UDC 517.9 + 531.19

D. Ya. Petrina (Inst. Math. Nat. Acad. Sci. Ukraine, Kiev)

SPECTRUM AND STATES OF BCS HAMILTONIAN IN FINITE DOMAIN. I. SPECTRUM

СПЕКТР ТА СТАНИ ГАМІЛЬТОНІАНА БКШ В СКІНЧЕННІЙ ОБЛАСТІ. I. СПЕКТР

The BCS Hamiltonian in a finite cube with periodic boundary condition is considered. The special subspace of pairs of particles with opposite momenta and spin is introduced. It is proved that, in this subspace, the spectrum of the BCS Hamiltonian is defined exactly for one pair, and for n pairs the spectrum is defined through the eigenvalues of one pair and a term that tends to zero as the volume of the cube tends to infinity. On the subspace of pairs, the BCS Hamiltonian can be represented as a sum of two operators. One of them describes the spectra of noninteracting pairs and the other one describes the interaction between pairs that tends to zero as the volume of the cube tends to infinity. It is proved that, on the subspace of pairs, as the volume of the cube tends to infinity, the BCS Hamiltonian tends to the approximating Hamiltonian, which is a quadratic form with respect to the operators of creation and annihilation.

Розглянуто гамільтоніан БКШ в скінченному кубі при періодичних граничних умовах. Введено спеціальний підпростір пар часток з протилежними імпульсами і спіном. Доведено, що в цьому підпросторі спектр гамільтоніана БКШ визначається точно для однієї пари, а у випадку n пар — через власні значення однієї пари з точністю до члена, що прямує до нуля, коли об'єм куба прямує до нескінченності. На підпросторі пар гамільтоніан БКШ може бути зображений як сума двох операторів. Один з них описує спектр невзаємодіючих пар, а другий — взаємодію між парами, що прямує до нуля, коли об'єм куба прямує до нескінченності. Доведено, що на підпросторі пар, коли об'єм куба прямує до нескінченності, гамільтоніан БКШ прямує до апроксимуючого гамільтоніана, що є квадратичною формою відносно операторів народження та знищення.

Introduction. Bogolyubov model Hamiltonian of superfluidity [1] and Bardeen – Cooper – Schrieffer model Hamiltonian of superconductivity [2] constantly attract attention of researcher during last fifty years. This is, on the one hand, due to the importance of the superfluidity and superconductivity phenomena they describe and, on the other hand, because of the fact that they admit exact solutions in the sense that their states can be exactly calculated in the thermodynamic limit. The latter is especially attractive from the viewpoint of mathematical physics because exactly solvable models with nontrivial interaction are very rare.

The problem of rigorous proof of exact solvability of these models in the thermodynamic limit has a long intriguing history and has not been solved completely so far. In his pioneering work on the theory of superfluidity [1], Bogolyubov gave several reasons why his model Hamiltonian can appear to be exactly solvable. He indicated that the operators of creation and annihilation of bosons with zero momenta in the thermodynamic limit commute with the whole algebra of observables and, therefore, they are multiples of the identity operator; in other words, they are c-numbers. The model Hamiltonian obtained after the replacement of such operators by c-numbers is called an approximating Hamiltonian. It can be reduced to quadratic forms, diagonalized with a u–v-transformation, and then exactly solved.

As to the BCS model Hamiltonian, Bardeen, Cooper, and Schrieffer [2] indicated that their solution, obtained by a variation method, is exact in the thermodynamic limit. Bogolyubov, Zubarev and Tserkovnikov [3] showed (within the framework of perturbation theory) that, in the thermodynamic limit, the BGS model Hamiltonian is equivalent to a certain approximating Hamiltonian obtained from the model one by the replacement of certain operator expressions by c-numbers and is a quadratic form with respect to the operators of creation and annihilation. (The equivalence of two Hamiltonians is understood as the coincidence of their states in the thermodynamic limit.)

This approximating Hamiltonian can be diagonalized, which gives exact expressions for its spectrum and states.

The equivalence of the model and approximating Hamiltonians (under nonzero temperature) was proved in the series of works by N. N. Bogolyubov [4] and N. N. Bogolyubov (Jr.) [5, 6] with the use of an equation for Green functions and certain very accurate estimates.

Haag [7] found out that the approximating Hamiltonian that corresponds to the BCS model can be obtained from the model one with the use of the fact that certain operator expressions commute in the thermodynamic limit with the algebra of all observables, and, thus, they are c-numbers.

For a simplified BCS Hamiltonian with spin operators instead of fermion ones, Thirring [8] rigorously calculated the states in the thermodynamic limit and showed that they coincide with the states of the corresponding approximating Hamiltonian.

Petrina [9, 10] suggested to consider the BCS Hamiltonian and the equation of states in the spaces of translation-invariant functions and thus established the equivalence of the model and approximating Hamiltonians in the thermodynamic limit. An analogous result was also obtained for the Bogolyubov superfluidity Hamiltonian [11].

At present, there are numerous works dealing with the models of superfluidity and superconductivity (we refer readers to [5, 6, 12] for references), but in none of them one can find the traditional scheme of quantum statistical mechanics: the calculation of spectrum and averages and subsequent thermodynamic limit transition. The only exception is the N. N. Bogolyubov's work [4] on the ground state in the BCS model.

This approach seemed to be pointless because these Hamiltonians are polynomials of the fourth degree in the operators of creation and annihilation and, seemingly, are as complex as the general Hamiltonians.

The hope of success appeared only after it was discovered that the model superfluidity and superconductivity Hamiltonians have invariant subspaces of the general Fock space, namely, the subspaces of pairs and condensate and subspaces of pairs, respectively [12].

We begin a series of papers, in which we investigate the spectra of the BCS and Bogolyubov Hamiltonians in the subspaces indicated above and, on this basis, we investigate the corresponding states in the thermodynamic limit and prove the thermodynamic equivalence of these Hamiltonians and their approximating Hamiltonians.

In the first paper of this series, we investigate the spectra of the BCS Hamiltonian.

It is established that, in these subspaces of pairs, the spectrum of the BCS Hamiltonian can be exactly obtained in the following sense: There is some basic Hamiltonian that describes only the interaction of two particles, which constitute a pair ("pair" Hamiltonian), and its spectrum can be exactly determined from a certain algebraic equation. The general BCS Hamiltonian is the sum of the Hamiltonian indicated, which leads to the creation of pairs, and the Hamiltonian that describes the interaction of pairs, the norm of which is proportional to $1/V^{1/2}$, where V is the volume of the system. This implies that the spectrum of the general BCS Hamiltonian is a small perturbation of the spectrum of the "pair" Hamiltonian, and this perturbation tends to zero as a certain power of $1/V^{1/2}$.

It is proved that, on the subspace of pairs, as the volume of the cube tends to infinity, the BCS Hamiltonian tends to an approximating Hamiltonian, which is a quadratic form with respect to the operators of creation and annihilation and which coincides with the first part of the BCS Hamiltonian, which describes the creation of pairs.

In the second part of this work, the excited eigenvectors will be introduced, the spectra of the BCS Hamiltonian on the excited eigenvectors will be investigated, and the states with certain temperature and density will be investigated in the thermodynamic limit.

This article have been completed during my stay at ESI in November, 1998 as a guest of the Austrian Academy of Sciences. I would like to express my gratitude to Prof. W. Thirring for the invitation and stimulating discussions.

I. Fock space of systems of fermions. *1. General Fock space.* Consider a system of fermions with spin $s = (1/2, -1/2)$ enclosed in a cube Λ in the three-dimensional Euclidean space E with periodic boundary conditions. Denote by L the length of the edge of the cube Λ centered at the origin. Denote by k the quasidiscrete momenta that take the values $k = (2\pi/L)n$, $n = (n_1, n_2, n_3)$, where the numbers n_i, $i = 1, 2, 3$, run through the entire set of integer numbers Z. Let $\bar{k} = (k, s)$ be the vector of momentum k and spin $1/2$ or $-1/2$.

Denote by $\mathcal{H}_n$ the Hilbert space of functions $f_n(\bar{k}_1, \ldots, \bar{k}_n) = f_n((\bar{k})_n)$ that depend on vectors $\bar{k}_1, \ldots, \bar{k}_n$ with the norm

$$\|f_n\| = \left\{ \sum_{\bar{k}_1,\ldots,\bar{k}_n} \left| f_n(\bar{k}_1,\ldots,\bar{k}_n) \right|^2 \right\}^{1/2} < \infty \tag{1.1}$$

and scalar product

$$(f_n, g_n) = \sum_{k_1,\ldots,k_n} \overline{f_n(\bar{k}_1,\ldots,\bar{k}_n)}\, g_n(\bar{k}_1,\ldots,\bar{k}_n). \tag{1.2}$$

For our fermion system, we need the subspace $\mathcal{H}_n^a$ of antisymmetric functions $f_n^a(k_1,\ldots,k_n)$ with the same norm (1.1) and scalar product (1.2).

We introduce the direct sum of the spaces $\mathcal{H}_n^a$,

$$\mathcal{H}^F = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n^a, \tag{1.3}$$

with obvious norm and scalar product. Note that $\mathcal{H}_0$ consists of complex numbers f_0. The sequences $f = (f_0, f_1(k_1), \ldots, f_n^a(k_1,\ldots,k_n), \ldots)$ of functions belonging to $\mathcal{H}_n^a$ with finite norm in $\mathcal{H}$ are elements of $\mathcal{H}^F$.

Denote by a_k and a_k^+ the operators of annihilation and creation, respectively. They satisfy the anticommutation relations

$$\left\{a_{\bar{k}}, a_{\bar{k}'}^+\right\} = \delta_{\bar{k},\bar{k}'}, \quad \left\{a_{\bar{k}}, a_{\bar{k}'}\right\} = 0, \quad \left\{a_{\bar{k}}^+, a_{\bar{k}'}^+\right\} = 0 \tag{1.4}$$

and the conditions of adjointness

$$(a_{\bar{k}})^* = a_{\bar{k}}^+, \quad (a_{\bar{k}}^+)^* = a_{\bar{k}}. \tag{1.5}$$

As is known, the operators $a_{\bar{k}}$ and $a_{\bar{k}}^+$ are bounded in $\mathcal{H}^F$.

We define the following state through a sequence $f \in \mathcal{H}^F$, the vacuum $|0\rangle = (1, 0, \ldots)$, and creation operators (for simplicity, we use the same notation f for it):

$$f = \sum_{n=0}^{\infty} \sum_{\bar{k}_1 \neq \ldots \neq \bar{k}_n}' f_n^a(\bar{k}_1, \ldots, \bar{k}_n) a_{\bar{k}_1}^+ \ldots a_{\bar{k}_n}^+ |0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\bar{k}_1 \neq \ldots \neq \bar{k}_n} f_n^a(\bar{k}_1,\ldots,\bar{k}_n) a_{\bar{k}_1}^+ \ldots a_{\bar{k}_n}^+ |0\rangle, \tag{1.6}$$

where

$$f_n^a(\bar{k}_1,\ldots,\bar{k}_n) = (-1)^{\eta} f_n^a(\bar{k}_{i_1}, \ldots, \bar{k}_{i_n}), \tag{1.7}$$

$i_1, \dots, i_n$ is an arbitrary permutation of the numbers $1, \dots, n$, and η is the parity of the permutation $i_1, \dots, i_n$; $\sum'_{\bar{k}_1 \neq \dots \neq \bar{k}_n}$ means that the summation is carried out over all $\bar{k}_1 \neq \dots \neq \bar{k}_n$ and the points $\bar{k}_1 \neq \dots \neq \bar{k}_n$ that differ only by permutations are identified.

The scalar product of two sequences f and g (1.6) is equal to

$$(f, g) = \sum_{n=0}^{\infty} \sum'_{k_1 \neq \dots \neq k_n} \overline{f_n^a(\bar{k}_1, \dots, \bar{k}_n)}\, g_n^a(\bar{k}_1, \dots, \bar{k}_n) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \neq \dots \neq k_n} \overline{f_n^a(\bar{k}_1, \dots, \bar{k}_n)}\, g_n^a(\bar{k}_1, \dots, \bar{k}_n). \tag{1.8}$$

2. Subspace of pairs. In what follows, we consider pairs of functions with opposite momenta and spin $(k, 1/2)$ and $(-k, -1/2)$ and the corresponding operators of annihilation $a_{k,1/2}$, $a_{-k,-1/2}$ and creation $a^+_{k,1/2}$, $a^+_{-k,-1/2}$. For the sake of simplicity, we denote them by $a_k \equiv a_{k,1/2}$, $a_{-k} \equiv a_{-k,-1/2}$ and $a^+_k \equiv a^+_{k,1/2}$, $a^+_{-k} \equiv a^+_{-k,-1/2}$, $k \equiv (k, 1/2)$, $-k \equiv (-k, -1/2)$. Consider the sequence

$$f = (f_0, 0, f_1(k_1)\delta_{k_1+k_1'}, 0, f_1^s(k_1, k_2)\delta_{k_1+k_1'}\delta_{k_2+k_2'},$$

$$\dots, f_n^s(k_1, \dots, k_n)\delta_{k_1+k_1'}\dots\delta_{k_n+k_n'}, 0 \dots), \tag{1.9}$$

which describes the state with an arbitrary random number of pairs of particles with opposite momenta and spin (δ is the Kronecker symbol).

The functions $f_n^s(k_1, \dots, k_n)$ belong to $\mathcal{H}_n$ and are odd with respect to each variable because the operators $a^+_{k_i}$ and $a^+_{-k_i}$ satisfy the anticommutation relations, and are symmetric because the operators $a^+_{k_i} a^+_{-k_i}$ and $a^+_{k_j} a^+_{-k_j}$ commute for $k_i \neq k_j$:

$$f_n^s(k_1, \dots, -k_i, \dots, k_n) = -f_n^s(k_1, \dots, k_i, \dots, k_n), \quad 1 \le i \le n,$$

$$f_n^s(k_1, \dots, k_n) = f_n^s(k_{i_1}, \dots, k_{i_n}).$$

We now define the states of indefinitely many pairs:

$$f = \sum_{n=0}^{\infty} \sum'_{k_1 \neq \dots \neq k_n} f_n^s(k_1, \dots, k_n)\, a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \neq \dots \neq k_n} f_n^s(k_1, \dots, k_n)\, a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n} f_n^s(k_1, \dots, k_n)\, a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle. \tag{1.10}$$

According to (1.10), the scalar product of two sequences f and g is equal to the following expression:

$$(f, g)' = \sum_{n=0}^{\infty} \sum'_{k_1 \neq \dots \neq k_n} \overline{f_n^s(k_1, \dots, k_n)}\, g_n^s(k_1, \dots, k_n) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \neq \dots \neq k_n} \overline{f_n^s(k_1, \dots, k_n)}\, g_n^s(k_1, \dots, k_n). \tag{1.11}$$

Remark. The antisymmetrization of the functions $f_n^s(k_1,\ldots,k_n)\delta_{k_1+k_1'}\ldots\delta_{k_n+k_n'}$ with $(k_1,-k_1)\neq\ldots\neq(k_n,-k_n)$ consists only of this term because the other terms obtained by the permutation of $(k_1,-k_1,\ldots,k_n,-k_n)$ are equal to zero.

We now consider an example of states of n pairs, which will be used in what follows. Assume that $f_1^1(k_1),\ldots,f_1^n(k_1)$ are odd functions, $f_1^i(k)=-f_1^i(-k)$, $i=1,\ldots$ $\ldots,n$, and some of them may coincide.

Consider the following state of n pairs:

$$\begin{aligned}
&\frac{1}{n!}\sum_{k_1} f_1^1(k_1)\,a_{k_1}^+\,a_{-k_1}^+\ldots\sum_{k_n} f_1^n(k_n)\,a_{k_n}^+\,a_{-k_n}^+\,|\,0\,\rangle = \\
&= \sum_{k_1\neq\ldots\neq k_n}{}' f_n^s(k_1,\ldots,k_n)\,a_{k_1}^+\,a_{-k_1}^+\ldots a_{k_n}^+\,a_{-k_n}^+\,|\,0\,\rangle = \\
&= \frac{1}{n!}\sum_{k_1\neq\ldots\neq k_n} f_n^s(k_1,\ldots,k_n)\,a_{k_1}^+\,a_{-k_1}^+\ldots a_{k_n}^+\,a_{-k_n}^+\,|\,0\,\rangle = \\
&= \frac{1}{n!}\sum_{k_1,\ldots,k_n} f_n^s(k_1,\ldots,k_n)\,a_{k_1}^+\,a_{-k_1}^+\ldots a_{k_n}^+\,a_{-k_n}^+\,|\,0\,\rangle,
\end{aligned} \tag{1.12}$$

where

$$f_n^s(k_1,\ldots,k_n) = \frac{1}{n!}\sum_{i_1,\ldots,i_n} f_1^1(k_{i_1})\ldots f_1^n(k_{i_n}) = \frac{1}{n!}\,\mathrm{sym}\left(f_1^1(k_1)\ldots f_1^n(k_n)\right),$$

and the sum is carried out over all permutations $i_1,\ldots,i_n$ of the numbers $1,\ldots,n$.

Note that, in the last expression in (1.12) and in (1.10), we add terms equal to zero if some momenta from $(k_1,\ldots,k_n)$ coincide.

In what follows, we shall also use a scalar product for two sequences f and g of indefinite (random) number of pairs (1.10) of the form

$$(f,g) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\ldots,k_n}\overline{f_n^s(k_1,\ldots,k_n)}\,g_n^s(k_1,\ldots,k_n) \tag{1.11$'$}$$

because the functions $f_n^s(k_1,\ldots,k_n)$ and $g_n^s(k_1,\ldots,k_n)$ are also defined if some momenta coincide. It is obvious that the norm of f defined according to (1.11′), $\|f\|=$ $=(f,f)^{1/2}$, is greater than that defined according to (1.11), $\|f\|'=\{(f,f)'\}^{1/2}$, i.e., $\|f\|\geq\|f\|'$.

The subspace of the space $\mathcal{H}^F$ that consists of sequences (1.8)–(1.10) is the subspace of pairs and we denote it by $\mathcal{H}_P^F$.

It is easy to prove that the entire Fock space $\mathcal{H}^F$ can be represented as the direct orthogonal sum of the subspace of pairs $\mathcal{H}_P^F$ and its complement:

$$\mathcal{H}^F = \mathcal{H}_P^F \oplus \left(\mathcal{H}^F/\mathcal{H}_P^F\right).$$

Indeed, the only sequences g from $\mathcal{H}^F$ that are not orthogonal to $f\in\mathcal{H}_P^F$ are the sequences with even number of particles that can be grouped into pairs with opposite momenta. This means that $g\in\mathcal{H}_P^F$.

In what follows, we omit the superscript F because we consider only the space $\mathcal{H}^F$, i.e., $\mathcal{H}^F\equiv\mathcal{H}$ and $\mathcal{H}_P^F\equiv\mathcal{H}^P$.

II. Hamiltonian and its action on the state of pairs. ***1.*** H_Λ ***on states of*** n ***pairs.*** Consider the BBC Hamiltonian for a system of fermi-particles without spin in a cube Λ with periodic boundary conditions

$$H_\Lambda = \sum_{\bar p}\left(\frac{p^2}{2m}-\mu\right)a^+_{\bar p}a_{\bar p} + \frac{g}{V}\sum_{p_1,p_2} V_{p_1,p_2}a^+_{p_1}a^+_{-p_1}a_{-p_2}a_{p_2}, \tag{2.1}$$

where $V = L^3$ is the volume of the cube Λ, m is the mass of a particle, and μ is the chemical potential. The potential V_{p_1,p_2} satisfies the following conditions:

$$\overline{V_{p_1,p_2}} = V_{p_2,p_1}, \quad V_{-p_1,p_2} = V_{p_1,-p_2} = -V_{p_1,p_2},$$
$$\frac{1}{V^2}\sum_{p_1,p_2}\left|V_{p_1,p_2}\right|^2 \le \gamma < \infty, \tag{2.2}$$

where the constant $\gamma < \infty$ does not depend on V.

Consider the action of the Hamiltonian H_Λ on the state of pairs (1.10). The Hamiltonian preserves the number of particles. We consider its action on the state of one pair and, finally, on the state of n pairs.

For the state of one pair, we have

$$H_\Lambda \sum_k f_1(k_1)a^+_{k_1}a^+_{-k_1}|0\rangle = \sum_{k_1}\left(\frac{2k_1^2}{2m}-2\mu\right)f_1(k_1)a^+_{k_1}a^+_{-k_1}|0\rangle +$$
$$+ \frac{g}{V}\sum_{k_1,p} V_{k_1,p}\,f_1(p)a^+_{k_1}a^+_{-k_1}|0\rangle$$

or, equivalently,

$$(H_\Lambda f_1)(k_1) = \left(\frac{2k_1^2}{2m}-2\mu\right)f_1(k_1) + \frac{g}{V}\sum_p V_{k_1,p}\,f_1(p). \tag{2.3}$$

Denote the operator on the right-hand side of (2.3) by $H_{2,\Lambda} = H_2$. Then formula (2.3) can be represented as follows*:

$$(H_\Lambda f_1)(k_1) = (H_2 f_1)(k_1). \tag{2.4}$$

For the state of n pairs, we have

$$H_\Lambda \sum_{k_1,\ldots,k_n} f_n(k_1,\ldots,k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle =$$
$$= \sum_{k_1,\ldots,k_n}\left(\frac{2k_1^2}{2m}+\ldots+\frac{2k_n^2}{2m}-2\mu n\right)f_n(k_1,\ldots,k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle +$$
$$+ \frac{g}{V}\sum_{i=1}^n \sum_{p_1,k_1,\overset{i}{\vee}\ldots,k_n}\ \sum_{k_i\ne k_1,\ldots,k_i\ne k_n} V_{p_1,k_i}f_n(k_1,\ldots,k_i,\ldots,k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{p_1}\overset{i}{a}{}^+_{-p_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle +$$
$$= \sum_{k_1,\ldots,k_n}\left(\frac{2k_1^2}{2m}+\ldots+\frac{2k_n^2}{2m}-2\mu n\right)f_n(k_1,\ldots,k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle +$$
$$+ \frac{g}{V}\sum_{i=1}^n \sum_{k_1,\ldots,k_n,p} V_{k_i,p}\,f_n(k_1,\ldots,\overset{i}{p},\ldots,k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_i}a^+_{-k_i}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle -$$
$$- \frac{g}{V}\sum_{i=1}^n \sum_{1=j\ne i}^n \sum_{k_1,\ldots,k_n} V_{k_i,k_j}\,f_n(k_1,\ldots,\overset{i}{k_j},\ldots,k_n)\ a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_i}a^+_{-k_i}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle. \tag{2.5}$$

* In what follows, we omit the sign s for the functions $f_n^s \equiv f_n$.

Note that we add and subtract the last term in (2.5) in order to replace the sum $\sum_{k_i \neq k_1, \ldots, k_i \neq k_n}$ by the sum $\sum_{k_i}$. We also add zero terms with some equal momenta.

From (2.5), we obtain (equating the coefficients of the same products of the operators of creation of pairs)

$$(H_\Lambda f_n)(k_1, \ldots, k_n) = \left(\frac{2k_1^2}{2m} + \ldots + \frac{2k_n^2}{2m} - 2\mu n \right) f_n(k_1, \ldots, k_n) +$$

$$+ \frac{g}{V} \sum_{i=1}^{n} \sum_{p} V_{k_i, p} f_n(k_1, \ldots, \overset{i}{p}, \ldots, k_n) -$$

$$- \frac{g}{V} \sum_{i=1}^{n} \sum_{1=j \neq i}^{n} V_{k_i, k_j} f_n(k_j, \ldots, \overset{i}{k_j}, \ldots, k_n). \tag{2.6}$$

By using the operator H_2 defined according to (2.3), (2.3′), we can represent operator (2.6) as follows:

$$(H_\Lambda f_n)(k_1, \ldots, k_n) = \left((H_2 \otimes I \ldots \otimes I + \ldots + I \otimes I \ldots \otimes H_2) f_n \right)(k_1, \ldots, k_n) -$$

$$- \frac{g}{V} \sum_{i=1}^{n} \sum_{1=j \neq i}^{n} V_{k_i, k_j} f_n(k_1, \ldots, \overset{i}{k_j}, \ldots, k_n). \tag{2.7}$$

In what follows, the function $f_n(k_1, \ldots, k_n)$ is called the wave function of n pairs. We have proved the following theorem:

Theorem 1. *The BCS Hamiltonian* (2.1) *is defined on the wave functions of n pairs $f_n(k_1, \ldots, k_n)$ according to formula* (2.7).

III. Domain of definition of the Hamiltonian. *1. Domain of definition of* $\boldsymbol{H_2}$. Consider the Hamiltonian in the one-pair subspace. According to (2.3) and (2.3′), we have

$$(H_2 f_1)(k_1) = \left(\frac{2k_1^2}{2m} - 2\mu \right) f_1(k_1) + \frac{g}{V} \sum_{p} V_{k_1, p} f_1(p). \tag{3.1}$$

The first term on the right-hand side of (3.1) is defined on the functions $f_1(k_1)$ that belong to $\mathcal{H}_1^P$ together with the functions $\frac{2k_1^2}{2m} f_1(k_1)$, i.e., $f_1(k_1) \in \mathcal{H}_1^P$, $\frac{2k_1^2}{2m} f_1(k_1) \in \mathcal{H}_1^P$. On these functions, the operator defined by the first term is self-adjoint. For the second term, we have the following estimates:

$$\left| \frac{g}{V} \sum_{p} V_{k_i, p} f_1(p) \right| \leq \frac{|g|}{V} \left| \left\{ \sum_{p} \left| V_{k_1, p} \right|^2 \right\}^{1/2} \left\{ \sum_{p} | f_1(p) |^2 \right\}^{1/2} \right|,$$

$$\sum_{k_1} \left| \frac{g}{V} \sum_{p} V_{k_i, p} f_1(p) \right|^2 \leq \frac{g^2}{V^2} \sum_{k_1, p_1} \left| V_{k_1, p} \right|^2 \sum_{p} | f_1(p) |^2 = g^2 \gamma \| f_1 \|^2. \tag{3.2}$$

It follows from (3.2) that the second term in (3.1) defines an operator bounded in $\mathcal{H}_1^P$. Thus, the operator H_2 is defined on functions $f_1(k)$ such that $f_1(k) \in \mathcal{H}_1^P$, $\frac{2k_1^2}{2m} f_1(k_1) \in \mathcal{H}_1^P$ and is self-adjoint as a sum of self-adjoint and bounded self-adjoint operators.

We now consider the operator

$$H_2 \otimes I \ldots \otimes I + \ldots + I \otimes \ldots \otimes H_2. \tag{3.3}$$

It is obvious that operator (3.3) is defined on functions $f_n(k_1,\ldots,k_n)$ that belong to $\mathcal{H}_n^P$ together with $\sum_{i=1}^{n} \frac{k_i^2}{2m} f_n(k_1,\ldots,k_n)$, i.e., $f_n(k_1,\ldots,k_n) \in \mathcal{H}_n^P$, $\sum_{i=1}^{n} \frac{k_i^2}{2m} f_n(k_1,\ldots,k_n) \in \mathcal{H}_n^P$, and it is self-adjoint.

Indeed, this follows from the estimates

$$\left|\frac{g}{V}\sum_p V_{k_i,p}\, f_n(p,k_2,\ldots,k_n)\right| \le \frac{g}{V}\left\{\sum_p \left|V_{k_1,p}\right|^2\right\}^{1/2}\left\{\sum_p |f_n(p,k_2,\ldots,k_n)|^2\right\}^{1/2},$$

$$\sum_{k_1,\ldots,k_n}\left|\frac{g}{V}\sum_p V_{k_1,p}\, f_n(p,k_2,\ldots,k_n)\right|^2 \le$$

$$\le \frac{g^2}{V^2}\sum_{k_1,p}\left|V_{k_1,p}\right|^2 \sum_{p,k_2,\ldots,k_n}|f_n(p,k_2,\ldots,k_n)|^2 = g^2\gamma\|f_n\|^2. \tag{3.4}$$

Here, we have used the norm $\|f\| = \{(f,f)\}^{1/2}$ defined according to (1.11′).

2. Domain of definition of the perturbation. According to formula (2.7), we represent the operator H_Λ in the subspace of n-pairs $\mathcal{H}_n^P$:

$$H_\Lambda = A + B, \tag{3.5}$$

where

$$\begin{aligned}(Af_n)(k_1,\ldots,k_n) &= \big((H_2 \otimes I \ldots \otimes I + \ldots + I \otimes I \ldots \otimes H_2)\, f_n\big)(k_1,\ldots,k_n),\\ (Bf_n)(k_1,\ldots,k_n) &= -\frac{g}{V}\sum_{i=1}^{n}\sum_{i=j\neq i}^{n} V_{k_i,k_j}\, f_n(k_1,\ldots,\overset{i}{k_j},\ldots,k_n).\end{aligned} \tag{3.6}$$

We now estimate the norm of the operator B in $\mathcal{H}_n^P$.

For simplicity, we assume that the potential V_{k_1,k_2} is separable, i.e.,

$$V_{k_1,k_2} = v_{k_1} v_{k_2}, \tag{3.7}$$

where $v_k = -v_{-k}$, $\overline{v_k} = v_k$, and v_k is a continuous function with compact support D.

The operator B can be represented as the sum over $i = 1,\ldots,n$ of the matrices with the following elements:

$$\delta_{k_1,k_1'}\delta_{k_2,k_2'}\ldots\frac{g}{V}V_{k_i,k_i'}\delta_{k_1,k_i'}\delta_{k_{i+1},k_{i+1}'}\ldots\delta_{k_n,k_n'},\ldots,$$

$$\delta_{k_1,k_1'}\delta_{k_2,k_2'}\ldots\frac{g}{V}V_{k_i,k_i'}\delta_{k_n,k_i'}\delta_{k_{i+1},k_{i+1}'}\ldots\delta_{k_n,k_n'},$$

where all quasimomenta k_l, k_l', $l = 1,\ldots,n$, satisfy the conditions $k_l \in D$, $k_l' \in D$. We regard the operator B as a perturbation of the operator A.

We have

$$\frac{g^2}{V^2}\sum_{k_1,\ldots,k_n} v_{k_i}v_{k_1}\overline{f_n(k_1,\ldots,\overset{i}{k_1},\ldots,k_n)}\, v_{k_i}v_{k_1} f_n(k_1,\ldots,\overset{i}{k_1},\ldots,k_n) \le$$

$$\le \frac{g^2 v^2}{V}\frac{1}{V}\sum_{k_i} v_{k_i}^2 \sum_{k_1,\overset{i}{\vee},k_n}\left|f_n(k_1\ldots\overset{i}{k_1}\ldots k_n)\right|^2 \le$$

$$\leq \frac{g^2 v^2}{V}\|v\|^2 \sum_{k_1,\ldots,k_i,\ldots,k_n} |f_n(k_1,\ldots,k_i,\ldots,k_n)|^2 = \frac{g^2 v^2}{V}\|v\|^2 \|f_n\|^2, \tag{3.8}$$

where

$$v = \sup|v(k)|, \quad \|v\|^2 = \frac{1}{V}\sum_k v_k^2.$$

It follows from inequality (3.8) and definition (3.6) of the operator B that

$$\|Bf_n\| \leq \frac{|g|v\|v\|}{V^{1/2}} n(n-1)\|f_n\|. \tag{3.9}$$

Note that we have used norm (1.11′).

This means that we have proved the following statement:

Theorem 2. *The perturbation B is a bounded operator in $\mathcal{H}_n^P$ and, according to* (3.9), *its norm tends to zero as $V \to \infty$ for arbitrary fixed n.*

3. H_Λ in the space of pairs with norm that depends on V. We introduce the following norm and scalar product equivalent to (1.11) and (1.11′) in the subspace of pairs:

$$\|f_n\|'_V = \frac{1}{V^n} \sum'_{k_1 \neq \ldots \neq k_n} |f_n(k_1,\ldots,k_n)|^2,$$

$$(f_n, g_n)'_V = \frac{1}{V^n} \sum'_{k_1 \neq \ldots \neq k_n} \overline{f_n(k_1,\ldots,k_n)}\, g_n(k_1,\ldots,k_n), \tag{3.10}$$

$$\|f_n\|_V^2 = \frac{1}{n!}\frac{1}{V^n} \sum_{k_1,\ldots,k_n} |f_n(k_1,\ldots,k_n)|^2,$$

$$(f_n, g_n)_V = \frac{1}{n!}\frac{1}{V^n} \sum_{k_1,\ldots,k_n} \overline{f_n(k_1,\ldots,k_n)}\, g_n(k_1,\ldots,k_n), \tag{3.10′}$$

$$\|f_n\|'_V \leq \|f_n\|_V.$$

Norms (3.10), (3.10′) are useful for performing the thermodynamic limit transition. In the thermodynamic limit, they turn into the norms introduced in [8]. We now recall a simple fact related to this.

The functions $f_n(k_1,\ldots,k_n)$ are defined as the Fourier coefficients of functions $f_n(x_1,\ldots,x_n)$ such that

$$f_n(x_1,\ldots,x_n) = \frac{1}{V^{n/2}} \sum_{k_1,\ldots,k_n} f_n(k_1,\ldots,k_n) e^{i(k_1x_1+\ldots+ik_nx_n)}, \quad x_i \in \Lambda, \quad i = 1,\ldots,n.$$

Namely,

$$f_n(k_1,\ldots,k_n) = \frac{1}{V^{n/2}} \int_{\Lambda^n} f_n(x_1,\ldots,x_n) e^{-i(k_1x_1+\ldots+ik_nx_n)}\, dx_1\ldots dx_n, \quad k_i = \frac{2\pi}{L} n.$$

We have the equality

$$\int_{\Lambda^n} |f_n(x_1,\ldots,x_n)|^2 dx_1\ldots dx_n = \sum_{k_1,\ldots,k_n} |f_n(k_1,\ldots,k_n)|^2 =$$

$$= \frac{(2\pi)^3}{V^n}\frac{1}{(2\pi)^3} \sum_{k_1,\ldots,k_n} \left|\tilde{f}_n(k_1,\ldots,k_n)\right|^2,$$

where

$$\tilde{f}_n(k_1,\ldots,k_n) = V^{n/2} f_n(k_1,\ldots,k_n).$$

Passing to the limit as $L \to \infty$ $(\Lambda \to R^3,\ V \to \infty)$, we finally obtain

$$\int |f_n(x_1,\ldots,x_n)|^2 dx_1 \ldots dx_n = \frac{1}{(2\pi)^3} \int \left| \tilde{f}_n(k_1,\ldots,k_n) \right|^2 dk_1 \ldots dk_n,$$

where

$$\tilde{f}_n(k_1,\ldots,k_n) = \int f_n(x_1,\ldots,x_n) e^{-i(k_1x_1+\ldots+k_nx_n)} dx_1 \ldots dx_n$$

is the Fourier transform that satisfies the Parseval equality. In what follows, for simplicity, we use the notation $f_n(k_1,\ldots,k_n)$ instead of $\tilde{f}_n(k_1,\ldots,k_n)$.

Remark. We might use only a Hilbert space with the norm and scalar product (3.11), (3.11′) in the entire article. But, in the direct sum of spaces with these scalar products, the operators of creation and annihilation do not satisfy the conditions of adjointness $(a_k)^* = a_k^+$ and $(a_k^+)^* = a_k$. The result of the action of the Hamiltonian H_Λ is again self-adjoint.

Note that we have the same estimates (3.2), (3.4), and (3.9) in the space $\mathcal{H}_n^P$ with norm (3.10′).

IV. Spectrum of the Hamiltonian in the subspace of n pairs. ***1. Spectrum of the operator A.*** It immediately follows from definition (3.6) of the operator A that its eigenvectors are products of n eigenvectors of the operator H_2, and the corresponding eigenvalues are sums of the eigenvalues of the eigenvectors of the operator H_2. If we denote by $f_1^1(k_1),\ldots, f_1^n(k_n)$ the sequence of eigenvectors of H_2, and by $E^1, \ldots, E^n$ the corresponding sequence of their eigenvalues, then the wave function of n pairs

$$f_n(k_1,\ldots,k_n) = \frac{1}{n!} \operatorname{sym} \left(f_1^1(k_1),\ldots, f_1^n(k_n) \right) \tag{4.1}$$

is the eigenvector of the operator A with the eigenvalue

$$E = E^1 + \ldots + E^n, \tag{4.2}$$

i.e.,

$$(Af_n)(k_1,\ldots,k_n) = \left(E^1 + \ldots + E^n \right) f_n(k_1,\ldots,k_n).$$

Thus, in order to determine the spectrum of the operator A, we must first determine the spectrum of the operator H_2.

2. Spectrum of the operator H_2. Consider the equation for the eigenvectors and eigenvalues of the operator H_2:

$$(H_2 f_1)(k_1) = \left(\frac{2k_1^2}{2m} - 2\mu \right) f_1(k_1) + \frac{g}{V} \sum_p v_{k_1} v_p f_1(p) = Ef(k_1). \tag{4.3}$$

Denote by D the support of v_k. For simplicity, we assume that D is the spherical layer centered at the origin $\left| \frac{p^2}{2m} - \mu \right| < \omega$, $\omega < \mu$. The Hilbert space $\mathcal{H}_1^P$ can be represented as the direct orthogonal sum of the subspace of functions with supports in D and in its complement.

Denote them by h_1^D and h_1^{E-D}, respectively, i.e.,

$$\mathcal{H}_1^P = h_1^D \oplus h_1^{E-D}. \tag{4.4}$$

It is obvious that an arbitrary function $f_1(k) \subset \mathcal{H}_1^P$ can be represented as the sum of two functions

$$f_1(k) = f_1^D(k) + f_1^{E-D}(k), \tag{4.5}$$

where $f_1^D(k) \in h_1^D$ and $f_1^{E-D}(k) \in h_1^{E-D}$.

If an eigenvector $f_1(k)$ belongs to h_1^D, then equation (4.3) yields

$$\left(\frac{2k_1^2}{2m} - 2\mu\right) f_1(k_1) + c v_{k_1} = E f_1(k_1),$$

$$c = \frac{g}{V}\sum_p v_p f_1(p), \qquad f_1(k_1) = \frac{c v_{k_1}}{-\frac{2k_1^2}{2m} + 2\mu + E}. \tag{4.6}$$

It is obvious that eigenvectors (4.6) belong to h_1^D.

From (4.6), we derive the following equation for the eigenvalues of eigenvectors from h_1^D:

$$1 = \frac{g}{V}\sum_p \frac{v_p^2}{-\frac{2p^2}{2m} + 2\mu + E}. \tag{4.7}$$

Note that the summation in (4.7) is carried out over the domain D. Denote by N the number of quasidiscrete momenta $p = (2\pi/L)\,n$, $n = (n_1, n_2, n_3)$, such that $p \subset D$. Equation (4.7) has N solutions $E_1, \ldots, E_N$ because it is equivalent to the equation

$$P_N(E) = 0, \tag{4.8}$$

where $P_N(E)$ is a polynomial of the Nth degree with respect to E. The eigenvalues $E_1, \ldots, E_N$ are real numbers because the operator H_2 is symmetric.

For the eigenvectors $f_1(k_1)$ from h_1^{E-D}, we have the equation

$$\left(\frac{2k_1^2}{2m} - 2\mu\right) f_1(k_1) = E f_1(k_1), \tag{4.9}$$

whence we determine the eigenvectors

$$f_1(k_1) = \delta_{k_1 - k}, \quad k \cup E - D, \tag{4.10}$$

and the corresponding eigenvalues

$$E = \frac{2k^2}{2m} - 2\mu.$$

3. Spectrum of the operator H_Λ *in* $\mathcal{H}_n^P$. In order to determine the spectrum of the operator H_Λ in $\mathcal{H}_n^P$, we use representation (3.5) and regard the operator B as a perturbation of the operator A. The spectrum of the operator A is known and is determined by (4.2). The norm of the matrix of the operator B is proportional to $\frac{1}{V^{1/2}}$ according to (3.9). From the well-known theorem of linear algebra (see, e.g., [13 – 15]), we conclude that the eigenvalues of the operator $A + B$ differ from the eigenvalues of the operator A (4.2) by values that are proportional to $\frac{1}{V^{1/2m}}$, where m is the multiplicity of an eigenvalue.

We summarize the results obtained above in the following statement:

Theorem 3. *The spectrum of the operator* H_Λ *in the space of pairs* $\mathcal{H}_n^P$ *is given by the formula*

$$\tilde{E} = E_1 + \ldots + E_n + \varepsilon(1/V),$$

where $E_1, \ldots, E_n$ *are the eigenvalues of the operator* H_Λ^2 *and can be determined exactly. The perturbation* $\varepsilon(1/V)$ *tends to zero as* $1/V^{1/2m}$*, where* m *is the multiplicity of the eigenvalue* $\tilde{E}$.

Note that the operator B differs from zero only on the wave functions with supports in D. Thus, $E_1, \ldots, E_n$ are determined by formula (4.7). For the wave functions with supports in $E - D$, the perturbation B is equal to zero.

V. Convergence of the Hamiltonian H^Λ **to the operator** A **on states of pairs and excited states.** *1. The number of quasimomenta in the layer of the Fermi sphere* $-\omega < \dfrac{k^2}{2m} - \mu < \omega$. We have proved above that the Hamiltonian H^Λ converges to the operator A on arbitrary states f with finite number of pairs $f = (f_0, 0, f_2(k_1, k_2), \ldots, f_n(k_1, \ldots, k_n), 0, 0, \ldots)$ (see inequalities (3.9)). We now want to prove that this assertion is true for states with arbitrary possible number of pairs. To do this, we use the fact that the potential v_k is different from zero in the layer of the Fermi sphere: $v_k \neq 0$ if $-\omega < k^2/2m - \mu < \omega$. We assume that all $f_n(k_1, \ldots, k_n)$ have supports in D^n, i.e., $f_n(k_1, \ldots, k_n) = 0$ if some k_i does not belong to D.

Denote by N the number of quasimomenta in the layer D of the Fermi sphere. For N, we have the following estimate:

$$N \leq \frac{2\frac{4}{3}\pi[2m(\mu+\omega)]^{3/2}}{(2\pi)^3/L^3} = \alpha V, \quad \alpha = \frac{8\pi}{3(2\pi)^3}[2m(\mu+\omega)]^{3/2}, \quad \frac{N}{V} \leq \alpha. \tag{5.1}$$

From N quasimomenta $k_1, k_2, \ldots, k_N$ we can choose $\dfrac{N!}{n!(N-n)!}$ different states (quasimomenta) consisting of n different quasimomenta $k_1 \neq k_2 \neq \ldots \neq k_N$ (we identify the states that differ only by permutations). We have the following trivial inequality:

$$\frac{N!}{n!(N-n)!} = \frac{N(N-1)\ldots(N-n+1)}{n!} \leq \frac{N^n}{n!}. \tag{5.2}$$

We now consider the direct sum of the Hilbert spaces with scalar product and norm (3.10) and denote it by $\mathcal{H}_V^P$. The scalar product of two sequences of pairs $f \in \mathcal{H}_V^P$ and $g \in \mathcal{H}_V^P$ is defined as

$$(f, g)'_V = \sum_{n=0}^{\infty} \frac{1}{V^n} \sum_{k_1 \neq k_2 \neq \ldots \neq k_n}{}' \overline{f_n(k_1, \ldots, k_n)} g_n(k_1, \ldots, k_n) \tag{5.3}$$

and the corresponding norm $\|f\|'$ is defined as

$$\{\|f\|'_V\}^2 = \sum_{n=0}^{\infty} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}{}' |f_n(k_1, \ldots, k_n)|^2 =$$

$$= \sum_{n=0}^{N} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}{}' |f_n(k_1, \ldots, k_n)|^2 < \infty. \tag{5.4}$$

Assume that

$$\sup_{k_1,\ldots,k_n} |f_n(k_1,\ldots,k_n)| \le f^n, \quad f > 0. \tag{5.5}$$

For example, if $f_n(k_1,\ldots,k_n) = f_1(k_1)\ldots f_1(k_n)$, then $f = \sup_k |f_1(k)|$.

With the use of (5.1), (5.2), and (5.5), the norm $\|f\|'_V$ can be estimated as follows:

$$\left\{\|f\|'_V\right\}^2 \le \sum_{n=0}^{N} \frac{1}{V^n}\frac{N^n}{n!} f^{2n} \le \sum_{n=0}^{N} \frac{\alpha^n f^{2n}}{n!} \le e^{\alpha f^2}. \tag{5.6}$$

It follows from from (5.6) that $\|f\|'_V$ is finite for $f < \infty$.

2. Estimate for the operator $H_\Lambda - A$ on states of pairs. Consider the expression

$$(f,(H_\Lambda - A)f)'_V = \sum_{n=0}^{N} \frac{1}{V^n} \sum_{k_1\ne\ldots\ne k_n}{}' \overline{f_n(k_1,\ldots,k_n)}((H_\Lambda - A)f_n)(k_1,\ldots,k_n) =$$

$$= \sum_{n=0}^{N} \frac{1}{V^n} \sum_{k_1\ne\ldots\ne k_n}{}' \overline{f_n(k_1,\ldots,k_n)}(Bf_n)(k_1,\ldots,k_n). \tag{5.7}$$

In order to estimate expression (5.7), we use representation (3.6) of the operator B in the subspace of n pairs $\mathcal{H}_n^P$ and estimates (3.9), (5.1), (5.2), and (5.5).

Then we obtain the following estimate:

$$\left|(f,(H_\Lambda - A)f)'_V\right| = \left|\sum_{n=2}^{N}(f_n, Bf_n)'_V\right| \le$$

$$\le \sum_{n=2}^{N} \frac{|g|}{V}\frac{1}{V^n} \sum_{k_1\ne\ldots\ne k_n}{}' |f_n(k_1,\ldots,k_n)| \sum_{i=1}^{n}\sum_{i=j\ne i}^{n} \left|V_{k_i,k_j}\right| \left|f_n(k_1,\ldots,\overset{i}{k_j},\ldots,k_n)\right| \le$$

$$\le \frac{|g|v^2}{V}\sum_{n=2}^{N}\frac{N^n}{V^n}\frac{n(n-1)}{n!} f^{2n} \le \frac{|g|v^2}{V}\sum_{n=2}^{N}\frac{\alpha^n f^{2n}}{(n-2)!} \le \frac{|g|v^2}{V}\alpha^2 f^4 e^{\alpha f^2}. \tag{5.8}$$

It follows from (5.8) that $(f,(H_\Lambda - A)f)'_V \to 0$ as $V \to \infty$.

We now estimate the following expression with $n_0 > V^\delta$, $\delta > 0$:

$$\left|\sum_{n=n_0+1}^{N}\frac{1}{V^n}\sum_{k_1\ne\ldots\ne k_n}{}' \overline{f_n(k_1,\ldots,k_n)}\left(H^\Lambda f_n\right)(k_1,\ldots,k_n)\right| =$$

$$= \left|\sum_{n=n_0+1}^{N}\left\{\frac{1}{V^n}\sum_{k_1\ne\ldots\ne k_n}{}' \overline{f_n(k_1,\ldots,k_n)}\sum_{i=1}^{n}\left[\left(\frac{2k_i^2}{2m} - 2\mu\right)f_n(k_1,\ldots,k_n) + \right.\right.\right.$$

$$\left.\left.\left. + \frac{g}{V}\sum_{p\ne k_1,\overset{i}{\ldots},p\ne k_n} V_{k_i,p} f_n\left(k_1,\ldots,\overset{i}{p},\ldots,k_n\right)\right]\right\}\right| \le$$

$$\le \sum_{n=n_0+1}^{N}\frac{1}{V^n}\frac{N^n}{n!} f^{2n}\left(2\omega n + |g|v^2\alpha n\right) =$$

$$= \sum_{n=n_0+1}^{N}\frac{\alpha^n f^{2n}}{(n-1)!}\left[2\omega + |g|v^2\alpha\right] \le \frac{1}{n_0}\alpha^2 f^4\left(2\omega + |g|v^2\alpha\right)e^{\alpha f^2} \le$$

$$\le \frac{1}{V^\delta}\alpha^2 f^4\left(2\omega + |g|v^2\alpha\right)e^{\alpha f^2}. \tag{5.9}$$

In (5.9), we have used the estimate

$$\frac{1}{V}\sum_p |V_{k,p}| \le \frac{N}{V}v^2 = \alpha v^2.$$

Consider the same expression with the operator A.
We have the following estimate:

$$\left| \sum_{n=n_0+1}^{N} \left\{ \frac{1}{V^n} \sum_{k_1 \ne \ldots \ne k_n}{}' \overline{f_n(k_1, \ldots, k_n)} \sum_{i=1}^{n} \left(\frac{2k_i^2}{2m} - 2\mu \right) f_n(k_1, \ldots, k_n) + \right.\right.$$

$$\left.\left. + \frac{g}{V} \sum_p V_{k_i,p} f_n(k_1 \ldots, \overline{p}, \ldots, k_n) \right\} \right| \le$$

$$\le \sum_{n=n_0+1}^{N} \frac{1}{V^n} \sum_{k_1 \ne \ldots \ne k_n}{}' f^{2n}\left(2\omega n + |g| v^2 \alpha n\right) \le \frac{1}{V^\delta} \alpha^2 f^4 \left(2\omega + |g| v^2 \alpha\right) e^{\alpha f^2}. \quad (5.10)$$

Estimates (5.9) and (5.10) obtained above show that, for both operators H^Λ and A, averages (5.9) and (5.11) tend to zero for sufficiently large n_0. By analogy with (5.8), we have the following estimate:

$$((H_\Lambda - A) f, (H_\Lambda - A) f)_V' \le \frac{g^2 v^4}{V^2} \sum_{n=2}^{N} \alpha^n \frac{(n(n-1))^2}{n!} f^{2n} \le$$

$$\le \frac{g^2 v^4}{V^2} \left(6\alpha^4 f^8 e^{\alpha f^2} + 2\alpha^2 f^4 + 6\alpha^3 f^6 \right). \quad (5.11)$$

This implies that $\|(H_\Lambda - A) f\|_V'$ tends to zero as $V \to \infty$.

It follows from (5.8) that the following theorem is true:

Theorem 4. *If the states of pairs* $f = (1, 0, f_1(k_1), 0, \ldots, f_n(k_1, \ldots, k_n), \ldots)$ *satisfy the conditions*

$$|f_n(k_1, \ldots, k_n)| < f^n, \quad f < \infty, \quad n \ge 1,$$

uniformly with respect to V, *and* $f_n(k_1, \ldots, k_n)$ *have supports in* D^n, *then* $\left(f, (H^\Lambda - A) f\right)_V'$ *and* $\|(H_\Lambda - A) f\|_V'$ *tend to zero as* $V \to \infty$ *and the estimates* (5.8)–(5.11) *hold.*

Corollary. If $f_1^0(k)$ is the eigenfunction corresponding to the lowest eigenvalue $E_0 < 0$ and there is a gap $\Delta = 2\omega + E_0 < 0$, $|\Delta| > \varepsilon_0 > 0$, uniform with respect to the volume V, then the ground state $f = (f_0, 0, f_1^0(k_1), 0, f_1^0(k_1) f_1^0(k_2), \ldots$ $\ldots, f_1^0(k_1) f_1^0(k_2) \ldots f_1^0(k_n), \ldots)$, which corresponds to indefinite (arbitrary) random number of pairs in the ground state, satisfies the condition of the theorem. Indeed, it follows from (4.6) that $f_1^0(k)$ is uniformly bounded:

$$|f_1^0(k_1)| \le \frac{|c v_k|}{\left| -\frac{2k_1^2}{2m} + 2\mu + E_0 \right|} \le \frac{|c| v}{\left| -\frac{2k_1^2}{2m} + 2\mu - 2\omega + \Delta \right|} < \frac{|c| v}{\varepsilon_0},$$

$$\|f_1^0\|_V \le \frac{|c| \|v\|}{\varepsilon_0}.$$

In the next section, we show that these conditions are satisfied.

Note that, for the other eigenfunctions of H_2, the conditions of Theorem 4 are not satisfied because the corresponding eigenvalues become continuous spectra in the thermodynamic limit.

VI. Equation for eigenvalues. *1. Existence of eigenvalues.* In Section 4, we have derived the equation for eigenvalues

$$1 = \frac{g}{V}\sum_p \frac{v_p^2}{-\frac{2p^2}{2m}+2\mu+E}, \quad V = L^3, \quad p = \frac{2\pi}{L}n, \quad n = (n_1, n_2, n_3), \quad n_i \subset Z, \tag{6.1}$$

where the summation is carried out over the layer of the Fermi sphere $\left|p^2/2m-\mu\right| \leq \omega$ because $v_p = 0$ for $\left|p^2/2m-\mu\right| > \omega$.

We consider attractive interaction, which means that the coupling constant is negative, i.e., $g < 0$.

We denote the expression on the right-hand side of (6.1) by $f_L(E)$ and investigate the properties of this function.

It is obvious that the function $f_L(E)$ has the poles $E = 2p^2/(2m) - 2\mu$. The lowest pole is -2ω. The function

$$f_L(E) = \frac{g}{V}\sum_p \frac{v_p^2}{-\frac{2p^2}{2m}+2\mu+E}$$

is increasing because

$$f_L'(E) = -\frac{g}{V}\sum_p \frac{v_p^2}{\left(-\frac{2p^2}{2m}+2\mu+E\right)^2} > 0.$$

Consider the function $f_L(E)$ on the interval $(-\infty, -2\omega)$. On this interval, it is continuous and increasing and takes values from the interval $(0, +\infty)$. This implies that equation (6.1) has a unique solution $E_0(L)$ on the interval $(-\infty, -2\omega)$ for arbitrary $-\infty < g < 0$ and $E_0(L) < -2\omega$.

The behavior of the function $f_L(E)$ is represented in Fig. 1.

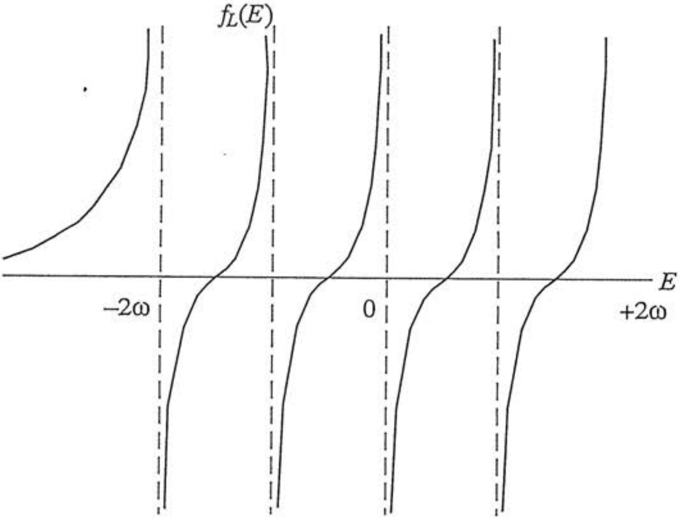

Fig. 1

It is obvious that equation (6.1) $f_L(E)=1$ has unique solutions that correspond to each point (vector) p from the layer $\left|\frac{p^2}{2m}-\mu\right|\le\omega$. If v_p^2 depends on $|p|$, then the same E may correspond to different p on the sphere $|p|=\text{const}$.

Consider, for example, the case where v_p^2 depends on $|p|$. We have $v_p=-v_{-p}$ and, thus, one can assume that v_p depends on $|p|$ for $p^1\ge 0$, $v_p=v_{|p|}$, and $v_p = -v_{-p}$ for $p^1\le 0$. Equation (6.1) reduces to the following:

$$1 = \frac{2g}{V}\sum_{p,\,p^1>0}\frac{v_p^2}{E-\frac{2p^2}{2m}+2\mu} = \frac{2g}{V}\sum_{n^0}\frac{v_{|p|}^2N(n^0)}{E-\frac{2p^2}{2m}+2\mu}\Bigg|_{p^2=\frac{2m(2\pi)^2}{L^2}(n^0)^2},$$

where the summation is carried out over the interval

$$\frac{L}{2\pi}\sqrt{2m(\mu-\omega)} \le n^0 \le \frac{L}{2\pi}\sqrt{2m(\mu+\omega)},$$

$N(n^0)$ is the number of vectors $n=(n^1,n^2,n^3)$ such that $n^2=(n^1)^2+(n^2)^2+(n^3)^2=(n^0)^2$, and n^0 are integer numbers from the interval mentioned above.

It is obvious that equation (6.1) has only simple roots, i.e., the eigenvalues are simple. The eigenvectors

$$f(k) = \frac{cv_k}{E-\frac{2k^2}{2m}+2\mu}$$

depend on $|k|$ for $k^1>0$ and $k^1<0$, and $f(k)=-f(k)$, $k^1>0$.

Denote by $\tilde{E}$ and E the corresponding eigenvalues of H^Λ and A in $\mathcal{H}_n^P$. The eigenvalues of A are simple and we have the following inequality [13–15]:

$$\left|\tilde{E}-E\right| \le \|B\| \le \frac{gv\|v\|}{V^{1/2}}n(n-1)$$

(see (3.10)).

We now consider the lowest solution of equation (6.1) — the lowest eigenvalue $E_0(L)$. This eigenvalue is known as the energy of the ground state of one pair. We are interested in the behavior of $E_0(L)$ as $V\to\infty$, i.e., in the thermodynamic limit. (For this purpose, we denote the lowest eigenvalue for given $V=L^3$ by $E_0(L)$.) Let L tend to infinity and let $L<L_1<L_2<\ldots<L_i$ be a sequence such that $\lim_{i\to\infty}L_i=\infty$. Consider the corresponding sequence

$$E_0(L),\ E_0(L_1),\ E_0(L_2),\ldots,\ E_0(L_i),\ldots. \tag{6.2}$$

The statement below describes the properties of sequence (6.2).

Theorem 5. *Sequence* (6.2) *is convergent*

$$\lim_{L_i\to\infty}E_0(L_i) = E_0. \tag{6.3}$$

There exists a nonzero gap $\Delta=2\omega+E_0>0$ *(for all* $-\infty<g<0$ *except, possibly, one point* $g=g_0$*).*

Proof. Consider the function

$$\varphi(p) = \frac{v_p^2}{\frac{2p^2}{2m} - 2\mu - E}, \quad E < -2\omega, \tag{6.4}$$

and assume that v_p^2 is nonincreasing function of $|p|$, $v_p^2 \le v_{p'}^2$ if $|p| > |p'|$. For a given fixed $E < -2\omega$, the function $\varphi(p)$ is decreasing with respect to $|p|$ and is continuous. This implies that $f_L(E)$ is a decreasing function of L because if $L_1 > L$, then the sums satisfy the following inequality:

$$f_L(E) = \frac{|g|}{V(L)} \sum_{p(L)} \frac{v_{p(L)}^2}{\frac{2p^2(L)}{2m} - 2\mu - E} \ge \frac{|g|}{V(L_1)} \sum_{p(L_1)} \frac{v_{p(L_1)}^2}{\frac{2p^2(L_1)}{2m} - 2\mu - E} = f_{L_1}(E), \tag{6.5}$$

where

$$p(L) = \frac{2\pi}{L} n, \quad p(L_1) = \frac{2\pi}{L_1} n, \quad n = (n_1, n_2, n_3), \quad n_i \in Z.$$

Indeed, if $L_1 > L$, then the number of points $p(L_1)$ in the layer $\left|\frac{p^2}{2m} - \mu\right| \le \omega$ is greater (at least not less) than the number of points $p(L)$. We can regard both sums in (6.5) as integral sums of the decreasing function $\varphi(p)$ (6.4) with infinitesimal volumes $1/V(L)$ and $1/V(L_1)$, respectively. Therefore, inequality (7.4) is obvious (see Fig. 2).

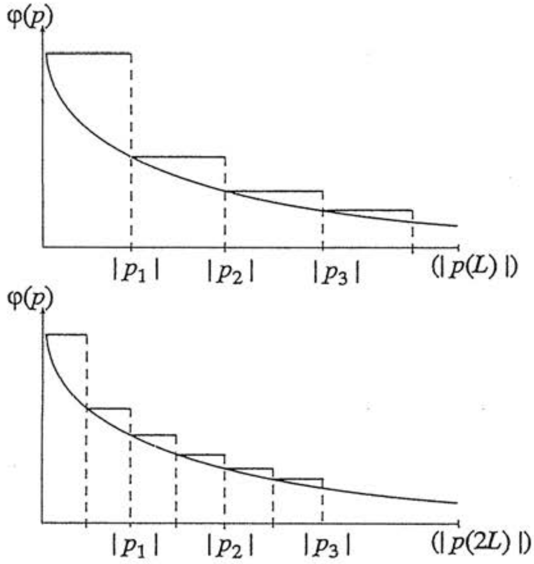

Fig. 2

Thus, the integral sums that represent the functions $f_L(E)$ are decreasing functions of L:

$$f_L(E) > f_{L_1}(E), \quad L < L_1.$$

The sequence of sums $f_L(E), f_{L_1}(E), \dots, f_{L_i}(E), \dots$ is decreasing and bounded from below, $f_{L_i}(E) > 0$. Therefore, there exists a unique limit and it is equal to the integral

$$\lim_{L_i\to\infty} f_{L_i}(E) = \frac{g}{(2\pi)^3}\int \frac{v_p^2\,dp}{-\frac{2p^2}{2m}+2\mu+E}, \qquad E < -2\omega. \tag{6.6}$$

Consider again the sequence of equalities (6.1). We have

$$1 = \frac{g}{V(L)}\sum_{p(L)}\frac{v_{p(L)}^2}{-\frac{2p^2(L)}{2m}+2\mu+E_0(L)} = \frac{|g|}{V(L)}\sum_{p(L)}\frac{v_{p(L)}^2}{\frac{2p(L)^2}{2m}-2\mu+|E_0(L)|},$$
$$1 = \frac{g}{V(L_i)}\sum_{p(L_i)}\frac{v_{p(L_i)}^2}{-\frac{2p^2(L_i)}{2m}+2\mu+E_0(L_i)} = \frac{|g|}{V(L_i)}\sum_{p(L_i)}\frac{v_{p(L_i)}^2}{\frac{2p(L_i)^2}{2m}-2\mu+|E_0(L_i)|}, \tag{6.7}$$

where $E_0(L_i)$ are the lowest solutions (the energies of ground states for given L_i).

As mentioned above, the sum $f_L(E)$ is a decreasing function of L for a given fixed E. This implies that the function $f_{L_1}(E)$ is equal to 1 for $E = E_0(L_1) > E_0(L)$. In the general case, $E_0(L_j) > E_0(L_i)$ if $L_j > L_i$. We have obtained the increasing sequence

$$E_0(L) < E_0(L_1) < \dots < E_0(L_i) < \dots, \qquad L < L_1 < \dots < L_i < \dots, \tag{6.8}$$

which is bounded from above, i.e., $E_0(L_i) < -2\omega$. This implies that the sequence $E_0(L_i)$ is convergent, i.e., there exists unique E_0 such that

$$\lim_{L_i\to\infty} E_0(L_i) = E_0. \tag{6.9}$$

Let us show that there is a gap $\Delta = 2\omega + E_0 < 0$. Assume that the gap is equal to zero, i.e. $E_0 = -2\omega$ for some $g < 0$. Then

$$1 = \frac{g_0}{(2\pi)^3}\int \frac{v_p^2\,dp}{-\frac{2p^2}{2m}+2\mu-2\omega} < \infty. \tag{6.10}$$

Equality (6.10) implies that the limiting equation (6.1) has the solution $E_0 = -2\omega$ only for unique $g = g_0 < 0$.

It follows from from (6.10) that the gap Δ may be equal to zero for unique g_0 determined according to (6.19). Then, for $g \neq g_0$, the gap Δ is different from zero. There exists a theorem [12] according to which the gap Δ is different from zero for sufficiently small $|g|$ for a general (not necessarily separable) potential. For a separable potential $V_{p_1,p_2} = v_{p_1}v_{p_2}$ and $v_p^2|p| = \text{const}$, the gap Δ is calculated explicitly; it is different from zero for $g > -\infty$ and has an essential singularity at $g = 0$. We have

$$E_0 = \frac{2\omega\left(e^{2/|g|a}+1\right)}{1-e^{2/|g|a}},$$

where a is a positive number. The theorem is proved.

VII. Coincidence of the BCS Hamiltonian with the approximating Hamiltonian. ***1. Operators A^I, A^+, and A^- on the ground state.*** We now want to show that the BCS Hamiltonian and the approximating Hamiltonian coincide in the thermodynamic limit on the ground state with an arbitrary (random) number of pairs with the lowest energy E_0. Consider the ground state

$$\Phi_0 = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i} f_1^0(k_i)a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f_1^0(k_n)a_{k_n}^+ a_{-k_n}^+ |0\rangle =$$

$$= \sum_{n=0}^{\infty} \sum_{k_1 \neq \ldots \neq k_n}{}' f_1^0(k_1)\ldots f_1^0(k_n)\, a_{k_1}^+ a_{-k_1}^+ \ldots a_{k_n}^+ a_{-k_n}^+ |0\rangle, \qquad (\Phi_0)_0 = 1, \tag{7.1}$$

where $f_1^0(k)$ is the eigenfunction of the operator H_2 with the lowest energy E_0 (the ground eigenfunction of the pair).

Denote by A^I the part of the operator A that describes the interaction of two particles with opposite momenta. According to the definition of the operator A (2.6), (3.6), we have

$$A^I\Phi_0 = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=1}^{n}\sum_{k_1} f_1^0(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots$$

$$\ldots \frac{g}{V}\sum_{k_i}^{i} v_{k_i}\sum_{p} v_p f_1^0(p) a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f_1^0(k_n)a_{k_n}^+ a_{-k_n}^+ |0\rangle =$$

$$= c\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=1}^{n}\sum_{k_1} f(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i} \overset{i}{v}_{k_i} a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f(k_n)a_{k_n}^+ a_{-k_n}^+ |0\rangle, \tag{7.2}$$

where

$$c = \frac{g}{V}\sum_{p} v_p f_1^0(p). \tag{7.3}$$

Consider the operator

$$A^+ = c\sum_{k} v_k\, a_k^+ a_{-k}^+, \tag{7.4}$$

where the constant c is defined according (7.3).

We have

$$A^+\Phi_0 = c\sum_{k} v_k\, a_k^+ a_{-k}^+ \Phi_0 =$$

$$= c\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k} v_k\, a_k^+ a_{-k}^+ \sum_{k_1} f_1^0(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_n} f_1^0(k_n)a_{k_n}^+ a_{-k_n}^+ =$$

$$= c\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{i=1}^{n+1}\sum_{k_1} f_1^0(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i}^{i} v_{k_i} a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_{n+1}} f_1^0(k_{n+1})a_{k_{n+1}}^+ a_{-k_{n+1}}^+ |0\rangle =$$

$$= c\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=1}^{n}\sum_{k_1} f_1^0(k_1)a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i}^{i} v_{k_i} a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f_1^0(k_n)a_{k_n}^+ a_{-k_n}^+ |0\rangle. \tag{7.5}$$

Formulas (7.2) and (7.5) imply that the operators A^+ and A^I coincide on the ground state Φ_0 (7.1):

$$A^+\Phi_0 = A^I\Phi_0.$$

Consider the operator

$$A^{-} = c \sum_{k} v_k a_{-k} a_k \tag{7.6}$$

and its action on Φ_0.

We have

$$A^{-}\Phi_0 = c \sum_{k} v_k a_{-k} a_k \Phi_0 = c \sum_{k} v_k f_1^0(k)|0\rangle + c \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots$$

$$\ldots \sum_{\substack{k_i \neq k_1 \\ \vdots \\ k_i \neq k_n}} v_{k_i} f_1^0(k_i) \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^{+} a_{-k_n}^{+} |0\rangle =$$

$$= c \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots \sum_{k_{n-1}} f_1^0(k_{n-1}) a_{k_{n-1}}^{+} a_{-k_{n-1}}^{+} |0\rangle \sum_{k} v_k f_1^0(k) -$$

$$- c \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots$$

$$\ldots \sum_{k_{n-1}} f_1^0(k_{n-1}) a_{k_{n-1}}^{+} a_{-k_{n-1}}^{+} |0\rangle \sum_{k=k_1,\ldots,k=k_{n-1}} v_k f_1^0(k) =$$

$$= g^{-1} c^2 V \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^{+} a_{-k_n}^{+} |0\rangle -$$

$$- c \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^{+} a_{-k_n}^{+} |0\rangle \sum_{k=k_1,\ldots,k=k_n} v_k f_1^0(k) =$$

$$= g^{-1} c^2 V \Phi_0 - c \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^{+} a_{-k_1}^{+} \ldots$$

$$\ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^{+} a_{-k_n}^{+} |0\rangle \sum_{k=k_1,\ldots,k=k_n} v_k f_1^0(k) =$$

$$= g^{-1} c^2 V \Phi_0 - c B_1 \Phi_0, \tag{7.7}$$

where the operator B_1 is defined by the second term in the last expression (7.7). The operator B_1 resembles the operator B (3.6), but the factor $1/V$ is absent.

Let us show that the operator B_1 can be neglected in the thermodynamic limit in the following sense:

consider the averages

$$\frac{1}{V}(\Phi_0, B_1\Phi_0)_V' = \frac{1}{V} \sum_{n=1}^{N} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' \left| f_1^0(k_1) \ldots f_1^0(k_n) \right|^2 \sum_{k=k_1,\ldots,k=k_n} v_k f_1^0(k),$$

$$\frac{1}{V}(B_1\Phi_0, B_1\Phi_0)_V' \leq \frac{1}{V} \sum_{n=1}^{N} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' \left| f_1^0(k_1) \ldots f_1^0(k_n) \right|^2 \left| \sum_{k=k_1,\ldots,k=k_n} v_k f_1^0(k_1) \right|^2.$$

We have the estimates

$$\left|\frac{1}{V}(\Phi_0, B_1\Phi_0)'\right| \leq \frac{1}{V}\sum_{n=1}^{N}\frac{N^n}{V^n}\frac{f^{2n+1}}{n!}v\cdot n \leq \frac{1}{V}v\alpha f^3 e^{\alpha f^2}, \quad f = \sup_k \left|f_1^0(k)\right|, \tag{7.8}$$

$$\frac{1}{V}(B_1\Phi_0, B_1\Phi_0)'_V \leq \frac{1}{V}\sum_{n=1}^{N}\frac{N^n}{V^n}\frac{f^{2(n+1)}v^2n^2}{n!} \leq \frac{1}{V}v^2\left(\alpha f^4 + 2\alpha^2 f^6 e^{\alpha f^2}\right),$$

which imply that averages (7.8) tend to zero as $V \to \infty$.

2. Approximating Hamiltonian. We define the following approximating Hamiltonian:

$$H_{\text{appr},\Lambda} = H_{a,\Lambda} =$$

$$= \sum_{\bar{p}}\left(\frac{p^2}{2m} - \mu\right)a_{\bar{p}}^{+}a_{\bar{p}} + c\sum_{p}v_p a_p^{+}a_{-p}^{+} + c\sum_{p}v_p a_{-p}a_p - g^{-1}c^2V, \tag{7.9}$$

where the constant c is defined in (7.3). (Note that the term $g^{-1}c^2V$ should be understood as the operator $g^{-1}c^2VI$, where I is the identity operator; we use the notation $g^{-1}c^2V$ accepted in mathematical physics.)

We are ready to prove the following theorem about the approximating Hamiltonian.

Theorem 6. *The averages*

$$\frac{1}{V}\left(\Phi_0, (H_\Lambda - H_{a,\Lambda})\Phi_0\right)'_V$$

and

$$\frac{1}{V}\left((H_\Lambda - H_{a,\Lambda})\Phi_0, (H_\Lambda - H_{a,\Lambda})\Phi_0\right)'_V$$

tend to zero in the thermodynamic limit as $V \to \infty$.

Proof. Consider the identity

$$\frac{1}{V}\left(\Phi_0, (H_\Lambda - H_{a,\Lambda})\Phi_0\right)'_V = \frac{1}{V}(\Phi_0, (H_\Lambda - A)\Phi_0)'_V + \frac{1}{V}\left(\Phi_0, (A - H_{a,\Lambda})\Phi_0\right)'_V. \tag{7.10}$$

In Section V, we have showed that the first term $(\Phi_0, (H-A)\Phi_0)'$ tends to zero in the thermodynamic limit even without the factor $1/V$. The operator $H_{a,\Lambda}$ can be represented in terms of the operators A^+ and A^- as follows:

$$H_{a,\Lambda} = \sum_{\bar{p}}\left(\frac{p^2}{2m} - \mu\right)a_{\bar{p}}^{+}a_{\bar{p}} + A^+ + A^- - g^{-1}c^2V.$$

The operator A can be represented as follows:

$$A = \sum_{\bar{p}}\left(\frac{p^2}{2m} - \mu\right)a_{\bar{p}}^{+}a_{\bar{p}} + A^I.$$

By using relations (7.3) – (7.7), we represent the second term in (7.10) in the form

$$\frac{1}{V}\left(\Phi_0, (A - H_{a,\Lambda})\Phi_0\right)'_V = \frac{c}{V}(\Phi_0, B_1\Phi_0)'.$$

(Here, we have used the fact that $A^I\Phi_0 = A^+\Phi_0$ and $A^-\Phi_0 = g^{-1}c^2V\Phi_0 - cB_1\Phi_0$.)

Estimate (7.8) implies that this term tends to zero as $V \to \infty$.

We also have

$$\frac{1}{V}\left\|(H_\Lambda - H_{a,\Lambda})\Phi_0\right\|_V' \le \frac{1}{V}\|(H_\Lambda - A)\Phi_0\|_V' + \frac{1}{V}\left\|(A - H_{a,\Lambda})\Phi_0\right\|_V' =$$

$$= \frac{1}{V}\|(H_\Lambda - A)\Phi_0\|_V' + \frac{1}{V}\|B_1\Phi_0\|_V'.$$

According to (5.11) and (7.8), the last two expressions tend to zero as $V \to \infty$. The theorem is proved.

Remark. It follows from (7.7) and (7.8) that the average

$$\frac{1}{V}\left(\Phi_0, \left(A^- - c^2V\right)\Phi_0\right)_V' = \frac{1}{V}(\Phi_0, cB_1\Phi_0)'$$

tends to zero in the thermodynamic limit.

On the other hand, we have

$$A\Phi_0 = \left[\sum_{\bar p}\left(\frac{p^2}{2m} - \mu\right)a_{\bar p}^+ a_{\bar p} + A^I\right] = \left[\sum_{\bar p}\left(\frac{p^2}{2m} - \mu\right)a_{\bar p}^+ a_{\bar p} + A^+\right]\Phi_0.$$

Then it follows from Theorem 4 that

$$(\Phi_0, (H_\Lambda - A)\Phi_0)_V' = \left(\Phi_0, \left(H_\Lambda - \left[\sum_{\bar p}\left(\frac{p^2}{2m} - \mu\right)a_{\bar p}^+ a_{\bar p} + A^+\right]\Phi_0\right)\right)' \qquad (7.11)$$

tends to zero in the thermodynamic limit. This implies that the average of the BCS Hamiltonian H_Λ coincides in the thermodynamic limit with the average of the following operator:

$$H_{a,\Lambda}' = \sum_{\bar p}\left(\frac{p^2}{2m} - \mu\right)a_{\bar p}^+ a_{\bar p} + c\sum_p \upsilon_p\, a_p^+ a_{-p}^+. \qquad (7.12)$$

The operator $H_{a,\Lambda}'$ is a quadratic form with respect to the operators of creation and annihilation as well as the operator $H_{a,\Lambda}$, but $H_{a,\Lambda}'$ is not self-adjoint. We shall use the self-adjoint operator $H_{a,\Lambda}$.

Remark. Theorem 6 is also true for arbitrary states

$$\Phi = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1} f_1(k_1)\, a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i} f_1(k_i)\, a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f_1(k_n)\, a_{k_n}^+ a_{-k_n}^+ |0\rangle$$

if the functions $f_1(k)$ satisfy the following conditions:

$$f = \sup_k |f_1(k)| < \infty.$$

For every state of this type, there exists the corresponding Hamiltonian with $c = \frac{g}{V}\sum_p \upsilon_p f_1(p)$.

We have not considered the states Φ in which certain functions $f^0(k)$ that correspond to the lowest eigenvalue E_0 of the operator H_2 are replaced by the eigenfunctions $f^i(k)$ that correspond to the eigenvalues $E_i > E_0$. These states, excited states, and the action of the operators H_Λ and $H_{a,\Lambda}$ on them will be investigated in the second part of this work.

Recently, we have received from Professors W. Thirring and N. Ilieva a copy of the article "A pair potential supporting a mixed mean-field/BCS phase" (to be published in

Nucl. Phys. B), where a Hamiltonian is considered that becomes equivalent to the approximating Hamiltonian in the scaling limit. The author expresses his gratitude to Professors W. Thirring and N. Ilieva for sending him their article before its publication.

1. *Bogolyubov N. N.* To the theory of superfluidity // Izv. Akad. Nauk SSSR. Ser. Fiz. – 1947. – **11**, № 1. – P. 77 – 90.
2. *Bardeen J., Cooper L. N., Schrieffer J. R.* Theory of superconductivity // Phys. Rev. – 1957. – **108**. – P. 1175 – 1204.
3. *Bogolyubov N. N., Zubarev D. N., Tserkovnikov Yu. A.* An asymptotically exact solution for the model Hamiltonian in the theory of superconductivity // Zh. Éksp. Teor. Fiz. – 1960. – **39**, № 1. – P. 120 – 129.
4. *Bogolyubov N. N.* On the model Hamiltonian in the Theory of Superconductivity. – 1960. – (Preprint / JINR. R-511;60).
5. *Bogolyubov N. N. (Jr.)* A method for investigation of model Hamiltonians. – Moscow: Nauka, 1974. – 176 p.
6. *Bogolyubov N. N. (Jr.), Brankov J. G., Zagrebnov V. A., Kurbatov A. N., Tonchev N. S.* Method of approximating Hamiltonian in statistical physics. – Sofia: Izd. Bolg. Akad. Nauk, 1981. – 246 p.
7. *Haag R.* The mathematical structure of the Bardeen – Cooper – Schrieffer model // Nuovo Cim. – 1962. – **25**. – P. 287 – 299.
8. *Thirring W.* On the mathematical structure of the BCS model // Communs Math. Phys. – 1968. – **7**. – P. 181 – 189.
9. *Petrina D. Ya.* On Hamiltonians of quantum statistics and on the model Hamiltonian in the theory of superconductivity // Teor. Mat. Fiz. – 1970. – **4**. – P. 394 – 405.
10. *Petrina D. Ya., Yatsyshin V. P.* On a model Hamiltonian in the theory of superconductivity // Teor. Mat. Fiz. – 1972. – **10**, № 2. – P. 283 – 305.
11. *Petrina D. Ya., Gredzhuk V. I.* On two models of interacting Bose gas: Bogolyubov model of superfluidity and Huang–Yang–Luttinger model // Mat. Fizika, Anal. Geometr. – 1996. – **3**, № 3/4. – P. 370 – 405.
12. *Petrina D. Ya.* Mathematical foundations of quantum statistical mechanics. Continuous systems. – Dordrecht: Kluwer, 1995. – 444 p.
13. *Voevodin V. V., Kuznetsov Yu. A.* Matrix and calculations. – Moscow: Nauka, 1984. – 318 p.
14. *Voevodin V. V.* Numerical foundations of linear algebra. – Moscow: Nauka, 1977. – 304 p.
15. *Kato T.* Perturbation theory of linear operators. – Berlin: Springer-Verlag, 1966. – 740 p.

Received 27.04.99