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A PROBABILISTIC REPRESENTATION FOR THE SOLUTION TO ONE PROBLEM OF MATHEMATICAL PHYSICS

ЙМОВІРНІСНЕ ЗОБРАЖЕННЯ РОЗВ'ЯЗКУ ОДНІЄЇ ЗАДАЧІ МАТЕМАТИЧНОЇ ФІЗИКИ

We consider a multidimensional Wiener process with a semipermeable membrane situated on a given hyperplane. The paths of this process are the solutions to a stochastic differential equation which can be regarded as a generalization of the well-known Skorokhod equation for diffusion process in a bounded domain with boundary conditions on the boundary. We change randomly the time in this process by using an additive functional of local time type. As a result, we obtain a probabilistic representation for solutions to some problem to mathematical physics.

Розглядається багатовимірний вінерів процес з напівпрозорою мембраною, що розташована на заданій гіперплощині. Траєкторії цього процесу є розв'язками стохастичного диференціального рівняння, яке є деяким узагальненням відомого рівняння Скорохода для дифузійного процесу в обмеженій області з граничними умовами на межі. З допомогою адитивного функціонала від процесу, що має характер локального часу, зроблено випадкову заміну часу в цьому процесі і, як результат, отримано ймовірнісне зображення розв'язків однієї задачі математичної фізики.

1. Introduction. Let a unit vector ν in a d-dimensional Euclidean space $\mathbb{R}^d$ be given and denote by S the hyperplane in $\mathbb{R}^d$ that is orthogonal to ν. We put $\mathfrak{D}_+ = \{x \in \mathbb{R}^d: (x, \nu) > 0\}$ and $\mathfrak{D}_- = \{x \in \mathbb{R}^d: (x, \nu) < 0\}$. Two continuous functions $q(x)$ and $r(x)$ on S with their values in $[-1, 1]$ and $[0, +\infty)$ respectively are assumed to be given. Denote by $\mathbb{B}$ the Banach space of all bounded measurable real-valued functions on $\mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^d} |\varphi(x)|$ and by $\mathbb{C}$ the subspace of $\mathbb{B}$ consisting of all continuous functions.

The goal of this paper is to give a probabilistic representation for the solution to the following problem of mathematical physics.

For an arbitrary $\varphi \in \mathbb{C}$, we seek a function $u(t, x, \varphi)$ of the arguments $t > 0$ and $x \in \mathbb{R}^d$ such that:

1) it satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u$$

in the region $t > 0$ and $x \notin S$ (Δ is the Laplace operator);

2) it satisfies the initial condition

$$\lim_{t \downarrow 0} u(t, x, \varphi) = \varphi(x)$$

at every point $x \in \mathbb{R}^d$;

3) it satisfies the condition

$$u(t, x+, \varphi) = u(t, x-, \varphi)$$

for all $t > 0$ and $x \in S$, where $u(t, x+, \varphi)$ (respectively $u(t, x-, \varphi)$) stands for the limit of the function $u(t, y, \varphi)$, as $y \to x$ and $y \in \mathfrak{D}_+$ (respectively $\mathfrak{D}_-$); this common value is denoted by $u(t, x, \varphi)$ for $t > 0$ and $x \in S$;

4) for $t > 0$ and $x \in S$, the condition

$$r(x) \frac{\partial u(t, x, \varphi)}{\partial t} = \frac{1 + q(x)}{2} \frac{\partial u(t, x+, \varphi)}{\partial \nu} - \frac{1 - q(x)}{2} \frac{\partial u(t, x-, \varphi)}{\partial \nu}$$

is held, where $\partial/\partial\nu$ means the derivative in the direction ν and $\frac{\partial u(t, x+, \varphi)}{\partial\nu}$ (respectively $\frac{\partial u(t, x-, \varphi)}{\partial\nu}$) stands for the limit of the function $\frac{\partial u(t, y, \varphi)}{\partial\nu}$, as $y \to \to x \in S$ along an arbitrary non-tangent curve situated in $\mathfrak{D}_+$ (respectively in $\mathfrak{D}_-$).

As it is shown below, the solution to this problem can be obtained by some simple transformation of a Wiener process in $\mathbb{R}^d$ with a semipermeable membrane situated on the hyperplane S. This is the name for a (generalized) diffusion process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ (the notation and terminology by E. B. Dynkin [1] will be used) with an identity operator as its diffusion operator and the function $a(x) = \nu q(x)\delta_S(x)$ as its drift vector, where δ_S is a generalized function on $\mathbb{R}^d$ that is determined by the relation

$$\langle \delta_S, \varphi \rangle = \int_S \varphi(x)\, d\sigma \tag{1}$$

valid for an arbitrary test function φ on $\mathbb{R}^d$ with a surface integral on the right hand side of (1). The existence of this process was proved in [2, 3] (see also [4]). Moreover, it is shown there that for $\varphi \in \mathbb{C}$ there exists an additive homogeneous continuous functional $\eta_t(\varphi)$ of the process that can be written in the form

$$\eta_t(\varphi) = \int_0^t \varphi(x(\tau))\delta_s(x(\tau))\, d\tau, \qquad t \ge 0. \tag{2}$$

It is clear that this functional does not depend on the values of the function φ at the points of $\mathfrak{D}_+ \cup \mathfrak{D}_-$. Therefore, for the function $r(x)$ defined on S only (see above), we can construct the functional $\eta_t(r)$ of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ as a continuous homogeneous non-negative additive functional. A brief description of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ and a functional of the type (2) will be given below in Section 2.

For $t > 0$, we now put $\zeta_t = \inf\{s \ge 0 : s + \eta_s(r) \ge t\}$, $\hat{x}(t) = x(\zeta_t)$ and $\hat{\mathcal{M}}_t = \mathcal{M}_{\zeta_t}$. Then the process $(\hat{x}(t), \hat{\mathcal{M}}_t, \mathbb{P}_x)$ is a continuous Markov process in $\mathbb{R}^d$ ([1], Chapter 10, §5). Our main result is as follows: for a given $\varphi \in \mathbb{C}$, the function

$$u(t, x, \varphi) = \mathbb{E}_x \varphi(\hat{x}(t)), \qquad t \ge 0, \qquad x \in \mathbb{R}^d,$$

is a solution to the problem 1) – 4). The proof of this statement is given below in Section 4. More precisely, in Section 4 we prove that the Laplace transform of the function u in the argument t is a solution to the problem 1) – 4) transformed respectively. It means that if we set

$$U_\lambda(x, \varphi) = \int_0^\infty e^{-\lambda t} u(t, x, \varphi)\, dt \tag{3}$$

for $\lambda > 0$ and $x \in \mathbb{R}^d$, then this function has the properties:

a) for $\lambda > 0$ and $x \notin S$ the equality

$$\lambda U_\lambda(x, \varphi) - \varphi(x) = \frac{1}{2}\Delta U_\lambda(x, \varphi)$$

is held;

b) for $\lambda > 0$ and $x \in S$ the relation

$$U_\lambda(x+, \varphi) = U_\lambda(x-, \varphi)$$

is true; we denote this common value by $U_\lambda(x, \varphi)$;

c) for $\lambda > 0$ and $x \in S$ the equality

$$r(x)\left[\lambda U_\lambda(x, \varphi) - \varphi(x)\right] = \frac{1+q(x)}{2}\frac{\partial U_\lambda(x+, \varphi)}{\partial \nu} - \frac{1-q(x)}{2}\frac{\partial U_\lambda(x-, \varphi)}{\partial \nu}$$

is fulfiled.

Our approach to the problem consists in direct calculating the function (3) (like that given in [5], Chapter II, §6) and making use of an integral equation for the function

$$\tilde{u}_\lambda(t, x, \varphi) = \mathbb{E}_x \varphi(x(t)) \exp\{-\lambda \eta_t(r)\}, \tag{4}$$

where $\lambda > 0$, $t > 0$, $x \in \mathbb{R}^d$, $\varphi \in \mathbb{C}$. This equation is an analogy to the Feynman–Kac formula for the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$. Some results of this type are given in Section 3.

Some analytical approaches to the problem 1)–4) (and a more general one) were proposed in [6]. As a martingale problem, it was solved in [7]. Diffusion processes in a bounded region with general Wentzel's boundary conditions were constructed by many authors (see, for example, [8–11]).

To conclude this section, we note that the paths of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ satisfy the following stochastic differential equation (see Section 2)

$$dx(t) = \nu q(x(t)) \delta_S(x(t))\, dt + dw(t), \tag{5}$$

where $w(t)$ is a standard Wiener process in $\mathbb{R}^d$. This equation can be considered as a generalization of well-known Skorokhod's equation for a diffusion process in a bounded region with some boundary conditions. The simplest version of this generalization can be formulated as follows.

A continuous strong Markov process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ in $\mathbb{R}^1$ is looking for, such that for it:

α) there exists a continuous additive homogeneous non-negative functional η_t that increases at the instants of the set $\{t \geq 0 : x(t) = 0\}$ only, and the Lebesgue measure of this set is equal to zero;

β) there exists a continuous additive homogeneous functional w_t that is a square integrable martingale with respect to $(\mathcal{M}_t, \mathbb{P}_x)$ (for any $x \in \mathbb{R}^1$) with the characteristic $\langle w \rangle_t = t$;

γ) the relation

$$x(t) = x(0) + \eta_t + w_t$$

is held for all $t \geq 0$ on a set of elementary events of full measure $\mathbb{P}_x$ for every $x \in \in \mathbb{R}^1$.

Under the additional assumption that $x(t) \geq 0$ for all $t \geq 0$, this problem is well-known as Skorokhod's problem. There exists only one solution to this problem and it is a Wiener process on $[0, +\infty)$ with the instantaneous reflection at the point $x = 0$.

If we do not assume any additional condition, then there exist many solutions to this problem. Namely, for any $q \in (0, 1]$, a continuous Markov process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ with the function

$$G(t, x, y) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{(y-x)^2}{2t}\right\} + q \operatorname{sign} y \exp\left\{-\frac{(|y|+|x|)^2}{2t}\right\}\right] \tag{6}$$

as its transition probability density (with respect to the Lebesgue measure on $\mathbb{R}^1$) solves this problem. It is not hard to show that each solution to this problem coincides with that given by (6) for some $q \in (0, 1]$. In particular, Skorokhod's solution corresponds to the case of $q = 1$.

If the word "increases" in α) is replaced with the word "decreases", then the solution of the problem is the same, but this time we have $q \in [-1, 0)$. All these processes have a commonn name: they are called skew Brownian motion. From various points of view they were described in [10, 12–15]. In particular, it is proved in [13] that the paths of a skew Brownian motion are the solutions to the following stochastic differential equation ($q \in [-1, 1]$ is a given parameter)

$$dx(t) = q\delta(x(t))\,dt + dw(t), \tag{7}$$

where $w(t)$ is a standard Wiener process in $\mathbb{R}^1$ and $\delta(x)$ is Dirac's δ-function. It is evident, that the equation (5) is a multidimensional analogy to the equation (7).

2. A Wiener process with a semipermeable membrane on S. Denote by $g(t, x, y)$ for $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ transition probability density of a Wiener process in $\mathbb{R}^d$

$$g(t, x, y) = (2\pi t)^{-d/2} \exp\left\{-\frac{|y-x|^2}{2t}\right\}$$

and put

$$G(t, x, y) = g(t, x, y) + \int_0^t d\tau \int_S g(\tau, x, z) \frac{\partial g(t-\tau, z, y)}{\partial \nu_z} q(z)\, d\sigma_z \tag{8}$$

for $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, where $\frac{\partial g(t-\tau, z, y)}{\partial \nu_z}$ stands for the derivative in the direction ν of the function $g(t-\tau, z, y)$ as a function of the argument z

$$\frac{\partial g(t-\tau, z, y)}{\partial \nu_z} = \frac{(y-z, \nu)}{t-\tau} g(t-\tau, z, y) \tag{9}$$

and the inner integral in (8) is a surface integral of the integrand as a function of the argument z. If $y \notin S$, the integrals in (8) are well defined because of the boundedness of the function (9) as a function of z (for each fixed $y \notin S$). If $y \in S$, it is reasonable to put $G(t, x, y) = g(t, x, y)$ because of the equality $\frac{\partial g(t-\tau, z, y)}{\partial \nu_z} = 0$ valid for all $z \in S$ and $y \in S$ according to (9). But, as it follows from well-known theorem on the jump of the normal derivative of a single-layer potential (see, for example, [4]), for $t > 0$, $x \in \mathbb{R}^d$ and $y \in S$, the relation

$$G(t, x, y\pm) = (1 \pm q(y))\, g(t, x, y) \tag{10}$$

holds true, where $G(t, x, y+)$ and $G(t, x, y-)$ are defined in a way similar to that given in Section 1.

It is known that there exists a continuous homogeneous Markov process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ such that for all $t \geq 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}$ the relation

$$\mathbb{E}_x \varphi(x(t)) = \int_{\mathbb{R}^d} G(t, x, y)\varphi(y)\, dy \tag{11}$$

is held (see [4], Chapter III, §3). In other words, this process has the function $G(t, x, y)$

as its transition probability density with respect to the Lebesgue measure in $\mathbb{R}^d$. It is not a difficult problem to verify that the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ is a generalized diffusion process with the identity operator as its diffusion operator and the function $a(x) = \nu q(x)\delta_S(x)$ as its drift vector. Namely (see [4], .Chapter III, §3), for an arbitrary compactly supported function $\varphi \in \mathbb{C}$, the relations

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \varphi(x)\frac{1}{t}\,\mathbb{E}_x(x(t)-x(0), \theta)\,dx = (\nu, \theta)\int_S q(x)\varphi(x)\,d\sigma,$$

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \varphi(x)\frac{1}{t}\,\mathbb{E}_x(x(t)-x(0), \theta)^2\,dx = |\theta|^2\int_{\mathbb{R}^d} \varphi(x)\,dx \tag{12}$$

are true for any $\theta \in \mathbb{R}^d$.

The function $q(x)$ can be thought of as a permeability coefficient: if $q(x) \equiv +1$, then the part of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ in $\mathcal{D}_+ \cup S$ coincides with a Wiener process in this region with an instantaneous reflection along the normal vector at the points of the hyperplane S (the membrane is impenetrable for getting into $\mathcal{D}_-$ from $\mathcal{D}_+$); if $q(x) \equiv -1$, the reflection is in the opposite direction. In the case of $q(x) \equiv 0$, there is no membrane on S, in the rest of cases a semipermeable membrane is situated on S.

For $t \geq 0$ and $x \in \mathbb{R}^d$ we now put

$$f_t(x) = \int_0^t \exp\left\{-\frac{(x, \nu)^2}{2\tau}\right\}\frac{d\tau}{\sqrt{2\pi\tau}} = \int_0^t d\tau \int_S g(\tau, x, y)\,d\sigma_y.$$

This function can be written in the form

$$f_t(x) = \int_0^t d\tau \int_{\mathbb{R}^d} G(\tau, x, y)\delta_S(y)\,dy, \tag{13}$$

if we accept the following rule for the function δ_S to act on a function φ having the limits $\varphi(y+)$ and $\varphi(y-)$ at any point $y \in S$:

$$\langle \delta_S, \varphi \rangle = \frac{1}{2}\int_S [\varphi(y+) + \varphi(y-)]\,d\sigma \tag{14}$$

(under the assumption that the function $\varphi(y+) + \varphi(y-)$ is integrable over S, of course). The relation (14) coincides with (1) in the case of a continuous function φ.

Taking into account (13), one can easily verify the validity of the relation

$$\int_{\mathbb{R}^d} f_s(y)G(t, x, y)\,dy = f_{t+s}(x) - f_t(x)$$

for $t \geq 0$, $s \geq 0$ and $x \in \mathbb{R}^d$. Besides, we have

$$\sup_{x\in\mathbb{R}^d} f_t(x) \to 0,$$

as $t \downarrow 0$. Hence (see [1], Chapter 6), there exists an additive homogeneous continuous and non-negative functional η_t of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ such that $\mathbb{E}_x\eta_t = f_t(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. Moreover, this functional can be constructed as a limit of some additive functionals of integral type, namely ([1], Chapter 6)

$$\eta_t = \operatorname*{l.i.m.}_{h\downarrow 0} \int_0^t h^{-1} f_h(x(t))\, d\tau. \tag{15}$$

Since $\lim_{h\downarrow 0} h^{-1} f_h(x) = \delta_S(x)$, we can write the functional η_t in the form

$$\eta_t = \int_0^t \delta_S(x(\tau))\, d\tau. \tag{16}$$

The paths of the functional η_t are non-decreasing continuous functions and one can easily observe that they do increase at those instants $t \geq 0$ only, for which $x(t) \in \in S$.

For $\varphi \in \mathbb{C}$, we put

$$\eta_t(\varphi) = \int_0^t \varphi(x(\tau))\, d\eta_\tau, \qquad t \geq 0, \tag{17}$$

where the integral is understood as a Riemann–Stieltjes integral. The functional $\eta_t(\varphi)$ is an additive homogeneous continuous functional of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$, it can be written in the form (2) if we take into account (16).

If a continuous bounded function ψ is defined on S, we can extend it to a function $\overline{\psi} \in \mathbb{C}$ and construct the functional $\eta_t(\overline{\psi})$ by the formula (17). It is clear, that this functional does not depend on the values of the function $\overline{\psi}$ at the points of the set $\mathfrak{D}_- \cup \mathfrak{D}_+$. So, we denote this functional by $\eta_t(\psi)$.

In particular, the functional $\eta_t(q)$ is well defined. The following assertion is proved in [4] (Chapter III, §4). If we put $w(t) = x(t) - x(0) - \nu \eta_t(q)$ for $t \geq 0$, then this process is a square integrable martingale with respect to $(\mathcal{M}_t, \mathbb{P}_x)$ for any $x \in \mathbb{R}^d$ and its characteristic is equal to tI, where I is the identity operator in $\mathbb{R}^d$. In other words, the paths of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ are the solution to stochastic differential equation (5).

3. The Feynman–Kac formula. Let a continuous bounded function $r(x)$ on S with non-negative values be given. For $\lambda \geq 0$, $t \geq 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}$, we define the function $\tilde{u}_\lambda(t, x, \varphi)$ by the formula (4). As shown in [16], this function can be given as follows

$$\tilde{u}_\lambda(t, x, \varphi) = \int_{\mathbb{R}^d} \varphi(y)\, G_\lambda(t, x, y)\, dy, \tag{18}$$

where $G_\lambda(t, x, y)$ is the solution to each one of the following pair of equations

$$\begin{gathered} G_\lambda(t, x, y) = G(t, x, y) - \lambda \int_0^t d\tau \int_S g(\tau, x, z)\, G_\lambda(t-\tau, z, y)\, r(z)\, d\sigma_z, \\ G_\lambda(t, x, y) = G(t, x, y) - \lambda \int_0^t d\tau \int_S G_\lambda(\tau, x, z)\, G(t-\tau, z, y)\, r(z)\, d\sigma_z. \end{gathered} \tag{19}$$

Besides, there is no more than one solution to each equation in (19) satisfying the inequality

$$G_\lambda(t, x, y) \leq 2g(t, x, y) \tag{20}$$

for all $\lambda \geq 0$, $t \geq 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

The function $G_\lambda(t, x, y)$ is a fundamental solution to the following problem (see [16]).

For a given $\varphi \in \mathbb{C}$, a function $\tilde{u}_\lambda(t, x, \varphi)$ of the arguments $t \geq 0$ and $x \in \mathbb{R}^d$ ($\lambda \geq 0$ is a fixed parameter) is looking for, such that it satisfies the conditions 1) –3) of Section 1 and the condition:

4a) for $t > 0$ and $x \in S$ the relation

$$\frac{1+q(x)}{2}\frac{\partial \tilde{u}_\lambda(t, x+, \varphi)}{\partial \nu} - \frac{1-q(x)}{2}\frac{\partial \tilde{u}_\lambda(t, x-, \varphi)}{\partial \nu} = \lambda r(x)\tilde{u}_\lambda(t, x, \varphi)$$

is fulfiled.

This problem is an analogy to the third boundary-value problem in the theory of partial differential equations, and formula (4) is a version of the Feynman–Kac formula for the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ and the functional $\eta_t(r)$ (see [16], a similar result is given in [17]).

4. A random change of time. A given continuous bounded function $r(x)$ on S with non-negative values is fixed. We define a continuous Markov process $(\hat{x}(t), \tilde{\mathcal{M}}_t, \mathbb{P}_x)$ and a function $u(t, x, \varphi)$ as in Section 1. Our purpose is to show that the function $U_\lambda(x, \varphi)$ defined by (3) for $\lambda > 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}$ is a solution to the problem a)–c). We have

$$U_\lambda(x, \varphi) = \mathbb{E}_x \int_0^\infty e^{-\lambda t}\varphi(x(\zeta_t))\,dt =: U_\lambda^{(1)}(x, \varphi) + U_\lambda^{(2)}(x, \varphi),$$

where

$$U_\lambda^{(1)}(x, \varphi) = \int_0^\infty e^{-\lambda t}\,\mathbb{E}_x\left[\varphi(x(t))e^{-\lambda\eta_t(r)}\right]dt,$$

$$U_\lambda^{(2)}(x, \varphi) = \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(r))}\,\varphi(x(t))\,d\eta_t(r).$$

The function $U_\lambda^{(1)}$ can be written in the form

$$U_\lambda^{(1)}(x, \varphi) = \int_0^\infty e^{-\lambda t}\,\tilde{u}_\lambda(t, x, \varphi)\,dt,$$

where $\tilde{u}_\lambda$ is given by (18) or (4). Taking into account the fact that the function $\tilde{u}_\lambda$ solves the problem 1)–3), 4a), we arrive at the conclusion that the function $U_\lambda^{(1)}$ satisfies the conditions a), b) and the following one:

c_1) for $\lambda > 0$ and $x \in S$ the relation

$$\frac{1+q(x)}{2}\frac{\partial U_\lambda^{(1)}(x+, \varphi)}{\partial \nu} - \frac{1-q(x)}{2}\frac{\partial U_\lambda^{(1)}(x-, \varphi)}{\partial \nu} = \lambda r(x)U_\lambda^{(1)}(x, \varphi)$$

holds.

To calculate $U_\lambda^{(2)}(x, \varphi)$, we make use of the following assertion proved in [18]. Denote by $\tilde{r}(y)$ an arbitrary extension of the function $r(y)$ defined on S to a continuous bounded non-negative function on $\mathbb{R}^d$, for example, one can set $\tilde{r}(y) = r(\Pi y)$ for $y \in \mathbb{R}^d$, where Πy is the orthogonal projection of y on S. For an arbitrary real-valued measurable function $\psi(\tau, y)$ on $[0, \infty) \times \mathbb{R}^d$ and for $h \in (0, h_0]$ ($h_0 > 0$ is a fixed constant), $t \geq 0$, $x \in \mathbb{R}^d$, we put

$$Q_h(t, x, \psi) = \int_0^t d\tau \int_{\mathbb{R}^d} G(t-\tau, x, y)\psi(\tau, y)\tilde{r}(y)h^{-1}f_h(y)\,dy.$$

Lemma 1. *For all given* $\varepsilon > 0$, $T > 0$ *and* $L > 0$ *there exists* $\delta > 0$ *such that the inequality*

$$\left|Q_h(t, x, \psi) - Q_h(t', x', \psi)\right| < \varepsilon$$

holds for all $h \in (0, h_0]$, $t \in [0, T]$, $t' \in [0, T]$, $x \in \mathbb{R}^d$, $x' \in \mathbb{R}^d$ *and* ψ *satisfying the conditions*

$$\sup_{(\tau, y) \in [0, T]\times\mathbb{R}^d} |\psi(\tau, y)| \leq L \quad \text{and} \quad |t-t'| + |x-x'| < \delta.$$

As a consequence from this statement, we have the following property of the functional $\eta_t(r)$:

$$\eta_t(r) = \underset{h\downarrow 0}{\text{l.i.m.}} \int_0^t \tilde{r}(x(\tau))h^{-1}f_h(x(\tau))\,d\tau. \tag{21}$$

Indeed, according to (10), we have

$$\lim_{h\downarrow 0} \int_0^t d\tau \int_{\mathbb{R}^d} G(\tau, x, y)\tilde{r}(y)h^{-1}f_h(y)\,dy =$$

$$= \int_0^t d\tau \int_S \frac{1}{2}[G(\tau, x, y+) + G(\tau, x, y-)]r(y)\,d\sigma_y =$$

$$= \int_0^t d\tau \int_S g(\tau, x, y)r(y)\,d\sigma_y = \mathbb{E}_x\eta_t(r)$$

and the convergence here is uniform in the argument $x \in \mathbb{R}^d$ and locally uniform in $t \geq 0$, as it follows from Lemma 1. Therefore, the relation

$$\lim_{h\downarrow 0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left|\mathbb{E}_x\eta_t^h(r) - \mathbb{E}_x\eta_t(r)\right| = 0$$

is true for an arbitrary $T < +\infty$, where we denote by $\eta_t^h(r)$ the integral on the right-hand side of (21). The Theorem 6.4 from [1], Chapter 6 now implies the relation (21).

Using (21), we can write

$$U_\lambda^{(2)}(x, \varphi) = \lim_{h\downarrow 0} \int_0^\infty e^{-\lambda t}\tilde{u}_\lambda\left(t, x, \varphi\tilde{r}h^{-1}f_h\right)dt \tag{22}$$

for $\lambda > 0$ and $x \in \mathbb{R}^d$. The formula (18) and the first equation in (19) imply the following integral equation for the function $\tilde{u}_\lambda\left(t, x, \varphi\tilde{r}h^{-1}f_h\right)$:

$$\tilde{u}_\lambda\left(t, x, \varphi\tilde{r}h^{-1}f_h\right) = \int_{\mathbb{R}^d} G(\tau, x, y)\varphi(y)\tilde{r}(y)h^{-1}f_h(y)\,dy -$$

$$- \lambda\int_0^t d\tau \int_S g(\tau, x, z)\tilde{u}_\lambda\left(t-\tau, z, \varphi\tilde{r}h^{-1}f_h\right)r(z)\,d\sigma_z. \tag{23}$$

For a fixed $h > 0$ this equation has no more than one bounded solution.

Lemma 2. *For all* $\lambda > 0$, $t \ge 0$ *and* $x \in \mathbb{R}^d$ *the relation*

$$\lim_{h \downarrow 0} \tilde{u}_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right) = \tilde{u}_\lambda(t, x, \varphi r \delta_S)$$

is fulfiled, where $\tilde{u}_\lambda(t, x, \varphi r \delta_S)$ *is the unique solution of the equation*

$$\tilde{u}_\lambda(t, x, \varphi r \delta_S) = \int_S g(t, x, y)\varphi(y) r(y)\, d\sigma_y -$$

$$- \lambda \int_0^t d\tau \int_S g(\tau, x, z) \tilde{u}_\lambda(t-\tau, z, \varphi r \delta_S) r(z)\, d\sigma_z \tag{24}$$

satisfying the inequality

$$\left|\tilde{u}_\lambda(t, x, \varphi r \delta_S)\right| \le K_T t^{-1/2} \sup_{x \in S} |\varphi(x)| \tag{25}$$

on each domain of the form $(t, x) \in (0, T] \times \mathbb{R}^d$ *with some constant* $K_T < +\infty$ *for* $T < +\infty$.

Proof. The unique bounded solution to the equation (23) can be obtained by the method of successive approximations. Namely, put

$$\tilde{u}^0_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right) = \int_{\mathbb{R}^d} G(\tau, x, y)\varphi(y)\tilde{r}(y) h^{-1} f_h(y)\, dy$$

and for $k = 0, 1, 2, \ldots$

$$\tilde{u}^{k+1}_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right) = \lambda \int_0^t d\tau \int_S g(\tau, x, z) \tilde{u}^k_\lambda\left(t-\tau, z, \varphi \tilde{r} h^{-1} f_h\right) r(z)\, d\sigma_z.$$

Making use of representation (13), we have the estimate

$$\left|\tilde{u}^0_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right)\right| \le \|\varphi\| \|\tilde{r}\| (2\pi t)^{-1/2}.$$

By induction on k we now arrive at the inequalities

$$\left|\tilde{u}^k_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right)\right| \le \|\varphi\| \lambda^k \left(\frac{\|\tilde{r}\|}{\sqrt{2}}\right)^{k+1} \frac{t^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} \tag{26}$$

valid for $\lambda > 0$, $t \ge 0$, $x \in \mathbb{R}^d$ and $k = 0, 1, 2, \ldots$. These estimates show that the unique bounded solution to the equation (23) can be represented by the series

$$\tilde{u}_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right) = \sum_{k=0}^{\infty} \tilde{u}^k_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right)$$

that is convergent uniformly in $h \in (0, h_0]$. Hence, we can pass to the limits, as $h \downarrow 0$, in each term of this series. Since

$$\lim_{h \downarrow 0} \tilde{u}^0_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right) = \int_S g(\tau, x, y)\varphi(y) r(y)\, d\sigma_y,$$

we arrive at the conclusion that $\lim_{h \downarrow 0} \tilde{u}^k_\lambda\left(t, x, \varphi \tilde{r} h^{-1} f_h\right)$ exists for all $k = 0, 1, \ldots$. Denote it by $\tilde{u}^k_\lambda(t, x, \varphi r \delta_S)$. So, we have the equality

$$\tilde{u}_\lambda^{k+1}(t, x, \varphi r \delta_S) = -\lambda \int_0^t d\tau \int_S g(\tau, x, z) \tilde{u}_\lambda^k(t-\tau, z, \varphi r \delta_S) r(z) d\sigma_z$$

valid for $t \geq 0$, $\lambda > 0$, $x \in \mathbb{R}^d$ and $k = 0, 1, 2, \ldots$, where

$$\tilde{u}_\lambda^0(t, x, \varphi r \delta_S) = \int_S g(t, x, y) \varphi(y) r(y) d\sigma_y.$$

From the other hand, the estimates (26) are held for functions $\tilde{u}_\lambda^k(t, x, \varphi r \delta_S)$, $k = 0, 1, \ldots$. Therefore, the series

$$\tilde{u}_\lambda(t, x, \varphi r \delta_S) = \sum_{k=0}^{\infty} \tilde{u}_\lambda^k(t, x, \varphi r \delta_S)$$

is convergent and it gives the solution to the equation (24). The inequality (25) and the uniqueness of a solution satisfying this inequality are simple consequences from the estimates (26). The lemma has been proved.

This lemma and the relation (22) imply the following equation for the function $U_\lambda^{(2)}(x, \varphi)$:

$$U_\lambda^{(2)}(x, \varphi) = \int_S \tilde{g}_\lambda(x, y) \varphi(y) r(y) d\sigma_y - \lambda \int_S \tilde{g}_\lambda(x, y) U_\lambda^{(2)}(y, \varphi) r(y) d\sigma_y \qquad (27)$$

valid for $x \in \mathbb{R}^d$, $\lambda > 0$, where

$$\tilde{g}_\lambda(x, y) = \int_0^{\infty} e^{-\lambda t} g(t; x, y) dt, \quad \lambda > 0, \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d.$$

From this equation we first arrive at the conslusion that the function $U_\lambda^{(2)}(x, \varphi)$ satisfies the conditions a)–b) of Section 1. Secondly, using the theorem on the jump of the normal derivative of a single-layer potential (mentioned above), from (27) we obtain the relations

$$\frac{\partial U_\lambda^{(2)}(x\pm, \varphi)}{\partial \nu} = \mp r(x) \varphi(x) \pm \lambda r(x) U_\lambda^{(2)}(x, \varphi)$$

valid for $\lambda > 0$ and $x \in S$. These relations imply the equality

$$\frac{1+q(x)}{2} \frac{\partial U_\lambda^{(2)}(x+, \varphi)}{\partial \nu} - \frac{1-q(x)}{2} \frac{\partial U_\lambda^{(2)}(x-, \varphi)}{\partial \nu} = r(x)\left[\lambda U_\lambda^{(2)}(x, \varphi) - \varphi(x)\right] \qquad (28)$$

that is satisfied by the function $U_\lambda^{(2)}$ at each point $x \in S$ for $\lambda > 0$. Now, the fact that the function $U_\lambda(x, \varphi)$ satisfies the condition c) of Section 1 is a consequence of the equality (28) and the equality c_1) above. We have thus proved the following statement.

Theorem. *The function $U_\lambda(x, \varphi)$ defined by* (3) *is a solution to the problem a)–c).*

Denote by S_r^+ the set of those points $x \in S$ for which $r(x) > 0$ and by $\mathbf{1}_\Gamma(x)$ the indicator function of a set $\Gamma \subset \mathbb{R}^d$. If the set S_r^+ is non-empty, then $U_\lambda(x, \mathbf{1}_{S_r^+}) = U_\lambda^{(2)}(x, \mathbf{1}_{S_r^+}) > 0$ for all $x \in \mathbb{R}^d$ and $\lambda > 0$, as it follows from the equation (28). Therefore, the inequality

$$0 < \lambda^{-1} U_\lambda(x, \mathbf{1}_{S_r^+}) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} \left(\int_0^t \mathbf{1}_{S_r^+}(\hat{x}(\tau))\, d\tau \right) dt$$

holds true for all $x \in \mathbb{R}^d$ and $\lambda > 0$ and this means that the Lebesgue measure of the set $\{t \geq 0 : \hat{x}(t) \in S_r^+\}$ is positive with a positive probability $\mathbb{P}_x$ for any $x \in \mathbb{R}^d$. In other words, for the process $(\hat{x}(t), \hat{\mathcal{M}}_t, \mathbb{P}_x)$ the points of the set S_r^+ have the property of delaying or, one can say, these points are sticky for this process. On the contrary, the points of the set $S \setminus S_r^+$ have no property of delaying because of the equality

$$\mathbb{E}_x \int_0^t \mathbf{1}_{S \setminus S_r^+}(\hat{x}(\tau))\, d\tau = 0$$

valid for all $t \geq 0$ and $x \in \mathbb{R}^d$.

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