UDC 517.5

K. A. Kopotun, post-grad. student (Inst. Math. National Akad. Sci. Ukraine, Kiev, and Univ. Alberta, Edmonton, Canada),
V. V. Listopad, post-grad. student (Pedagogical Inst., Kiev)

REMARKS ON MONOTONE AND CONVEX APPROXIMATION BY ALGEBRAIC POLYNOMIALS

ПРО МОНОТОННУ ТА ВИПУКЛУ АПРОКСИМАЦІЮ АЛГЕБРАЇЧНИМИ ПОЛІНОМАМИ

The following results are obtained: If $\alpha > 0$, $\alpha \neq 2$, $\alpha \overline{\in} [3, 4]$, and f is a nondecreasing (convex) function on $[-1, 1]$ such that $E_n(f) \leq n^{-\alpha}$ for any $n > \alpha$, then $E_n^{(1)}(f) \leq Cn^{-\alpha}$ ($E_n^{(2)}(f) \leq \leq Cn^{-\alpha}$) for $n > \alpha$ and $C = C(\alpha)$, where $E_n(f)$ is the value of the best uniform approximation of a continuous function by polynomials of degree $(n-1)$, and $E_n^{(1)}(f)$ and $E_n^{(2)}(f)$ are the values of the best monotone and convex approximation, respectively. For $\alpha = 2$ and $\alpha \in [3, 4]$, this result is not true.

Одержано такі результати: якщо $\alpha > 0$, $\alpha \neq 2$, $\alpha \overline{\in} [3, 4]$, f неспадна (випукла) на $[-1, 1]$ функція така, що для кожного $n > \alpha$ $E_n(f) \leq n^{-\alpha}$, то $E_n^{(1)}(f) \leq Cn^{-\alpha}$ ($E_n^{(2)}(f) \leq Cn^{-\alpha}$) для $n > \alpha$, $C = C(\alpha)$, де $E_n(f)$ — величина найкращого рівномірного наближення неперервної функції многочленами степеня $(n-1)$, а $E_n^{(1)}(f)$ та $E_n^{(2)}(f)$ — відповідно величини найкращого монотонного та випуклого наближення. Для $\alpha = 2$, $\alpha \in [3, 4]$, цей висновок не вірний.

1. Introduction and main results. Recall that the coapproximation (or shape preserving approximation) is the approximation of functions f such that $\overline{\Delta}_h^q(f, x) \geq 0$ for given $q \in N$, all $0 \leq h \leq 2/q$, and $x \in [-1, 1]$, by polynomials with nonnegative qth derivatives. Here,

$$\overline{\Delta}_h^q(f, x) = \begin{cases} \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} f(x + (i - q/2)h), & \text{if } |x \pm qh/2| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a qth symmetric difference.

Let Δ^q be a set of such functions f. Note that if $f \in C^q[a, b]$, then $f \in \Delta^q$ if and only if $f^{(q)}(x) \geq 0$, $x \in [-1, 1]$.

In the present paper, we consider the monotone and convex approximations by algebraic polynomials, i.e., the cases where $q = 1$ and $q = 2$, respectively. These kinds of coapproximation have been investigated extensively in recent years. Many estimates of the degree of coapproximation were obtained both in uniform metric and in L_p-metric, $0 < p < \infty$. These estimates often turn out to be of the same order as in the case of unconstrained approximation.

The following theorem is one of the results of this type:

Theorem A. *Let* $\alpha > 0$. *If, for a nondecreasing (convex) function* $f = f(x)$ *on* $[-1, 1]$ *and any integer* $n > \alpha$, *there exists an algebraic polynomial* $p_{n-1} = p_{n-1}(x)$ *of* $(n-1)$ *th degree such that*

$$|f(x) - p_{n-1}(x)| \leq \left(\frac{\sqrt{1 - x^2}}{n} + \frac{1}{n^2} \right)^{\alpha}, \quad x \in [-1, 1],$$

then, for any $n > \alpha - 1$ *there is a nondecreasing (convex) polynomial* $p_{n-1}^* = p_{n-1}^*(x)$ *such that the inequality*

$$|f(x) - p_{n-1}^*(x)| \le C\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{\alpha}, \quad C = C(\alpha),$$

holds for $x \in [-1, 1]$, *i.e., the constant* C *depends on* α *and is independent of* n *and* f.

Theorem A is a consequence of classical inverse theorems (see, e.g., V. K. Dzyadyk [1, p. 263]) and R. A. De Vore and X. M. Yu [2] (in the case of $f \in \Delta^1$ and $0 < \alpha < 2$), I. A. Shevchuk [3] ($f \in \Delta^1$ and $\alpha \ge 2$), D. Leviatan [4] ($f \in \Delta^2$ and $0 < \alpha < 2$), S. P. Manya and I. A. Shevchuk (see, e.g., [5]) ($f \in \Delta^2$ and $\alpha > 2$), and K. A. Kopotun [6] ($f \in \Delta^2$ and $\alpha = 2$).

It is clear that one of the crucial arguments of Theorem A is the application of the inverse results. It is well known that, for the algebraic polynomial approximation in uniform metric, the inverse theorems (in terms of the usual modulus of smoothness given by

$$\omega_k(f, t) := \sup_{0 < h \le t} \| \overline{\Delta}_h^k(f, x)\|_{\infty})$$

should be pointwise. This explains why we have pointwise estimates in Theorem A.

After appearance of the new Ditzian – Totik modulus of smoothness [7]

$$\omega_{\varphi}^k(f, t)_p := \sup_{0 < h \le t} \| \overline{\Delta}_{h\varphi(x)}^k(f, x)\|_p,$$

which measures differently the behavior of the functions near the endpoints of an interval and inside the interval, it became possible to establish inverse results in terms of the uniform estimates on the basis of these new moduli. Together with some direct results for the shape preserving approximation in terms of ω_{φ}^k (see, e.g., [6, 8]), this allows one to characterize functions by using uniform estimates of the algebraic polynomial approximation.

In this paper, we are interested in the correspondence between the rates of shape preserving and unconstrained approximation when considering uniform estimates instead of pointwise as in Theorem A.

Let $E_n^{(1)}(f)$ and $E_n^{(2)}(f)$ be the best $(n-1)$th degree monotone and convex polynomial approximation, respectively, of monotone or convex functions on $[-1, 1]$. Let $E_n(f)$ be the best $(n-1)$th degree unconstrained polynomial approximation of f, i.e.,

$$E_n(f) := \inf_{p_{n-1} \in P_{n-1}} \|f - p_{n-1}\|_{\infty}$$

and

$$E_n^{(q)}(f) := \inf_{p_{n-1} \in P_{n-1} \cap \Delta^q} \|f - p_{n-1}\|_{\infty}, \quad q = 1 \text{ or } 2,$$

where P_n is the set of algebraic polynomials whose degree does not exceed n.

The main results of the paper are the following theorems:

Theorem 1. *If* $\alpha > 0$, $\alpha \ne 2$, *and* f *is a nondecreasing function on* $[-1, 1]$ *such that, for each* $n > \alpha$, *the inequality* $E_n(f) \le n^{-\alpha}$ *holds, then* $E_n^{(1)}(f) \le Cn^{-\alpha}$ *for* $n > \alpha$ *with* $C = C(\alpha)$.

Theorem 2. *For* $\alpha = 2$, *the assertion of Theorem* 1 *is not true.*

Theorem 3. *If* $\alpha \in (0,3) \cup (4, +\infty)$ *and* f *is a convex function on* $[-1, 1]$ *such that, for each* $n > \alpha$, *the inequality* $E_n(f) \le n^{-\alpha}$ *holds, then* $E_n^{(2)}(f) \le \le Cn^{-\alpha}$ *for* $n > \alpha$ *with* $C = C(\alpha)$.

Theorem 4. *For* $\alpha \in [3, 4]$, *the assertion of Theorem* 3 *is not true.*

For example, Theorem 2 means that, for $\alpha = 2$, the constant C in Theorem 1 cannot be independent of n and f. The same situation exists with convex approximation for $\alpha \in [3, 4]$. We do not know whether these results in the negative direction will hold if we weaken the conditions imposed on C, for example, if we allow C to be dependent on α and f, and be independent of n.

We now recall some useful definitions and notation (see [7, 5]).

Let $\varphi(x) := \sqrt{1 - x^2}$ and let B^r, $r \in \mathbb{N}$, be the space of all functions f continuous on $[-1, 1]$ and such that their $(r-1)$th derivatives $f^{(r-1)}$ are absolutely locally continuous on $(-1, 1)$ and $\left| (\varphi(x))^r f^{(r-1)}(x) \right| < \infty$ almost everywhere on $(-1, 1)$.

For a function $f \in C(-1, 1)$, the weighted Ditzian–Totik modulus of smoothness is defined as follows:

$$\overline{\omega}^k_{\varphi,r}(f, t) := \sup_{0 < h \le t} \max_{x \in (-1,1)} \left| \left(1 - \frac{k}{2} h\varphi(x) - x \right)^{r/2} \times \right.$$

$$\left. \times \left(1 - \frac{k}{2} h\varphi(x) + x \right)^{r/2} \overline{\Delta}^k_{h\varphi(x)}(f, x) \right|.$$

Obviously, $\overline{\omega}^k_{\varphi,0}(f, t) = \omega^k_{\varphi}(f, t)_\infty$. For $k = 0$, let

$$\overline{\omega}^0_{\varphi,r}(f, t) := \operatorname*{ess\,sup}_{x \in (-1,1)} \left| (\varphi(x))^r f(x) \right|.$$

Of course, the function $\overline{\omega}^k_{\varphi,r}(f, t)$ can be unbounded. As was shown in [5], the necessary and sufficient condition for $\overline{\omega}^k_{\varphi,r}(f, t)$ to be bounded for all $t > 0$ is $\left| (\varphi(x))^r f(x) \right| < M$, $x \in (-1, 1)$, where $M = \text{const} < \infty$. This implies that

$$\overline{\omega}^k_{\varphi,r}(f^{(r)}, t) < \infty, \quad t > 0, \quad \Leftrightarrow \quad f \in B^r.$$

For a function $f \in B^r \cap C^r(-1, 1)$, $r \ge 1$, and $k \ge 0$, the following inequality holds (see, e.g., [5]):

$$\overline{\omega}^{k+r-l}_{\varphi,l}(f^{(l)}, t) \le C t^{r-l} \overline{\omega}^k_{\varphi,r}(f^{(r)}, t), \; t > 0, \tag{1}$$

where $0 \le l \le r - 1$.

Let $B^r \overline{H}[k, \psi]$ be a set of functions $f \in B^r \cap C^r(-1, 1)$ such that $\overline{\omega}^k_{\varphi,r}(f^{(r)}, t) \le \psi(t)$, where $\psi \in \Phi^k$ (we have $\psi \in \Phi^k$ if $\psi(0) = 0$, $\psi = \psi(t)$ is a continuous and nondecreasing function for $t \ge 0$, and $t^{-k}\psi(t)$ is nonincreasing).

We can now define an analog of the class $\text{Lip}^*\alpha := \left\{ f \,|\, \omega_2(f^{(r)}, t) = O(t^\beta), \right.$ where $\alpha > 0$, $\alpha = r + \beta$, $r \in N \cup \{0\}$, and $\left. 0 < \beta \le 1 \right\}$, in terms of the Ditzian–Totik weighted moduli of smoothness as follows:

$$\hat{H}^{\alpha} = \begin{cases} B^r\, \overline{H}\left[1, t^{\beta}\right], & \text{if} \quad \alpha \notin N, \quad \text{where } r := [\alpha] \text{ and } \beta := \alpha - r, \\ B^r\, \overline{H}[2, t], & \text{if} \quad \alpha \in N, \quad \text{where } r := \alpha - 1. \end{cases}$$

Note that the equivalence $f \in \hat{H}^{\alpha} \Leftrightarrow E_n(f) \leq Cn^{-\alpha}$ for any $\alpha > 0$ and $n > \alpha$, where $C = C(\alpha)$ is a constant dependent only on α, is a consequence of the following direct and inverse theorems:

Direct theorem (*see, e.g.*, [7, 5]). *Let $k \in N$, $(r+1) \in N$, and $\psi \in \Phi^k$. Then, for a given function $f \in B^r\, \overline{H}\,[k, \psi]$ on $[-1, 1]$ and each $n \geq k + r$, $E_n(f) \leq \leq Cn^{-r}\psi(n^{-1})$, $C = C(r, k)$.*

Inverse theorem ([7, 5]). *Let $k \in N$, $(r+1) \in N$, and $\psi \in \Phi^k$. If, for a given function f on $[-1, 1]$ and each $n \geq k + r$, the inequality $E_n(f) \leq n^{-r}\psi(n^{-1})$ holds, then*

$$\overline{\omega}^{k}_{\varphi, r}(f^{(r)}, t) = C\left\{ r \int_0^t \psi(u)\, u^{-1}\, du + t^k \int_t^1 \psi(u)\, u^{-k-1}\, du \right\}, \quad C = C(r, k).$$

Taking this into account, one concludes that Theorems 1 – 4 are consequences of the following theorems:

Theorem 5. *Let $\alpha > 0$, $\alpha \neq 2$. Then, for a given nondecreasing function $f \in \in \hat{H}^{\alpha}$ on $[-1, 1]$ and each $n \in N$, $n > \alpha$, the following inequality holds: $E_n^{(1)}(f) \leq Cn^{-\alpha}$, $C = C(\alpha)$. For $\alpha = 2$, this implication is false.*

Theorem 6. *Let $\alpha \in (0, 3) \cup (4, +\infty)$. Then, for a given convex function $f \in \hat{H}^{\alpha}$ on $[-1, 1]$ and each $n \in N$, $n > \alpha$, the following inequality holds: $E_n^{(2)}(f) \leq Cn^{-\alpha}$, $C = C(\alpha)$. For $\alpha \in [3, 4]$, this implication is false.*

2. Proofs of the negative results. For arbitrary n, $\alpha = 2$, and $\alpha \in [3, 4]$, in the monotone and convex cases, respectively, we shall construct the sequences of functions $\{g_b\} \subset \hat{H}^{\alpha}$ and $\{f_b\} \subset \hat{H}^{\alpha}$ such that $E_n^{(1)}(g_b) \to \infty$, $E_n^{(2)}(f_b) \to \infty$ as $b \to \infty$. This will prove the negative parts of Theorems 5 and 6.

We need the following lemma:

Lemma 1 [9]. *For arbitrary $n \in N$ and $M =$ const, there exists a convex function f_b, $f_b''(x) = bx + b - \ln b - \ln(1 + x)$, $b \in R$, on $[-1, 1]$ such that, for any convex polynomial p_n of degree n on $[-1, 1]$, the inequality $\|f - - p_n\| > M$ holds.*

By using the same method, one can easily prove a similar result for monotonic case (see also [9]).

Lemma 2. *For arbitrary $n \in N$ and $M =$ const, there exists a nondecreasing function g_b on $[-1, 1]$, $g_b'(x) = bx + b - \ln b - \ln(1 + x)$, $b \in R$, such that, for any nondecreasing polynomial $\tilde{p}_n$ of degree n, the inequality $\|f - \tilde{p}_n\| > M$ holds on $[-1, 1]$.*

Let us determine classes that contain the functions f_b and g_b.

Lemma 3. *For any $b \in R$, the functions f_b and g_b belong to the classes $B^3\, \overline{H}\,[1, Ct]$ and $B^1\, \overline{H}\,[2, Ct]$, respectively, where C is an absolute constant.*

Proof. For any real number b and functions $f_b \in B^4$ and $g_b \in B^2$, it follows from inequality (1) that

$$\overline{\omega}^1_{\varphi,3}\left(f_b^{(3)}, t \right) \le Ct\,\overline{\omega}^0_{\varphi,4}\left(f_b^{(4)}, t \right) =$$
$$= Ct \operatorname*{ess\,sup}_{x\in(-1,1)} \left| (\varphi(x))^4 (1+x)^{-2} \right| \le Ct, \quad t>0,$$

and

$$\overline{\omega}^1_{\varphi,2}\left(g_b'', t \right) \le \sup_{0<h\le t}\ \sup_{x\pm(1/2)h\varphi(x)\in(-1,1)} \left| \left(1 - \frac{1}{2}h\varphi(x) - x \right) \times \right.$$
$$\left. \times \left(1 - \frac{1}{2}h\varphi(x) + x \right) \left(g_b''\left(x - \frac{1}{2}h\varphi(x) \right) - g_b''\left(x + \frac{1}{2}h\varphi(x) \right) \right) \right| \le$$
$$\le 2 \sup_h \sup_x \left| \frac{h\varphi(x)}{1 + x + (1/2)h\varphi(x)} \right| \le 4.$$

This implies $\overline{\omega}^2_{\varphi,1}\left(g_b', t \right) \le Ct\,\overline{\omega}^1_{\varphi,2}\left(g_b'', t \right) \le Ct,\ t>0$. Thus, Lemma 3 is proved.

To complete the proof of negative results, it suffices to note that $B^1\overline{H}[2,t] = \hat{H}^2$ and $B^3\overline{H}[1,t] \subset \hat{H}^\alpha$ for $\alpha \in [3,4]$.

For $0 < \alpha < 2$, Theorems 5 and 6 are consequences of the work of D. Leviatan [8], and, for $2 \le \alpha < 3$, Theorem 6 follows from [6].

The proof of the direct results in Theorems 5 and 6 for other α is similar to [3, 9, 10 – 12]. It involves nonlinear techniques and is rather nontrivial. At the same time, these theorems are intermediate steps on the path of investigation of degrees of coapproximation of functions from the classes $B^k\overline{H}[k,\psi]$ being, thus, only relatively valuable. This is why, we omit the details of their proofs in this paper.

Acknowledgement. The authors are indebted to Professor I. A. Shevchuk for useful discussions on the subject and valuable suggestions that helped much to make this paper more readable.

1. *Дзядык В. К.* Введение в теорию равномерного приближения функций полиномами. – М.: Наука, 1977. – 512 с.
2. *De Vore R. A., Yu X. M.* Pointwise estimates for monotone polynomial approximation // Constr. Approx. – 1985. – **1**. – P. 323–331.
3. *Шевчук И. А.* О коприближении монотонных функций // Докл. АН СССР. – 1989. – **308**. – № 3. – С. 537 – 541.
4. *Leviatan D.* Pointwise estimates for convex polynomial approximation // Proc. Amer. Math. Soc. – 1986. – **98**, № 3. –P. 471 – 474.
5. *Шевчук И. А.* Приближения многочленами и следы непрерывных на отрезке функций. – Киев: Наук. думка, 1992. – 225 с.
6. *Kopotun K. A.* Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials // Constr. Approx. – 1994. – **10**, № 2. – P. 153 – 178.
7. *Ditzian Z., Totik V.* Moduli of smoothness. – Berlin: Springer-Verlag, 1987. – 300 p.
8. *Leviatan D.* Monotone and comonotone approximation revisited // J. Approx. Theory. – 1988. – **53**. – P. 1 – 16.
9. *Копотун К. А.* Равномерные оценки ковыпуклого приближения функций полиномами // Мат. заметки. – 1992. – **51**, № 3. – С. 35 – 46.
10. *De Vore R. A.* Monotone approximation by polynomials // SIAM J. Anal. – 1977. – **8**, № 5. – P. 905–921.
11. *De Vore R. A.* Monotone approximation by splines // Ibid. – P. 891 – 905.
12. *Дзюбенко Г. А., Листопад В. В., Шевчук И. А.* Равномерные оценки для монотонной полиномиальной аппроксимации // Укр. мат. журн. – 1993. – **45**, № 1. – С. 38 – 45.

Получено 22.12.92