the related optimal splitting*

$$\gamma_{\mathrm{MP}}^2 \leq \frac{5}{12}. \tag{15}$$

Although the statements of Lemma 4.1. look very simply, the midterm derivations are rather technical, which is just illustrated by the following expressions of one of the similarly looking solutions for (b,c):

$$b = \frac{-1}{70(-729+2240p)}\left(24786 - 76160p + 2658\sqrt{\phi(p)} - 7280p\sqrt{\phi(p)} + \sqrt{\phi^3(p)}\right)$$

$$c = \frac{1}{70}\left(6 - \sqrt{\phi(p)}\right)$$

where $\phi(p) = -1329 + 3640p - 140\sqrt{63 - 32p + 420p^2}$.

Variant MV: The same approach is applied to get the estimates below.

Lemma 4.2 *There exists a two-level splitting satisfying the condition* (ii)*, if and only if,* $p \geq \frac{2}{5}$. *Then, the solutions for* (b,c) *are invariant with respect to the local CBS constant* $\gamma_E^2 = 1 - \frac{1}{4p}$*, and for the related optimal splitting*

$$\gamma_{\mathrm{MV}}^2 \leq \frac{3}{8}. \tag{16}$$

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