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ON COMPENSATED COMPACTNESS FOR NONLINEAR ELLIPTIC PROBLEMS IN PERFORATED DOMAINS

ПРО КОМПЕНСОВАНУ КОМПАКТНІСТЬ ДЛЯ НЕЛІНІЙНИХ ЕЛІПТИЧНИХ ЗАДАЧ У ПЕРФОРОВАНИХ ОБЛАСТЯХ

We consider a sequence of Dirichlet problems for a nonlinear divergent operator $A: W_m^1(\Omega_s) \to \left[W_m^1(\Omega_s)\right]^*$ in a sequence of perforated domains $\Omega_s \subset \Omega$. Under the condition on the local capacity of a set $\Omega \setminus \Omega_s$, we prove the following principle of the compensated compactness: $\lim\limits_{s\to\infty} \langle Ar_s, z_s \rangle = 0$, where $r_s(x)$ and $z_s(x)$ are sequences weakly converging in $W_m^1(\Omega)$ and such that $r_s(x)$ is analogous to a corrector for homogenization problem, $z_s(x)$ is an arbitrary sequence from $\overset{\circ}{W}{}_m^1(\Omega_s)$ whose weak limit is equal to zero.

Розглядається послідовність задач Діріхле для нелінійного дивергентного еліптичного оператора $A: W_m^1(\Omega_s) \to \left[W_m^1(\Omega_s)\right]^*$ в послідовності перфорованих областей $\Omega_s \subset \Omega$. За умови на локальну ємність множини $\Omega \setminus \Omega_s$ доведено такий принцип компенсованої компактності: $\lim\limits_{s\to\infty} \langle Ar_s, z_s \rangle = 0$, де $r_s(x)$, $z_s(x)$ — слабко збіжні в $W_m^1(\Omega)$ послідовності такі, що $r_s(x)$ — аналог коректора для задачі усереднення, $z_s(x)$ — довільна послідовність в $\overset{\circ}{W}{}_m^1(\Omega_s)$, слабка границя якої дорівнює нулю.

1. Introduction. Let Ω be a bounded open set in the n-dimensional Euclidean space R^n and let $\Omega_s \subset \Omega$, $s = 1, 2, \ldots$, be a sequence of subdomains. In Ω_s we consider a nonlinear elliptic boundary-value problem

$$\sum_{j=1}^{n} \frac{d}{dx_j} a_j\left(x, \frac{\partial u}{\partial x}\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j(x), \qquad x \in \Omega_s, \tag{1.1}$$

$$u(x) = f(x), \qquad x \in \partial\Omega_s. \tag{1.2}$$

The asymptotic behaviour of solutions of such type problem for $s \to \infty$ was studied in papers [1 – 3], monographs [4, 5] and in papers of another authors (see [6 – 9] and references in [1]) for nonlinear equations satisfying strong monotonicity assumptions.

A new monotonicity approach for the study of the asymptotic behaviour of solutions of the problem (1.1), (1.2) for the equations (1.1) satisfying weak monotonicity condition was developed in the paper [10]. In this paper we assumed that C_m — capacity of the part of the holes $\Omega \setminus \Omega_s$, $s = 1, 2, \ldots$, in small cubes is estimated by Lebesgue measure of the cubes. This approach was based on new Convergence Theorem that is analogous to well known compensated compactness principle [11, 12] for linear equations with periodic coefficients.

The aim of this paper is to establish analogous Convergence Theorem under very weak assumptions on the sets Ω_s that are coincided with corresponding conditions in [1]. Our main hypothesis is the following condition B_1 where $K(x, r)$ denotes the closed cube at centre x and side $2r$, and $C_m(F)$ is the m — capacity of a closed set $F \subset \Omega$ with respect to a fixed bounded open set Ω_0 such that $\Omega \subset \Omega_0$, $\rho(\partial\Omega_0, \Omega) \geq 1$, where $\rho(\partial\Omega_0, \Omega)$ is the distance from $\partial\Omega_0$ to Ω.

Condition B_1. There exist a non-negative bounded measure $\nu(B)$, defined for every Borel set $B \subset \Omega$, and a sequence $\rho_s > 0$, tending to zero as $s \to \infty$ such that the inequality

$$C_m(K(x,r) \setminus \Omega_s) \leq \nu(K(x, r+\rho_s))$$

holds for every $x \in \Omega$ and for every $r \geq \rho_s$ with $K(x, r+\rho_s) \subset \Omega$.

We take the attention of the reader that the Convergence Theorem of this paper gives us a possibility to make principal modification in the construction of the corrector in the paper [1]. In the paper [1] the definition of the subdivision of the domain and consequently the construction of the asymptotic expansion was connected with the sequence of solutions $u_s(x)$ of the problem (1.1), (1.2). Using the Convergence Theorem of this paper we can construct corresponding subdivision and the asymptotic expansion without the connection with $u_s(x)$.

Using the Convergence Theorem of this paper we are able to analyse the asymptotic behaviour of solutions of the problem (1.1), (1.2) with weak monotonicity assumption for $a_j(x,p)$, $j = 1, \dots, n$, in the sequence of domains Ω_s satisfying the condition B_1. This result will be published in forthcoming paper of the author.

2. Statement of the main result. We assume that the functions $a_j(x,p)$, $j = 1, \dots, n$, are defined for $x \in R^n$, $p \in R^n$, and satisfy the following conditions:

Condition A_1. The functions $a_j(x,p)$ are continuous in p for almost all $x \in R^n$ and measurable in x for all $p \in R^n$.

Condition A_2. There exist positive constants ν_1, ν_2 and $m \in [2, n)$ such that for $x \in R^n$, $p, q \in R^n$ the inequalities

$$\sum_{j=1}^{n} a_j(x,p) p_j \geq \nu_1 |p|^m, \tag{2.1}$$

$$\sum_{j=1}^{n} \left[a_j(x,p) - a_j(x,q)\right](p_j - q_j) \geq 0, \tag{2.2}$$

$$\left|a_j(x,p)\right| \leq \nu_2 |p|^{m-1}, \quad j = 1, \dots, n, \tag{2.3}$$

hold.

***Remark* 2.1.** The inequality (2.2) means the weak monotonicity assumption for the equation (1.1). The strong monotonicity condition from [1 – 10] is the following inequality

$$\sum_{j=1}^{n} \left[a_j(x,p) - a_j(x,q)\right](p_j - q_j) \geq \nu_1 |p-q|^m.$$

***Remark* 2.2.** We can replace in right-hand sides of inequalities (2.1), (2.3) $|p|^m$, $|p|^{m-1}$ by $(1+|p|)^{m-2}|p|^2$, $(1+|p|)^{m-1}|p|$ respectively.

Our main assumption on the sequence Ω_s is condition B_1 which was formulated in the introduction in terms of the m-capacity $C_m(F)$. For every compact set F contained in Ω_0 the m-capacity $C_m(F)$ of F with respect Ω_0 is defined by equality

$$C_m(F) = \inf \int_{\Omega_0} \left|\frac{\partial \varphi(x)}{\partial x}\right|^m dx, \tag{2.4}$$

where the infimum is taken over all functions $\varphi(x) \in C_0^\infty(\Omega_0)$ which satisfy the equality $\varphi(x) = 1$ for $x \in F$.

For the proof of the Convergence Theorem we need also the following additional assumption on the measure ν.

Condition B_2. There exists an increasing continuous function $\omega(\rho)$, such that

$$\nu(K(x,\rho)\cap\Omega) \le \omega(\rho) \tag{2.5}$$

for arbitrary $x\in\Omega$, $\rho>0$ and

$$\int_0^1 \frac{\omega(\rho)}{\rho^{n-m+1}}\,d\rho < +\infty. \tag{2.6}$$

Remark **2.3.** It is simple to check (see [1]) that the condition (2.6) implies

$$\lim_{\rho\to 0}\frac{\omega(\rho)}{\rho^{n-m}} = 0.$$

Remark **2.4.** We can assume that for an arbitrary Borel set $B\subset\Omega$ an inequality $\nu(B)\ge \operatorname{meas} B$ holds where $\operatorname{meas} B$ is the Lebesgue measure of B. For this it is sufficient to change the measure ν on the measure $\tilde\nu$ such that $\tilde\nu(B)=\nu(B)+\operatorname{meas} B$.

Let us fix a function $\psi(x)$ of class $C_0^\infty(\Omega_0)$ equal to 1 on $\overline{\Omega}$. A crucial role in this paper belongs to special auxiliary function $v(x,F,q)$ that is defined as a maximal solution of a boundary-value problem

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} a_j\left(x,\frac{\partial v}{\partial x}\right) = 0, \qquad x\in\Omega_0\setminus F, \tag{2.7}$$

$$v(x) = q\psi(x), \qquad x\in\partial(\Omega_0\setminus F). \tag{2.8}$$

Here F is an arbitrary closed subset of Ω, q is an arbitrary real number. The solvability of the problem (2.7), (2.8) in $W_m^1(\Omega_0\setminus F)$ is followed easy from the theory of monotone operators. In the paper [13] it is proved the existence of such solution $\overline{v}(x)$ of the problem (2.7), (2.8) that $v(x)\le\overline{v}(x)$ for an arbitrary solution $v(x)$ of this problem. The function $\overline{v}(x)$ is called the maximal solution of the problem (2.7), (2.8). We extend $v(x,F,q)$ to R^n by setting $v(x,F,q)=q$ in F and $v(x,F,q)=0$ outside Ω_0.

Let us introduce a special decomposition of the domain Ω depending on a sequence λ_s. Let t_s be a solution of the equation

$$t_s^{n+1}\left(\frac{\omega(t_s)}{t_s^{n-m}}\right)^{\frac{1}{m-1}} = \rho_s^{n+1}, \tag{2.9}$$

where ρ_s is the sequence from the condition B_1.

We define λ_s to be the odd integer number which satisfies an inequality

$$\lambda_s \le \frac{t_s}{\rho_s} < \lambda_s+2. \tag{2.10}$$

It is easy to check following properties of λ_s:

$$\lim_{s\to\infty}\lambda_s = +\infty, \qquad \lim_{s\to\infty}\lambda_s\rho_s = 0 \tag{2.11}$$

(see Lemma 4.1 [1]).

For a given point $x_0^{(s)}\in K(0,\lambda_s\rho_s)$ we consider the cubic lattice composed of the points $x_\alpha^{(s)}=x_0^{(s)}+2\lambda_s\rho_s\alpha$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index with integer coordinates and we denote

$$F_s = \bigcup_\alpha \left\{K\left(x_\alpha^{(s)},\lambda_s\rho_s\right)\setminus K\left(x_\alpha^{(s)},(\lambda_s-6)\rho_s\right)\right\}, \tag{2.12}$$

where the union is taken over all possible multi-indices α with integer coordinates.

From Lemma 4.2 in [1] it is followed that there exists a point $x_0^{(s)} \in K(0, \lambda_s \rho_s)$ such that

$$\nu(F_s \cap \Omega) \le \frac{7n}{\lambda_s} \nu(\Omega). \tag{2.13}$$

The domain Ω will be decomposed

$$\Omega = \left\{ \bigcup_{\alpha \in I_s} K\left(x_\alpha^{(s)}, \lambda_s \rho_s\right) \right\} \bigcup U_s, \tag{2.14}$$

where I_s is the set of all multi-indices α such that $K\left(x_\alpha^{(s)}, 2\lambda_s \rho_s\right) \subset \Omega$ and U_s is the complement in Ω of the set $\bigcup\limits_{\alpha \in I_s} K\left(x_\alpha^{(s)}, \lambda_s \rho_s\right)$.

Moreover we introduce the notations

$$K_s(\alpha) = K\left(x_\alpha^{(s)}, \lambda_s \rho_s\right), \qquad K_s'(\alpha) = K\left(x_\alpha^{(s)}, (\lambda_s - 2)\rho_s\right). \tag{2.15}$$

Let us define the function

$$v_\alpha^{(s)}(x, q) = v(x, K_s'(\alpha) \setminus \Omega_s, q), \tag{2.16}$$

where $v(x, F, q)$ is the solution of the problem (2.7), (2.8) which was introduced above.

We define new sequence μ_s by the equality

$$\mu_s = \max\left\{ \lambda_s^n \left[\frac{\omega(\lambda_s \rho_s)}{(\lambda_s \rho_s)^{n-m}} \right]^{\frac{1}{m-1}}, \lambda_s \rho_s \right\}. \tag{2.17}$$

We have from the Lemma 4.1 [1]

$$\lim_{s \to \infty} \mu_s = 0. \tag{2.18}$$

Denote by $L_p(\Omega, \nu)$ the space of functions $v(x)$ defined on Ω measurable with respect to measure ν and such that

$$\|v\|_{L_p(\Omega,\nu)}^p = \int_\Omega |v(x)|^p d\nu < \infty.$$

Let $q_s(x)$ be an arbitrary sequence in $L_m(\Omega, \nu)$ that converges strongly in $L_m(\Omega, \nu)$ to some function $q_0(x)$ and we denote

$$q_\alpha^{(s)} = \frac{1}{\nu(K_s(\alpha))} \int_{K_s(\alpha)} q_s(x) d\nu. \tag{2.19}$$

We introduce subsets I_s', I_s'' of multi-indices α:

$$I_s' = \left\{\alpha \in I_s : \left|q_\alpha^{(s)}\right| > 2\mu_s\right\}, \qquad I_s'' = \left\{\alpha \in I_s : \left|q_\alpha^{(s)}\right| \le 2\mu_s\right\}. \tag{2.20}$$

Define the functions

$$\bar{v}_\alpha^{(s)}(x) = v_\alpha^{(s)}\left(x, \bar{q}_\alpha^{(s)}\right), \tag{2.21}$$

where

$$\bar{q}_\alpha^{(s)} = q_\alpha^{(s)}, \quad \text{for} \quad \alpha \in I_s', \qquad \bar{q}_\alpha^{(s)} = 2\mu_s \quad \text{for} \quad \alpha \in I_s''. \tag{2.22}$$

For an arbitrary function $g(x)$ we denote its positive part by $[g(x)]_+ = \max\{g(x), 0\}$. We define the cut-off functions $\varphi_\alpha^{(s)}(x)$ by the equality

$$\varphi_\alpha^{(s)}(x) = \frac{2}{\mu_\alpha^{(s)}} \min\left\{\left[\left|\overline{v}_\alpha^{(s)}(x)\right| - \frac{\mu_\alpha^{(s)}}{2}\right]_+, \frac{\mu_\alpha^{(s)}}{2}\right\}, \tag{2.23}$$

where

$$\mu_\alpha^{(s)} = \mu_s \max\left\{1, \left|q_\alpha^{(s)}\right|\right\}. \tag{2.24}$$

Let us construct the following sequence which is fundamental in the analysis of asymptotic behavior of solutions of the problem (1.1), (1.2)

$$r_s(x) = \sum_{\alpha \in I_s} v_\alpha^{(s)}\left(x, q_\alpha^{(s)}\right)\varphi_\alpha^{(s)}(x). \tag{2.25}$$

Remark that $r_s(x)$ is analogous to the corrector which was constructed in $[1-5]$. In particular $r_s(x)$ is analogous to principal term $r_s^{(3)}(x)$ of asymptotic expansion of the sequence of solutions in [1].

Our main result is the following theorem.

Theorem 2.1 (Convergence Theorem). *Assume that conditions A_1, A_2, B_1, B_2 are satisfied and let $q_s(x)$ be some sequence converging strongly in $L_m(\Omega, \nu)$. Let $z_s(x)$ be an arbitrary sequence of functions such that $z_s(x) \in \overset{\circ}{W}{}_m^1(\Omega_s)$ and $z_s(x)$ converges weakly to zero in $W_m^1(\Omega)$, $z_s(x) = 0$ on $\Omega \setminus \Omega_s$. Then the following equality*

$$\lim_{s\to\infty} \sum_{j=1}^{n} \int_\Omega a_j\left(x, \frac{\partial r_s(x)}{\partial x}\right)\frac{\partial z_s(x)}{\partial x_j}\, dx = 0 \tag{2.26}$$

holds.

***Remark* 2.5.** We take the attention of the reader that in conditions of the Theorem 2.1 the sequence $r_s(x)$ converges in $W_m^1(\Omega)$ only weakly (see Lemma 4.3 below) and this convergence is not strong.

3. Estimates for potentials. In this section we formulate some integral and pointwise estimates for the potential function $v(x, F, q)$ introduced in Section 2 as solution of the problem (2.7), (2.8).

Let us fix a compact set F contained in Ω and let $v(x, q) = v(x, F, q)$. For $0 < \mu \le |q|$ we define the set

$$E(\mu) = \{x \in \Omega_0 : |v(x, q)| \le \mu\}. \tag{3.1}$$

We shall assume that conditions A_1, A_2 are satisfied. We shall use integral and pointwise estimates for $v(x, q)$ that are proved in [1] (lemmas 2.1 and 2.5 respectively).

There exists a constant K_1 depending only on ν_1, ν_2, n, m such that the estimate

$$\int_{E(\mu)} \left|\frac{\partial v(x, q)}{\partial x}\right|^m dx \le K_1 \mu |q|^{m-1} \tag{3.2}$$

holds for every $q \in R^1$ and for every μ with $0 < \mu \le |q|$.

It is easy to see that the inequality $0 \le \frac{1}{q} v(x, q) \le 1$ holds for every $q \ne 0$. So we obtain an estimate of the norm of the function $v(x, q)$ in $\overset{\circ}{W}{}_m^1(\Omega_s)$ if we put $\mu = |q|$ in (3.2).

Assume that $K(x_0, 2r) \subset \Omega_0$ and set F is contained in a cube $K(x_0, r)$. Then there exists a constant K_2 depending only on ν_1, ν_2, n, m such that the following estimate

$$|v(x,q)| \leq K_2 |q| \left[\frac{r}{\rho(x, K(x_0, r))}\right]^{n-1} \left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}} \tag{3.3}$$

holds for $x \in K(x_0, 2r) \backslash K(x_0, r)$, where $\rho(x, K(x_0, r))$ is the distance from the point x to the cube $K(x_0, r)$.

Let us introduce auxiliary function $w(x, q, F)$ as a solution of following boundary-value problem

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\{\left|\frac{\partial w}{\partial x}\right|^{m-2} \frac{\partial w}{\partial x_j}\right\} = 0, \quad x \in K(x_0, 2r) \backslash F, \tag{3.4}$$

$$w(x) = q\varphi(x), \quad x \in \partial[K(x_0, 2r) \backslash F], \tag{3.5}$$

where $q \in R^1$, F is a closed subset of $K(x_0, r)$ and $\varphi(x)$ is a function of class $C_0^\infty(K(x_0, 2r))$ equal to one in $K(x_0, r)$. Extend $w(x, q, F)$ on F by the equality $w(x, q, F) = q$ for $x \in F$.

This function $w(x, q, F)$ satisfies estimates analogous to estimates (3.2), (3.3).

Theorem 3.1. *Let λ be some number from the interval $(3/2, 2)$. Then there exists a constant K_3 depending only on m, n, λ such that the estimate*

$$\left|\frac{\partial w(x)}{\partial x}\right| \leq \frac{K_3 |q|}{r} \left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}} \quad \text{for} \quad x \in K(x_0, \lambda r) \backslash K\left(x_0, \frac{3}{2} r\right) \tag{3.6}$$

holds, where $w(x)$ is the solution of the problem (3.4), (3.5).

Proof is analogous to the proof of the Theorem 5 in [13].

Theorem 3.2. *Let $w(x)$ be the solution of the problem* (3.4), (3.5). *Then there exist positive constants α, K_4 depending only on n, m such that the estimate*

$$|w(x)| \leq K_4 |q| \left[\frac{\rho(x, \partial K(x_0, 2r))}{r}\right]^{\alpha} \left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}} \tag{3.7}$$

holds for $x \in K(x_0, 2r) \backslash K\left(x_0, \frac{3}{2} r\right)$.

Proof. Let x' be an arbitrary point on the boundary of $K(x_0, 2r)$ and denote $M' = \max\left\{|w(x)| : x \in B\left(x', \frac{r}{2}\right)\right\}$, where $B(x_0, \rho)$ is a ball at centre x_0 and radius ρ. Using Moser method for the proof of Holder continuity of $w(x, q, F)$ (see for example [14], Chapter IX, §5) we obtain the estimate

$$|w(x) - w(x')| \leq C_1 \left(\frac{|x - x'|}{r}\right)^{\alpha} M' \quad \text{for} \quad x \in B\left(x', \frac{r}{2}\right) \tag{3.8}$$

with constants α, C_1 depending only on n, m.

Now the inequality (3.7) is followed from (3.8) and the analog of the estimate (3.3) for $w(x)$.

Theorem 3.3. *Let $w(x)$ be the solution of the problem* (3.4), (3.5) *and let γ be some number from the interval* (1, 2). *Then there exists a constant K_5 depending only on n, m, γ such that the inequality*

$$\min\left\{|w(x)|:\, x\in\partial K(x_0,\gamma r)\right\} \ \ge\ K_5|q|\left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}} \tag{3.9}$$

holds.

Proof. It suffices to consider the case $q>0$. Define $r_j=\gamma r+j(2-\gamma)r/5$, $j=1,2,3,4$, and functions $\psi_1(x),\ \psi_2(x)\in C_0^\infty(K(x_0,2r))$ such that $\psi_1(x)=1$ for $x\in K(x_0,r_3)$, $\psi_1(x)=0$ for $x\notin K(x_0,r_3)$, $\psi_2(x)=1$ for $x\in K(x_0,r_3)\setminus K(x_0,r_2)$ and $\psi_2(x)=0$ for $x\in K(x_0,r_1)$ or $x\notin K(x_0,r_4)$. We can assume that

$$\left|\frac{\partial\psi_k(x)}{\partial x}\right| \le C_2\frac{1}{r}, \qquad k=1,2.$$

Constants C_j in the proof of the proof of Theorem 3.3 depend only on n, m, γ.

We substitute in the integral identity

$$\sum_{j=1}^{n}\int\limits_{K(x_0,2r)}\left|\frac{\partial w}{\partial x}\right|^{m-2}\frac{\partial w}{\partial x_j}\frac{\partial\varphi}{\partial x_j}\,dx = 0, \qquad \varphi(x)\in \overset{\circ}{W}{}^1_m(K(x_0,2r)\setminus F) \tag{3.10}$$

a test function $\varphi(x)=[q-w(x)]\,\psi_1^m(x)$. Using Holder inequality we obtain

$$\int\limits_{K(x_0,2r)}\left|\frac{\partial w(x)}{\partial x}\right|^m\psi_1^m(x)\,dx \le C_3\frac{q}{r}\int\limits_{\mathcal{D}}\left|\frac{\partial w}{\partial x}\right|^{m-1}dx \le$$

$$\le C_3\frac{q}{r}\left\{\int\limits_{\mathcal{D}}w^{\sigma m}(x)\,dx\right\}^{\frac{1}{m}}\left\{\int\limits_{\mathcal{D}}\left|\frac{\partial w(x)}{\partial x}\right|^m[w(x)]^{-\sigma m/(m-1)}\,dx\right\}^{\frac{m-1}{m}} \le$$

$$\le C_4\,q[M(r_2)]^{\sigma}r^{(n-m)/m}\left\{\int\limits_{K(x_0,2r)}\left|\frac{\partial w(x)}{\partial x}\right|^m[w(x)]^{-\sigma m/(m-1)}\psi_2^m(x)\,dx\right\}^{\frac{m-1}{m}}, \tag{3.11}$$

where $\mathcal{D}=K(x_0,r_3)\setminus K(x_0,r_2)$, $\sigma=\dfrac{m-1}{2m}$,

$$M(\rho) = \max\{w(x):\, x\in\partial K(x_0,\rho)\}. \tag{3.12}$$

Using the Harnack inequality for the equation (3.4) (see [15]) we have $\min\{w(x):\, x\in K(x_0,r_4)\setminus K(x_0,r_1)\}>0$. Substitute in the identity (3.10) new test function $\varphi(x)=w^{1-\sigma m/(m-1)}(x)\psi_2^m(x)$ and we obtain

$$\int\limits_{K(x_0,2r)}\left|\frac{\partial w(x)}{\partial x}\right|^m[w(x)]^{-\sigma m/(m-1)}\psi_2^m(x)\,dx \le$$

$$\le C_5\frac{1}{r}\int\limits_{K(x_0,2r)}\left|\frac{\partial w(x)}{\partial x}\right|^{m-1}[w(x)]^{1-\sigma m/(m-1)}\psi_2^{m-1}(x)\,dx.$$

Estimating the last integral by Young inequality we have

$$\int\limits_{K(x_0,2r)}\left|\frac{\partial w(x)}{\partial x}\right|^m[w(x)]^{-\sigma m/(m-1)}\psi_2^m(x)\,dx \le$$

$$\le C_6\frac{1}{r^m}\int\limits_{K(x_0,r_4)\setminus K(x_0,r_1)}[w(x)]^{m-\sigma m/(m-1)}\,dx \le C_7[M(r_1)]^{m-\sigma m/(m-1)}r^{n-m}. \tag{3.13}$$

From inequalities (3.11), (3.13) we obtain the estimate

$$\int_{K(x_0,2r)} \left|\frac{\partial(w\psi_1)}{\partial x}\right|^m dx \le C_8[M(r_1)]^m r^{n-m} +$$

$$+ C_8 q[M(r_1)]^{m-1} r^{n-m} \le C_9 q[M(r_1)]^{m-1} r^{n-m}. \tag{3.14}$$

By the definition of the capacity we have the following estimate for the integral on the left-hand side of (3.14)

$$\int_{K(x_0,2r)} \left|\frac{\partial(w\psi_1)}{\partial x}\right|^m dx \ge q^m C_m(F). \tag{3.15}$$

The inequalities (3.14), (3.15) imply the estimate

$$M(r_1) \ge C_{10}\, q\left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}}. \tag{3.16}$$

From the Harnack inequality [15] and (3.16) we have the estimate

$$\min\{w(x) : x\in \partial K(x_0, r_1)\} \ge C_{11}M(r_1) \ge C_{12}\, q\left\{\frac{C_m(F)}{r^{n-m}}\right\}^{\frac{1}{m-1}}$$

and the proof of the Theorem 3.2 is completed.

Denote

$$\tau(r) = \int_0^r \frac{\omega(\rho)}{\rho^{n-m+1}}\, d\rho + \frac{\omega(r)}{r^{n-m}}, \tag{3.17}$$

where $\omega(\rho)$ is the function introduced in the condition B_2.

Lemma 3.1. *Assume that conditions B_1, B_2 are satisfied. Then there exists a constant K_6 depending only on n, m such that the inequality*

$$\int_{K(x_0,r)} \frac{d\nu_y}{|x-y|^{n-m}} \le K_6\,\tau(r) \tag{3.18}$$

holds for $x \in K(x_0, 2r)$, where $K(x_0, 2r)$ is an arbitrary cub satisfying an inclusion $K(x_0, 2r) \subset \Omega$.

Proof. Denote $\omega(r,\rho) = \omega(\min(r,\rho))$. From the properties of the function ω we have

$$\sup_{x\in R^n} \nu(K(x_0,r)\cap\Omega\cap B(x,\rho)) \le \omega(r,\rho).$$

Using the Theorem 6.1 of [16] we obtain the inequality

$$\int_{K(x_0,r)} \frac{d\nu_y}{|x-y|^{n-m}} \le C_{13}\int_0^\infty \frac{\omega(r,\rho)}{\rho^{n-m+1}}\, d\rho =$$

$$= C_{13}\left\{\int_0^r \frac{\omega(\rho)}{\rho^{n-m+1}}\, d\rho + \frac{\omega(r)}{n-m}\frac{1}{r^{n-m}}\right\} \le C_{13}\left(1+\frac{1}{n-m}\right)\tau(r)$$

that gives us the estimate (3.18).

Lemma 3.2. *Let ν be the measure introduced in the condition B_1 and assume that condition B_2 is satisfied. Then for any function $u(x) \in W_m^1(K(x_0,2r))$ and any cub $K(x_0,2r) \subset \Omega$ the inequality*

$$\int\limits_{K(x_0,2r)\setminus K(x_0,r)} \left|u(x)-u_{\nu,r}\right|^m dx \le K_7\tau(r)\frac{r^n}{\nu(K(x_0,r))}\int\limits_{K(x_0,r)}\left|\frac{\partial u(x)}{\partial x}\right|^m dx \tag{3.19}$$

holds with a constant K_7 *depending only on* n, m. *Here*

$$u_{\nu,r} = \frac{1}{\nu(K(x_0,r))}\int\limits_{K(x_0,r)} u(x)\,d\nu. \tag{3.20}$$

Proof. Remark that from the conditions B_1, B_2 and from the Theorems of § 8.6, 8.8 in [17] it is followed the compact embedding

$$W_m^1(K(x_0,2r)) \subset L_m(K(x_0,2r),\nu) \quad \text{for} \quad K(x_0,2r)\subset\Omega. \tag{3.21}$$

Consequently the integral in (3.20) is well defined and it suffices to prove the estimate (3.19) for $u(x)\in C^1(K(x_0,2r))$.

Let $x\in K(x_0,2r)\setminus K(x_0,r)$, $y\in K(x_0,r)$ be such points that

$$\frac{x-x_0}{|x-x_0|} = \frac{y-y_0}{|y-y_0|} = \omega.$$

Using an quality

$$u(x) - u(y) = \int\limits_{|y-y_0|}^{|x-x_0|}\frac{\partial u}{\partial t}(x_0+\omega t)\,dt$$

and a straight-forward computation we obtain

$$|u(x) - u_{\nu,r}| \le$$

$$\le \frac{1}{\nu(K(x_0,r))}\int\limits_{K(x_0,r)}|y-x_0|^{1-n/m}\,d\nu_y\left\{\int\limits_0^{|x-x_0|}\left|\frac{\partial u(x_0+\omega t)}{\partial t}\right|^m t^{n-1}\,dt\right\}^{\frac{1}{m}}. \tag{3.22}$$

Evaluate the first integral on the right-hand side of (3.22) by using Holder inequality and the estimate (3.18) and we obtain

$$\int\limits_{K(x_0,r)}|y-x_0|^{1-n/m}\,d\nu_y \le C_{14}\left[\nu(K(x_0,r))\right]^{(m-1)/m}\tau^{1/m}(r). \tag{3.23}$$

Representing the integral on the left-hand side of (3.19) in spherical coordinates centered at x_0 with respect to variables

$$\omega = \frac{x-x_0}{|x-x_0|}\in S(0,1), \qquad \rho = |x-x_0|\in[\rho_1(\omega),\rho_2(\omega)]$$

we have from (3.22), (3.23)

$$\int\limits_{K(x_0,2r)\setminus K(x_0,r)}\left|u(x)-u_{\nu,r}\right|^m dx = \int\limits_{S(0,1)}\int\limits_{\rho_1(\omega)}^{\rho_2(\omega)}\left|u(x_0+\rho\omega)-u_{\nu,r}\right|^m\rho^{n-1}\,d\rho\,d\omega \le$$

$$\le C_{15}\left[\nu(K(x_0,r))\right]^{-1}\tau(r)r^n\int\limits_{S(0,1)}\int\limits_{\rho_1(\omega)}^{\rho_2(\omega)}\left|\frac{\partial u(x_0+t\omega)}{\partial t}\right|^m t^{n-1}\,dt\,d\omega =$$

$$= C_{15}\frac{r^n}{\nu(K(x_0,r))}\tau(r)\int\limits_{K(x_0,r)}\left|\frac{\partial u(x)}{\partial x}\right|^m dx. \tag{3.24}$$

The proof of the Lemma 3.2 is completed.

4. Proof of the Convergence Theorem. Denote by $G_\alpha^{(s)}$ the support of the function $\varphi_\alpha^{(s)}(x)$. In this section we shall use the notation C_j, $j = 17, 18, \ldots$, for constants which depend only on $n, m, \nu_1, \nu_2, \nu(\Omega)$.

Lemma 4.1. *Assume that conditions A_1, A_2, B_1, B_2 are satisfied. Then there exists an integer s_1 such that the inclusion*

$$G_\alpha^{(s)} \subset K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big) \quad \text{for} \quad \alpha \in I_s \tag{4.1}$$

holds for $s \geq s_1$.

Proof. Using the pointwise estimate (3.8) and the conditions B_1, B_2 we obtain the inequality

$$\left|\overline{v}_\alpha^{(s)}(x)\right| \leq C_{16} \max\left\{\left|q_\alpha^{(s)}\right|, 2\mu_s\right\} \lambda_s^{n-1} \left\{\frac{C_m(K_s'(\alpha) \setminus \Omega_s)}{(\lambda_s \rho_s)^{n-m}}\right\}^{\frac{1}{m-1}} \leq$$

$$\leq C_{17} \max\left\{\left|q_\alpha^{(s)}\right|, 2\mu_s\right\} \lambda_s^{n-1} \left\{\frac{\omega(\lambda_s \rho_s)}{(\lambda_s \rho_s)^{n-m}}\right\}^{\frac{1}{m-1}} \tag{4.2}$$

for $x \in \partial K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big)$. By the maximum principle the same inequality holds for every $x \notin K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big)$. From (2.11), (2.18), (2.17) we have the inequality

$$C_{17} \lambda_s^{n-1} \left\{\frac{\omega(\lambda_s \rho_s)}{(\lambda_s \rho_s)^{n-m}}\right\}^{\frac{1}{m-1}} \leq \frac{C_{17}}{\lambda_s} \mu_s \leq \frac{\mu_s}{2}$$

for sufficiently large s. Consequently from (4.2), (2.24) we have

$$\left[\left|\overline{v}_\alpha^{(s)}(x)\right| - \frac{\mu_\alpha^{(s)}}{2}\right]_+ = 0 \quad \text{for} \quad x \in K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big) \tag{4.3}$$

which implies (4.1).

Lemma 4.2. *Assume that conditions A_1, A_2, B_1, B_2 are satisfied. Then the inequality*

$$\operatorname{meas} G_\alpha^{(s)} \leq C_{18} (\lambda_s \rho_s)^m \mu_s^{1-m} \nu\Big(K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big)\Big) \tag{4.4}$$

holds.

Proof. We introduce an auxiliary function

$$\overline{\varphi}_\alpha^{(s)}(x) = \frac{4}{\mu_\alpha^{(s)}} \min\left\{\left[\overline{v}_\alpha^{(s)}(x) - \frac{\mu_\alpha^{(s)}}{4}\right]_+, \frac{\mu_\alpha^{(s)}}{4}\right\}.$$

As in the proof of the Lemma 4.1 we can prove that $\overline{\varphi}_\alpha^{(s)}(x) = 0$ for $x \notin K_s(\alpha)$ and s large enough. Using Poincaré's inequality, the estimate (3.2) and the condition B_1 we obtain

$$\int\limits_{K_s(\alpha)} \left|\overline{\varphi}_\alpha^{(s)}(x)\right|^m dx \leq C_{19} (\lambda_s \rho_s)^m \int\limits_{K_s(\alpha)} \left|\frac{\partial \overline{\varphi}_\alpha^{(s)}(x)}{\partial x}\right|^m dx \leq$$

$$\leq C_{20} \mu_s^{1-m} (\lambda_s \rho_s)^m \nu\Big(K\big(x_\alpha^{(s)}, (\lambda_s - 1)\rho_s\big)\Big).$$

Observing that $\overline{\varphi}_\alpha^{(s)}(x) = 1$ for $x \in G_\alpha^{(s)}$ we obtain the estimate (4.4) from the last inequality.

Lemma 4.3. *Assume that conditions* A_1, A_2, B_1, B_2 *are satisfied and let* $q_s(x)$ *be an arbitrary sequence that converges strongly in* $L_m(\Omega, \nu)$. *Then the sequence* $r_s(x)$ *defined by the equality* (2.25) *converges strongly to zero in* $W_p^1(\Omega)$ *for any* $p < m$ *and converges weakly in* $W_m^1(\Omega)$ *as* $s \to \infty$.

Proof. We can assume that $s \geq s_1$, where s_1 is defined in Lemma 4.1. Then from the inclusion (4.1) we have

$$G_\alpha^{(s)} \cap G_\beta^{(s)} = \varnothing \quad \text{for} \quad \alpha \neq \beta, \quad \alpha, \beta \in I_s. \tag{4.5}$$

Let us estimate the norm of the gradient of $r_s(x)$ in $L_m(\Omega)$ for s large enough such that $\mu_s \leq 1$. We have

$$\left\| \frac{\partial r_s(x)}{\partial x} \right\|_{L_m(\Omega)}^m \leq C_{21} \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} \left| \frac{\partial v_\alpha^{(s)}(x, q_\alpha^{(s)})}{\partial x} \right|^m dx \, +$$

$$+ \, C_{21} \sum_{\alpha \in I_s} \left[\mu_\alpha^{(s)} \right]^{-m} \int_{\tilde{E}_\alpha^{(s)}} \left| v_\alpha^{(s)}(x, q_\alpha^{(s)}) \right|^m \left| \frac{\partial v_\alpha^{(s)}(x, \overline{q}_\alpha^{(s)})}{\partial x} \right|^m dx, \tag{4.6}$$

where $\tilde{E}_\alpha^{(s)} = \left\{ x \in \Omega_0 : \frac{\mu_\alpha^{(s)}}{2} < \left| v_\alpha^{(s)}(x) \right| < \mu_\alpha^{(s)} \right\}$.

We evaluate the first summand on the right-hand side of (4.6) by using the inequality (3.2) and the condition B_1:

$$\sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} \left| \frac{\partial v_\alpha^{(s)}(x, q_\alpha^{(s)})}{\partial x} \right|^m dx \leq C_{22} \sum_{\alpha \in I_s} \left| q_\alpha^{(s)} \right|^m \nu(K_s(\alpha)). \tag{4.7}$$

From the Holder inequality we have

$$\left| q_\alpha^{(s)} \right| = \frac{1}{\nu(K_s(\alpha))} \left| \int_{K_s(\alpha)} q_s(x)\, d\nu \right| \leq \left\{ \frac{1}{\nu(K_s(\alpha))} \int_{K_s(\alpha)} |q_s(x)|^m d\nu \right\}^{\frac{1}{m}} \tag{4.8}$$

and we estimate the sum on the right-hand side of the inequality (4.7)

$$\sum_{\alpha \in I_s} \left| q_\alpha^{(s)} \right|^m \nu(K_s(\alpha)) \leq \int_\Omega |q_s(x)|^m d\nu. \tag{4.9}$$

Remarking that the inequality

$$\left| v_\alpha^{(s)}(x, q_\alpha^{(s)}) \right| \leq 2\mu_s \quad \text{for} \quad \alpha \in I_s'' \tag{4.10}$$

holds we can evaluate the second summand on the right-hand side of (4.6) analogously to (4.7), (4.9) and we obtain the estimate

$$\int_\Omega \left| \frac{\partial r_s(x)}{\partial x} \right|^m dx \leq C_{23} \int_\Omega |q_s(x)|^m d\nu. \tag{4.11}$$

Remarking that the function $r_s(x)$ vanishes outside $\bigcup_{\alpha \in I_s} G_\alpha^{(s)}$ and using the Holder inequality we get

$$\left\| \frac{\partial}{\partial x} r_s(x) \right\|_{L_p(\Omega)} \leq \left\| \frac{\partial}{\partial x} r_s(x) \right\|_{L_m(\Omega)} \left\{ \sum_{\alpha \in I_s} \operatorname{meas} G_\alpha^{(s)} \right\}^{\frac{1}{p} - \frac{1}{m}} \quad \text{for} \quad 1 < p < m. \tag{4.12}$$

The second factor on the right-hand side of last inequality tends to zero by (4.4), (2.17), (2.18). This completes the proof of the Lemma.

Lemma 4.4. *Assume that conditions of the Lemma 4.3 are satisfied. Then the sequence*

$$r_s''(x) = \sum_{\alpha \in I_s''} v_\alpha^{(s)}(x, q_\alpha^{(s)}) \varphi_\alpha^{(s)}(x) \tag{4.13}$$

converges strongly to zero in $W_m^1(\Omega)$.

Proof. Analogously to the proof of the Lemma 4.3 we have the estimate

$$\int_\Omega \left| \frac{\partial r_s''(x)}{\partial x} \right|^m dx \le C_{24} \sum_{\alpha \in I_s''} \left| q_\alpha^{(s)} \right|^m \nu(K_s(\alpha))$$

and the right-hand side of the last inequality tends to zero by the definition of I_s'' and (2.18). The proof of the Lemma is completed.

Let ζ_s be an arbitrary sequence of real numbers satisfying an equality

$$\lim_{s \to \infty} \zeta_s = 0. \tag{4.14}$$

Let us define the subsets $I_{1,s}'$, $I_{2,s}'$ of multi-indices α by the equalities

$$I_{1,s}' = \left\{ \alpha \in I_s' : \zeta_s \left| q_\alpha^{(s)} \right|^{m-1} \le 1 \right\}, \qquad I_{2,s}' = \left\{ \alpha \in I_s' : \zeta_s \left| q_\alpha^{(s)} \right|^{m-1} > 1 \right\}$$

and denote

$$r_{i,s}'(x) = \sum_{\alpha \in I_{i,s}'} v_\alpha^{(s)}(x, q_\alpha^{(s)}) \varphi_\alpha^{(s)}(x), \qquad i = 1, 2. \tag{4.15}$$

Lemma 4.5. *Assume that conditions of the Lemma 4.3 are satisfied and let* ζ_s *be an arbitrary sequence satisfying the condition* (4.14). *Then the sequence* $r_{2,s}'(x)$ *defined by* (4.15) *converges strongly to zero in* $W_m^1(\Omega)$.

Proof. Denote $Q_s = \bigcup_{\alpha \in I_{2,s}'} K_s(\alpha)$. From the inequality

$$\zeta_s^{-m/m-1} \nu(Q_s) \le \sum_{\alpha \in I_{2,s}'} \left| q_\alpha^{(s)} \right|^m \nu(K_s(\alpha))$$

and (4.8) we have

$$\nu(Q_s) \le \zeta_s^{m/m-1} \int_\Omega |q_s(x)|^m d\nu. \tag{4.16}$$

Analogously to the proof of the inequality (4.11) we obtain

$$\int_\Omega \left| \frac{\partial r_{2,s}'(x)}{\partial x} \right|^m dx \le C_{25} \int_{Q_s} |q_s(x)|^m d\nu$$

and the convergence of the right-hand side of last inequality to zero is followed from (4.14), (4.16) and the assumption on the sequence $q_s(x)$. The proof of the Lemma is completed.

***Proof of the Theorem* 2.1.** Define the sequence ζ_s by the equality

$$\zeta_s = \max \left\{ \| z_s(x) \|_{L_m(\Omega, \nu)}^{1/2}, \lambda_s \rho_s, [\tau(\lambda_s \rho_s)]^{1/2m} \right\}, \tag{4.17}$$

where $z_s(x)$ is the sequence introduced in the Theorem 1.1. This sequence ζ_s satisfies the condition (4.14) and let $r'_{1,s}(x)$, $r'_{2,s}(x)$ be sequences defined by the equality (4.15) for considered choice of ζ_s.

Using the condition A_2, Lemmas 4.1, 4.3, 4.5 and assumptions on $z_s(x)$ we obtain

$$\lim_{s\to\infty}\sum_{j=1}^{n}\int_{\Omega}\left\{a_j\left(x,\frac{\partial r_s(x)}{\partial x}\right)-a_j\left(x,\frac{\partial r'_{1,s}(x)}{\partial x}\right)\right\}\frac{\partial z_s(x)}{\partial x_j}dx = 0 \tag{4.18}$$

and it is sufficient to study the behaviour of the term

$$J_s = \sum_{j=1}^{n}\int_{\Omega} a_j\left(x,\frac{\partial r'_{1,s}(x)}{\partial x}\right)\frac{\partial z_s(x)}{\partial x_j}dx. \tag{4.19}$$

Denote

$$\delta_\alpha^{(s)} = K_5\left(\frac{3}{2}\right)\left\{\frac{C_m(K'_s(\alpha)\setminus\Omega_s)}{[\lambda_s\rho_s]^{n-m}}\right\}^{\frac{1}{m-1}}, \tag{4.20}$$

where the constant $K_5\left(\frac{3}{2}\right)$ is defined in the Lemma 3.2.

Define a function

$$\zeta_\alpha^{(s)}(x) = \frac{2}{\delta_\alpha^{(s)}}\min\left\{\left[w_\alpha^{(s)}(x,K'_s(\alpha)\setminus\Omega_s,1)-\frac{\delta_\alpha^{(s)}}{2}\right]_+;\frac{\delta_\alpha^{(s)}}{2}\right\}, \tag{4.21}$$

where $w_\alpha^{(s)}(x,F,q)$ is the solution of the problem (3.4), (3.5) with $x_0 = x_\alpha^{(s)}$, $r = \lambda_s\rho_s$. The cub $K'_s(\alpha)$ in (4.20) is defined by (2.15).

Using the estimates (3.7), (3.9) and the choice of $\delta_\alpha^{(s)}$ we have

$$\zeta_\alpha^{(s)}(x) \equiv 1 \quad \text{for} \quad x\in K\left(x_\alpha^{(s)},\frac{3}{2}\lambda_s\rho_s\right), \tag{4.22}$$

$$\zeta_\alpha^{(s)}(x) \equiv 0 \quad \text{for} \quad x\in K\left(x_\alpha^{(s)},2\lambda_s\rho_s\right)\setminus K\left(x_\alpha^{(s)},\gamma\lambda_s\rho_s\right), \tag{4.23}$$

with some number γ depending only on n, m.

We rewrite J_s in the from

$$J_s = \sum_{i=1}^{5} J_s^{(i)}, \tag{4.24}$$

where

$$J_s^{(1)} = \sum_{\alpha\in I'_{1,s}}\sum_{j=1}^{n}\int_{\bar K_s(\alpha)}\left[a_j\left(x,\frac{\partial}{\partial x}\left(v_\alpha^{(s)}\varphi_\alpha^{(s)}\right)\right)-a_j\left(x,\frac{\partial v_\alpha^{(s)}}{\partial x}\right)\right]\frac{\partial z_s(x)}{\partial x_j}dx,$$

$$J_s^{(2)} = \sum_{\alpha\in I'_{1,s}}\sum_{j=1}^{n}\int_{\bar K_s(\alpha)} a_j\left(x,\frac{\partial v_\alpha^{(s)}}{\partial x}\right)\frac{\partial}{\partial x_j}\left[\zeta_\alpha^{(s)}z_s(x)\right]dx,$$

$$J_s^{(3)} = \sum_{\alpha\in I'_{1,s}}\sum_{j=1}^{n}\int_{\bar K_s(\alpha)} a_j\left(x,\frac{\partial v_\alpha^{(s)}}{\partial x}\right)\left(1-\zeta_\alpha^{(s)}(x)\right)\frac{\partial}{\partial x_j}z_s(x)\,dx, \tag{4.25}$$

$$J_s^{(4)} = -\sum_{\alpha\in I'_{1,s}}\sum_{j=1}^{n}\int_{\bar K_s(\alpha)} a_j\left(x,\frac{\partial v_\alpha^{(s)}}{\partial x}\right)\frac{\partial\zeta_\alpha^{(s)}(x)}{\partial x_j}\cdot\left[z_s(x)-z_s(\alpha)\right]dx,$$

$$J_s^{(5)} = -\sum_{\alpha\in I'_{1,s}} \sum_{j=1}^{n} z_s(\alpha) \int\limits_{\tilde K_s(\alpha)} a_j\left(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\right) \frac{\partial \zeta_\alpha^{(s)}}{\partial x_j}\, dx,$$

where

$$z_s(\alpha) = \frac{1}{\operatorname{meas}(K_s(\alpha))} \int\limits_{K_s(\alpha)} z_s(x)\, dx,$$

$$v_\alpha^{(s)} = v_\alpha^{(s)}\left(x, q_\alpha^{(s)}\right), \qquad \tilde K_s(\alpha) = K\left(x_\alpha^{(s)}, 2\lambda_s \rho_s\right).$$

Define a set

$$E_\alpha^{(s)}(\mu) = \left\{ x \in \tilde K_s(\alpha) : \left| v_\alpha^{(s)}\left(x, q_\alpha^{(s)}\right)\right| \le \mu \right\}.$$

The function $\varphi_\alpha^{(s)}(x)$ is equal to one if $\left|v_\alpha^{(s)}\left(x, q_\alpha^{(s)}\right)\right| \ge \mu_\alpha^{(s)}$, $\alpha \in I'_s$. Using (2.3) and Holder inequality we obtain the estimate

$$\left|J_s^{(1)}\right| \le$$

$$\le C_{26} \left\{ \sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\mu_\alpha^{(s)})} \left[\left| \frac{\partial}{\partial x}\left(v_\alpha^{(s)} \varphi_\alpha^{(s)}\right)\right| + \left|\frac{\partial v_\alpha^{(s)}}{\partial x}\right| \right]^m dx \right\}^{\frac{m-1}{m}} \times$$

$$\times \left\{ \int\limits_{\Omega} \left| \frac{\partial z_s(x)}{\partial x}\right|^m dx \right\}^{\frac{1}{m}}. \tag{4.26}$$

We estimate the second factor on the right-hand side of (4.26) by using inequalities (3.2), (4.8). We obtain

$$\sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\mu_\alpha^{(s)})} \left[\left| \frac{\partial}{\partial x}\left(v_\alpha^{(s)} \varphi_\alpha^{(s)}\right)\right| + \left|\frac{\partial v_\alpha^{(s)}}{\partial x}\right| \right]^m dx \le$$

$$\le C_{27} \mu_s \sum_{\alpha\in I'_{1,s}} \left|q_a^{(s)}\right|^m \nu(K_s(\alpha)) \le C_{27} \mu_s \int\limits_{\Omega} |q_s(x)|^m d\nu$$

and the right-hand side of the last inequality tends to zero as $s \to \infty$. Taking into account the assumption on $z_s(x)$ we obtain

$$\lim_{s\to\infty} J_s^{(1)} = 0. \tag{4.27}$$

In the same way as for $J_s^{(1)}$ we obtain the equality

$$\lim_{s\to\infty} J_s^{(3)} = 0. \tag{4.28}$$

The equality

$$J_s^{(2)} = 0 \tag{4.29}$$

is followed from the definition of the functions $v_\alpha^{(s)}\left(x, q_\alpha^{(s)}\right)$ and from the properties of $\zeta_\alpha^{(s)}(x)$, $z_s(x)$.

In order to estimate $J_s^{(4)}$ we remark that from the Theorem 3.1 the inequality

$$\left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right| \le \frac{C_{28}}{\lambda_s \rho_s} \tag{4.30}$$

holds for $\alpha \in I'_{1,s}$. Using the condition A_2 and Holder inequality we obtain the estimate

$$\left| J_s^{(4)} \right| \le C_{29} \frac{1}{\lambda_s \rho_s} \sum_{\alpha \in I'_{1,s}} \left\{ \int_{\mathcal{D}_s(\alpha)} \left| \frac{\partial v_\alpha^{(s)}}{\partial x} \right|^m dx \right\}^{\frac{m-1}{m}} \left\{ \int_{\mathcal{D}_s(\alpha)} |z_s(x) - z_s(\alpha)|^m dx \right\}^{\frac{1}{m}}, \tag{4.31}$$

where $\mathcal{D}_s(\alpha)$ is the support of the function $\left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|$.

We estimate integrals with $v_\alpha^{(s)}$ in (4.31) by using the estimate (3.2) and integral with $z_s(x)$ by using the Lemma 3.2. We obtain

$$\left| J_s^{(4)} \right| \le C_{30} [\tau(\lambda_s \rho_s)]^{1/m} \sum_{\alpha \in I'_{1,s}} \left| q_\alpha^{(s)} \right|^{m-1} [\nu(K_s(\alpha))]^{(m-1)/m} \left\{ \int_{\tilde{K}_s(\alpha)} \left| \frac{\partial z_s}{\partial x} \right|^m dx \right\}^{\frac{1}{m}}. \tag{4.32}$$

Estimating the right-hand side of (4.32) by Holder inequality we get

$$J_s^{(4)} \le C_{31} [\tau(\lambda_s \rho_s)]^{1/m} \left\{ \int_\Omega |q_s(x)|^m d\nu \right\}^{\frac{m-1}{m}} \left\{ \int_\Omega \left| \frac{\partial z_s}{\partial x} \right|^m dx \right\}^{\frac{1}{m}}. \tag{4.33}$$

Taking into account that $\tau(r)$ tends to zero as $r \to 0$ we get from (4.33)

$$\lim_{s \to \infty} J_s^{(4)} = 0. \tag{4.34}$$

Let us consider the behaviour of $J_s^{(5)}$ as $s \to \infty$. Remarking that the support of $\left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|$ is contained in $K\left(x_\alpha^{(s)}, \gamma \lambda_s \rho_s\right) \setminus K\left(x_\alpha^{(s)}, \frac{3}{2} \lambda_s \rho_s\right)$ and using the inequality (3.2) we have the estimate

$$\left| \sum_{j=1}^n \int_{\tilde{K}_s(\alpha)} a_j\left(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\right) \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x_j} dx \right| \le$$

$$\le C_{32} \frac{\left| q_\alpha^{(s)} \right|^{m-1}}{\delta_\alpha^{(s)}} \left\{ \frac{C_m(K'_s(\alpha) \setminus \Omega_s)}{[\lambda_s \rho_s]^{n-m}} \right\}^{\frac{1}{m-1}} \nu(K_s(\alpha)). \tag{4.35}$$

From the Hölder inequality and the Lemma 3.2 we have the estimate

$$\left| z_s(\alpha) - \frac{1}{\nu(K_s(\alpha))} \int_{K_s(\alpha)} z_s(x) d\nu \right| \le C_{33} \left\{ \tau(\lambda_s \rho_s) \frac{1}{\nu(K_s(\alpha))} \int_{K_s(\alpha)} \left| \frac{\partial z_s(x)}{\partial x} \right|^m dx \right\}^{\frac{1}{m}}.$$

Using Remark 2.4 and the inequality $\left| q_\alpha^{(s)} \right|^{m-1} \le \frac{1}{\zeta_s}$ for $\alpha \in I'_{1,s}$ we obtain from (4.35), (4.17) and the last inequality

$$\left| J_s^{(5)} \right| \le C_{34} \| z_s \|_{L_m(\Omega, \nu)}^{1/2} + C_{34} \tau(\lambda_s \rho_s) \left\| \frac{\partial z_s}{\partial x} \right\|_{L_m(\Omega)}. \tag{4.36}$$

From compactness of embedding $W_m^1(\Omega) \subset L_m(\Omega, \nu)$ and (4.36) following equality

$$\lim_{s \to \infty} J_s^{(5)} = 0. \tag{4.37}$$

holds. Now the equality (1.26) is followed from (4.18), (4.19), (4.24), (4.27)–(4.29), (4.34), (4.37) and the proof of the Theorem 1.1 is completed.

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