doi: https://doi.org/10.15407/dopovidi2019.05.011
UDK 517.9:519.46

**A.F. Barannyk [1], T.A. Barannyk [2], I.I. Yuryk [3]**

[1] Institute of Mathematics, Pomeranian University, Słupsk, Poland
[2] Poltava V.G. Korolenko National Pedagogical University
[3] National University of Food Technologies, Kiev
E-mail: anatolij.barannyk@apsl.edu.pl, barannykt@rambler.ru, i.yu@ukr.net

# On exact solutions of the nonlinear heat equation

*Presented by Corresponding Member of the NAS of Ukraine A.G. Nikitin*

*A method for construction of exact solutions to the nonlinear heat equation $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$, which is based on the ansatz $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$, is proposed. The function $p(x)$ is a solution of the equation $(p')^2 = Ap^2 + B$, and the functions $\omega_1(t)$, $\omega_2(t)$ and $\varphi(u)$ can be found from the condition that this ansatz reduces the nonlinear heat equation to a system of two ordinary differential equations with unknown functions $\omega_1(t)$ and $\omega_2(t)$.*

***Keywords***: *group-theoretic methods, exact solutions, nonlinear heat equation, generalized variable separation.*

**1. Introduction.** The paper is devoted to the construction of exact solutions of the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right] + G(u)\frac{\partial u}{\partial x} + H(u). \qquad (1)$$

This equation for $G(u) \equiv \text{const}$ describes the unsteady state heat transfer in a medium that is moving with a constant velocity, where the thermal conductivity coefficient and the reaction speed coefficient are arbitrary functions of the temperature. The soliton solutions of Eq. (1) are presented in [1].

In the case $G(u) \equiv 0$, we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right] + H(u), \qquad (2)$$

which describes the unsteady state heat transfer in an unmovable medium. The group classification of a class of equations of this type and exact solutions for different functions $F(u)$ and $H(u)$ are presented in [1–5].

In this paper, we propose a method for constructing new exact solutions of Eqs. (1) and (2). To solve these equations, we use the ansatz

$$p(x) = \omega_1(t)\varphi(u) + \omega_2(t), \qquad (3)$$

which contains the unknown functions $\omega_1(t)$, $\omega_2(t)$, and $\varphi(u)$, whereas the function $p(x)$ is *a priori* predefined. Assume that $p(x)$ is a solution of the equation

$$(p')^2 = Ap^2 + B,$$

and then determine the functions $\omega_1(t)$, $\omega_2(t)$, and $\varphi(u)$, using the reduction idea. Namely, assume that ansatz (3) reduces the given equation to a system of two ordinary differential equations with unknown functions $\omega_1(t)$ and $\omega_2(t)$. This approach gives a description of a class of equations of the forms (1) and (2) that have solutions of the form (3), as well as an effective technique for constructing such solutions. An ansatz of the form (3) is used in [6, 7] for constructing exact solutions of nonlinear wave equations and Korteweg—de Vries equations.

**2. Exact solutions of Eq. (1).** In this section, we determine the functions $F(u)$, $G(u)$, and $H(u)$, for which Eq. (1) has solutions of the form

$$x = \omega_1(t)\varphi(u) + \omega_2(t), \tag{4}$$

i.e. admits ansatz (4). This ansatz contains the three unknown functions $\omega_1(t)$, $\omega_2(t)$, and $\varphi(u)$. These functions will be determined from the condition that ansatz (4) reduces Eq. (1) to a system of two ordinary differential equations with the unknown functions $\omega_1(t)$ and $\omega_2(t)$. In order to obtain this system, we substitute relation (4) into Eq. (1):

$$-\frac{\omega_1'}{\omega_1}\frac{\varphi}{\varphi'} - \frac{\omega_2'}{\omega_1}\frac{1}{\varphi'} = \left(-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2}\right)\frac{1}{\omega_1^2} + \frac{1}{\omega_1}\frac{G}{\varphi'} + H(u), \tag{5}$$

If there exists a solution of Eq. (1) of the form (4), then the obtained Eq. (5) means that the functions

$$\frac{\varphi}{\varphi'},\ \frac{1}{\varphi'},\ -F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2},\ \frac{G}{\varphi'}, H \tag{6}$$

are linearly dependent. The functions $\frac{\varphi}{\varphi'}, \frac{1}{\varphi'}$ are linearly independent, so all other functions (6) should obey the condition that they are representable as a linear combination of the functions $\frac{\varphi}{\varphi'}, \frac{1}{\varphi'}$. We have

$$-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2} = \lambda_1\frac{\varphi}{\varphi'} + \mu_1\frac{1}{\varphi'}, \tag{7}$$

$$G = \lambda_2\varphi + \mu_2, \tag{8}$$

$$H = \lambda_3\frac{\varphi}{\varphi'} + \mu_3\frac{1}{\varphi'}, \tag{9}$$

for some $\lambda_i, \mu_i \in R$. Substitute (7)—(9) into Eq. (5):

$$\left(-\frac{\omega_1'}{\omega_1} - \frac{\lambda_1}{\omega_1^2} - \frac{\lambda_2}{\omega_1} - \lambda_3\right)\frac{\varphi}{\varphi'} + \left(-\frac{\omega_2'}{\omega_1} - \frac{\mu_1}{\omega_1^2} - \frac{\mu_2}{\omega_1} - \mu_3\right)\frac{1}{\varphi'} = 0. \tag{10}$$

The functions $\frac{\varphi}{\varphi'}$ and $\frac{1}{\varphi'}$ are linearly independent, so Eq. (10) splits into a system of equations

 ISSN 1025-6415. Dopov. Nac. akad. nauk Ukr. 2019. № 5

$$\frac{\omega_1'}{\omega_1}+\frac{\lambda_1}{\omega_1^2}+\frac{\lambda_2}{\omega_1}+\lambda_3=0, \tag{11}$$

$$\frac{\omega_2'}{\omega_1}+\frac{\mu_1}{\omega_1^2}+\frac{\mu_2}{\omega_1}+\mu_3=0. \tag{12}$$

Let $F'(u)\neq 0$. Integrating Eq. (7), which is linear with respect to the function $F=F(u)$, we find

$$F=(\lambda_1\int\varphi du+\mu_1 u+A)\,\varphi', \tag{13}$$

where A is an arbitrary constant. As a result, we can formulate the following theorem.

**Theorem 1.** *Let* $F'(u)\neq 0$ *in Eq.* (1). *If Eq.* (1) *admits ansatz* (4), *then the functions F*(*u*), *G*(*u*), *and H*(*u*) *are defined by formulas* (13), (8), *and* (9), *respectively, whereas* $\omega_1(t)$ and $\omega_2(t)$ *are solutions of the system of equations* (11), (12).

In accordance with Theorem 1, the function $\varphi(u)$ in ansatz (4) is arbitrary, whereas the functions $F(u)$, $G(u)$, and $H(u)$ can be represented via the function $\varphi(u)$. Finding solutions of the form (4) of Eq. (1) is reduced to integrating the system of equations (11), (12). Rewrite this system in terms of new functions $v_1$ and $v_2$,

$$v_1=\frac{1}{\omega_1},\ v_2=\frac{\omega_2}{\omega_1}.$$

Then this system transforms to

$$v_1'=\lambda_1 v_1^3+\lambda_2 v_1^2+\lambda_3 v_1, \tag{14}$$

$$v_2'=(\lambda_1 v_1^2+\lambda_2 v_1+\lambda_3)v_2+\mu_1 v_1^2+\mu_2 v_1+\mu_3. \tag{15}$$

Consider three possible cases.

a) Case $\lambda_1\neq 0$, $\lambda_2=\lambda_3=0$. Equation (1) has the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right]+\mu_2\frac{\partial u}{\partial x}+\mu_3\frac{1}{\varphi'}, \tag{16}$$

where the function $F(u)$ is defined by formula (13). The general solution of system (14), (15) for $\lambda_2=\lambda_3=0$ is

$$v_1=[-2\lambda_1(t+c_1)]^{-\frac{1}{2}},$$

$$v_2=-\frac{\mu_1}{\lambda_1}+\mu_2 t[-2\lambda_1(t+c_1)]^{-\frac{1}{2}}-\frac{\mu_3}{3\lambda_1}[-2\lambda_1(t+c_1)]+c_2[-2\lambda_1(t+c_1)]^{-\frac{1}{2}},$$

where $c_1$ and $c_2$ are arbitrary constants. As a result, we have the following solution of Eq. (16):

$$\varphi(u)=[-2\lambda_1(t+c_1)]^{-\frac{1}{2}}x+\frac{\mu_1}{\lambda_1}-\mu_2 t[-2\lambda_1(t+c_1)]^{-\frac{1}{2}}+\frac{\mu_3}{3\lambda_1}[-2\lambda_1(t+c_1)]-c_2[-2\lambda_1(t+c_1)]^{-\frac{1}{2}}. \tag{17}$$

Setting $\mu_1=\mu_2=\mu_3=0$ and $c_2=0$ in (17), we obtain automodel solutions

$$\varphi(u)=[-2\lambda_1(t+c_1)]^{-\frac{1}{2}} \tag{18}$$

to the equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right], \qquad (19)$$

where the function $F(u)$ is defined by formula (13). Solutions of the form (18) are studied in [5].

b) Case $\lambda_1=\lambda_2=0$, $\lambda_3\neq 0$. The general solution of system (14), (15) is defined by the formulas

$$v_1=c_1\exp(\lambda_3 t),$$

$$v_2=\frac{\mu_1 c_1}{\lambda_3}\exp(2\lambda_3 t)+\mu_2 c_1 t\exp(\lambda_3 t)-\frac{\mu_3}{\lambda_3}+c_2\exp(\lambda_3 t).$$

Equation (1) in this case becomes

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(\mu_1 u+A)\varphi'\frac{\partial u}{\partial x}\right]+\mu_2\frac{\partial u}{\partial x}+\frac{1}{\varphi'}(\lambda_3\varphi+\mu_3), \qquad (20)$$

and has the following family of solutions:

$$\varphi(u)=c_1 x\exp(\lambda_3 t)-\frac{\mu_1 c_1^2}{\lambda_3}\exp(2\lambda_3 t)-\mu_2 c_1 t\exp(\lambda_3 t)+\frac{\mu_3}{\lambda_3}-c_2\exp(\lambda_3 t). \qquad (21)$$

When we set $\mu_2=0$ in (21), we obtain a family of solutions

$$\varphi(u)=c_1 x\exp(\lambda_3 t)-\frac{\mu_1 c_1^2}{\lambda_3}\exp(2\lambda_3 t)+\frac{\mu_3}{\lambda_3}-c_3\exp(\lambda_3 t).$$

to the equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(\mu_1 u+A)\varphi'\frac{\partial u}{\partial x}\right]+\frac{1}{\varphi'}(\lambda_3\varphi+\mu_3), \qquad (22)$$

c) Case $\lambda_1=0$, $\lambda_2\neq 0$, $\lambda_3\neq 0$. In this case, Eq. (1) has the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(\mu_1 u+A)\varphi'\frac{\partial u}{\partial x}\right]+(\lambda_3\varphi+\mu_2)\frac{\partial u}{\partial x}+\frac{1}{\varphi'}(\lambda_3\varphi+\mu_3), \qquad (23)$$

and the general solution of system (14), (15) is defined as

$$v_1=\frac{\lambda_3 c_1\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)},$$

$$v_2=-\frac{\mu_1\lambda_3 c_1}{\lambda_2}\frac{\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}\cdot\ln\left|c_2-\lambda_2 c_1\exp(\lambda_3 t)\right|+$$

$$+(\mu_2\lambda_3-\mu_3\lambda_2)c_1 t\frac{\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}-$$

$$-\frac{\mu_3 c_2}{\lambda_3(c_2-\lambda_2 c_1\exp(\lambda_3 t))}+\frac{\lambda_3 c_3\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}.$$

Substituting these expressions for $v_1$ and $v_2$ into $\varphi(u) = v_1 x - v_2$, we obtain solutions of Eq. (23).

**3. Exact solutions of Eq. (2).** In order to construct exact solutions of Eq. (2), we can use the substitution

$$p(x)=\omega_1(t)\varphi(u), \tag{24}$$

where $p(x)$ is a solution of the equation

$$(p')^2 = Ap^2 + B,\ \ A\neq 0,\ B\neq 0.$$

Determine the functions $\omega_1(t)$ and $\varphi(u)$ from the condition that ansatz (24) reduces Eq. (2) to an ordinary differential equation with the unknown function $\omega_1(t)$. Then we obtain the following system of equations for determining the functions $F(u)$, $\varphi(u)$

$$-F\frac{\varphi''}{(\varphi')^3}+F'\frac{1}{(\varphi')^2}=\lambda_2\frac{\varphi}{\varphi'}, \tag{25}$$

$$-FA\frac{\varphi^2\varphi''}{(\varphi')^3}+F'A\frac{\varphi^2}{(\varphi')^2}+FA\frac{\varphi}{\varphi'}+H=\lambda_3\frac{\varphi}{\varphi'}, \tag{26}$$

where $\lambda_2,\lambda_3\in R$. Suppose that $F'(u)\neq 0$. Then, integrating Eq. (25), which is linear with respect to the function $F=F(u)$, we have

$$F=(\lambda_2\int\varphi du+c_1)\,\varphi', \tag{27}$$

where $c_1$ is a constant. The function $\omega_1(t)$ can be determined from the equation

$$\frac{\omega_1'}{\omega_1}+\lambda_2 B\frac{1}{\omega_1^{\,2}}+\lambda_3=0. \tag{28}$$

In the case $\lambda_3\neq 0$, the solution of (28) is the function

$$\omega_1^2=\frac{c_2}{\lambda_3}\exp(-2\lambda_3 t)-\frac{\lambda_2}{\lambda_3}B,$$

where $c_2$ is a constant. In the case $\lambda_3=0$, the solution is

$$\omega_1^2=-2\lambda_2 Bt+c_2,$$

where $c_2$ is a constant, and $\lambda_2\neq 0$.

From Eqs. (25) and (26), we have

$$H=\frac{1}{\varphi'}(-\lambda_2 A\varphi^3-AF\varphi+\lambda_3\varphi). \tag{29}$$

As a result, we formulate the next theorem.

**Theorem 2.** *If Eq.* (2) *admits ansatz* (24) *and if* $F'(u)\neq 0$, *then the functions* $F(u)$ *and* $H(u)$ *are defined by formulas* (27) *and* (29), *respectively, whereas the function* $\omega_1(t)$ *is a solution of Eq.* (28).

The obtained solutions of Eq. (2) can be generalized by means of the substitutions

$$\varphi(u)=\omega_1(t)\operatorname{ch}[k(x+c_3)]+\omega_2(t)\operatorname{sh}[k(x+c_3)], \tag{30}$$

if $A=k^2>0$,

and

$$\varphi(u)=\omega_1(t)\cos[k(x+c_3)]+\omega_2(t)\sin[k(x+c_3)],$$

if $A=-k^2<0$.

For example, consider substitution (30). If the functions $F(u)$ and $H(u)$ are defined by formulas (27) and (29), respectively, and $A=k^2>0$, then substitution (30) reduces Eq. (2) to the system

$$\omega_1'=(-\lambda_2k^2\omega_1^2+\lambda_2k^2\omega_2^2)\omega_1+\lambda_3\omega_1, \quad (31)$$

$$\omega_2'=(-\lambda_2k^2\omega_1^2+\lambda_2k^2\omega_2^2)\omega_2+\lambda_3\omega_2, \quad (32)$$

Let $\omega_1\neq 0$. From Eqs. (31) and (32), we derive that $\omega_2=c\omega_1$, $c$ is a constant. Equation (31) becomes

$$\omega_1'==\lambda_2k^2(c^2-1)\omega_1^3+\lambda_3\omega_1. \quad (33)$$

If $\lambda_3\neq 0$, then the solution of Eq. (33) is

$$\omega_1^2=\left[\frac{c_2}{\lambda_3}\exp(-2\lambda_3t)-\frac{\lambda_2}{\lambda_3}k^2(c^2-1)\right]^{-1},$$

where $c_2\neq 0$ is a constant.

The solution of Eq. (2) is

$$\varphi(u)=\left[\frac{c_2}{\lambda_3}\exp(-2\lambda_3t)-\frac{\lambda_2}{\lambda_3}k^2(c^2-1)\right]^{-\frac{1}{2}}[\mathrm{ch}[k(x+c_3)]+c\cdot\mathrm{sh}[k(x+c_3)]].$$

If $\lambda_3=0$, then the solution of Eq. (33) is

$$\omega_1^2=[-2\lambda_2k^2(c^2-1)t+c_2]^{-1},$$

where $c_2$ is a constant and $\lambda_2\neq 0$.

As a result, we have the following solution of Eq. (2):

$$\varphi(u)=[-2\lambda_2k^2(c^2-1)t+c_2]^{-\frac{1}{2}}[\mathrm{ch}[k(x+c_3)]+c\cdot\mathrm{sh}[k(x+c_3)]].$$

The case $\omega_1=0$ reduces to integrating the equation

$$\omega_2'=\lambda_2k^2\omega_2^3+\lambda_3\omega_2.$$

**4. Conclusion.** We have described equations of the form (1) that admit ansatz (4). The functions $F(u)$, $G(u)$, and $H(u)$ in Eq. (1) can be represented in terms of $\varphi(u)$, and the corresponding system for finding $\omega_1(t)$ and $\omega_2(t)$ can be integrated. A voluntary choice of the function $\varphi(u)$ in ansatz (4) allows one to find solutions of Eq. (1), that should satisfy predefined conditions. All this is true also for Eq. (2), which is a special case of Eq. (1). Moreover, ansatz (24) gives essentially new solutions of Eq. (2).

The techniques of constructing the solutions of Eqs. (1) and (2) described in Sections 2 and 3 can also be efficiently applied for constructing the solutions of plenty more equations, for example, nonlinear wave equations [7].

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Received 08.02.2019

*А.Ф. Баранник [1], Т.А. Баранник [2], І.І. Юрик [3]*

[1] Інститут математики Поморського університету, Слупськ, Польща
[2] Полтавський національний педагогічний університет ім. В.Г. Короленка
[3] Національний університет харчових технологій, Київ
E-mail: anatolij.barannyk@apsl.edu.pl, barannykt@rambler.ru, i.yu@ukr.net

## ПРО ТОЧНІ РОЗВ'ЯЗКИ НЕЛІНІЙНОГО РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ

Запропоновано метод побудови точних розв'язків нелінійного рівняння теплопровідності $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$, який ґрунтується на використанні підстановки $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$, де функція $p(x)$ є розв'язком рівняння $(p')^2 = Ap^2 + B$, а функції $\omega_1(t)$, $\omega_2(t)$ та $\varphi(u)$ знаходяться з умови, що дана підстановка редукує рівняння до системи двох звичайних диференціальних рівнянь з невідомими функціями $\omega_1(t)$ та $\omega_2(t)$.

***Ключові слова***: *теоретико-групові методи, точні розв'язки, нелінійне рівняння теплопровідності, узагальнене розділення змінних.*

*А.Ф. Баранник [1], Т.А. Баранник [2], И.И. Юрик [3]*

[1] Институт математики Поморского университета, Слупск, Польша
[2] Полтавский национальный педагогический университет им. В.Г. Короленко
[3] Национальный университет пищевых технологий, Киев
E-mail: anatolij.barannyk@apsl.edu.pl, barannykt@rambler.ru, i.yu@ukr.net

## О ТОЧНЫХ РЕШЕНИЯХ НЕЛИНЕЙНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

Предложен метод построения точных решений нелинейного уравнения теплопроводности $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$, основанный на использовании подстановки $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$, где функция $p(x)$ является решением уравнения $(p')^2 = Ap^2 + B$, а функции $\omega_1(t)$, $\omega_2(t)$ и $\varphi(u)$ находятся из условия, что данная подстановка редуцирует уравнение к системе двух обыкновенных дифференциальных уравнений с неизвестными функциями $\omega_1(t)$ и $\omega_2(t)$.

***Ключевые слова***: *теоретико-групповые методы, точные решения, нелинейное уравнение теплопроводности, обобщённое разделение переменных.*