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BOGOLYUBOV AVERAGING AND NORMALIZATION PROCEDURES IN NONLINEAR MECHANICS. I*

УСЕРЕДНЕННЯ ЗА БОГОЛЮБОВИМ ТА ПРОЦЕДУРИ НОРМАЛІЗАЦІЇ У НЕЛІНІЙНІЙ МЕХАНІЦІ. I*

A new method of asymptotic analysis of nonlinear dynamical systems is developed with extensive use of group–theoretical methods. The technique of normalization, which is named an "asymptotic decomposition" by the authors, is developed in the context of Bogolyubov averaging. In this paper, we also discuss how this technique helps to understand and develop the averaging method for systems in standard form and systems with several fast variables. The new method treats a centralized system as a direct analog of an averaged system according to Bogolyubov. The operation of averaging is interpreted as a Bogolyubov projector in the construction of the projection of an operator onto the algebra of the centralizer.

Запропоновано новий метод дослідження нелінійних динамічних систем, розроблений на основі широкого застосування теоретико-групових методів. Розроблено техніку нормалізації у тісному зв'язку з методом усереднення М. М. Боголюбова, яка названа авторами асимптотичною декомпозицією. Розглянуто питання інтерпретації та розвитку методу усереднення щодо систем у стандартній формі та систем з кількома швидкими змінними. Новий метод інтерпретує централізовану систему як прямий аналог усередненої системи за Боголюбовим. Операція усереднення інтерпретується як проектор Боголюбова для побудови проекції будь-якого оператора на алгебру централізатора.

1. Introduction. Asymptotic methods of nonlinear mechanics suggested by N. M. Krylov and N. N. Bogolyubov (Krylov N. M., Bogolyubov N. N. [1, 2], Bogolyubov N. N. [3]) and described and developed in the famous books of Bogolyubov N. N., Mitropolsky Yu. A. [4], Bogolyubov N. N., Mitropolsky Yu. A., Samoilenko A. M. [5], and Mitropolsky Yu. A. [6, 7] originated a new big trend in perturbation theory. They deeply penetrated into various applied branches (theoretical physics, mechanics, applied astronomy, dynamics of space flights, and others) and laid the foundation of numerous generalizations and various modifications of these methods. There exist a large number of approaches and techniques, and different classes of mathematical objects are considered (ordinary differential equations, partial differential equations, delay differential equations, and others).

An up-to-date survey of averaging methods is given in [8]. It connects the asymptotic theory with the geometric ideas which have been important in modern dynamics.

A survey of development of Bogolyubov's averaging method is given in the papers of Mitropolsky Yu. A. [9] and Samoilenko A. M. [10].

For the last two decades, new generalizations of asymptotic methods of nonlinear mechanics, which tend to elaborate general conceptions in the development of these methods, appeared. First of all, this is a trend called the Lie series and transformations averaging method. For the first time, Lie series were applied in perturbation theory by G. Hori [11] for canonical systems and transferred by G. Hori [12] and A. Kamel [13] to noncanonical systems. The perturbation theory based on Lie series and transformations has some advantages in comparison with the existing method. One of these is the simplicity of algorithms. One can get acquainted with the ideas of these methods and the bibliography in the papers of Giacaglia G. E. O. [14], Nayfeh A. H. [15], Kirchgraber U. and Stiefel E. [16], and Kirchgraber U. [17].

The approach in which Lie series with respect to a parameter are used as transformations was proposed by A. Ya. Povzner [18]. Special assumptions concerning spectral properties of an operator associated with a system of zero approximation made it possible to give a constructive algorithm, formulate several sharp theorems on

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separation of fast and slow variables in a transformed system, and obtain some other results (Bogaevsky V. N., Povzner A. Ya. [19 – 21]).

A single-frequency method of averaging based on the Campbell – Hausdorff formula was developed by V. F. Zhuravlev (V. F. Zhuravlev [22], V. F. Zhuravlev, D. N. Klimov [23]). This method was applied successfully to the investigation of multi-frequency systems, construction of a normal form, and a number of other problems.

All the works cited above in connection with the Lie series and transformations actually use the well-known Campbell – Hausdorff formula. The Campbell – Hausdorff formula gives the exact form of a vector field under the action of a one-parameter group generated by an arbitrary vector field. This formula was either used in an explicit form or derived in the course of calculation.

Among the authors mentioned above, V. F. Zhuravlev used the group theoretical principles most consistently.

Another approach consists in determining the procedure of normalization without employing the Campbell – Hausdorff formula. This approach was used in the theory of normal forms in the classical works by Delaunay, Poincaré, Dulac, and Birkhoff.

The development of normalization techniques in the context of the N. N. Bogolyubov averaging method is characteristic to the second approach.

A. M. Molchanov's paper [24] is a pioneer work in this direction.

The connection between the averaging method and the theory of normal forms was considered in the papers of A. D. Brjuno [25, 26].

An axiomatic approach characterizing general properties of an asymptotic method is described in the paper of Yu. A. Mitropolsky and A. M. Samoilenko [27].

A connection between the A. S. Lomov regularization method [28], the averaging method, and the normal forms has been investigated by Gubin Yu. P. [29] and Lomov S. A. and Safonov V. F. [30].

In the monograph of J. A. Sanders and F. Verhulst [8], a definition of a normal form is given in a close connection with the method of averaging. Normal forms for slowly varying systems are given. Also the theory of a Hamiltonian normal form is presented. New results due to the authors and their Dutch colleagues have been obtained for a Hamiltonian normalized near equilibrium points.

The results of the authors in developing the methods of asymptotic analysis of nonlinear dynamical systems with wide use of group–theoretical methods were summarized in monograph [31]. The papers of Mitropolsky Yu. A. [32] and Lopatin A. K. [33 – 35] deal with the same problems.

The present series of three papers treats the technique of normalization which was called by the authors the asymptotic decomposition in the context of averaging by N. N. Bogolyubov and realizations of this technique in different Hilbert spaces. The space of homogeneous polynomials and the space of representation of rotation group on the plane are considered. The general ideas of approach are illustrated on well-known models. The essential aim of such a study is to compare the new and existing approaches to the problem.

2. General scheme of the algorithm of asymptotic decomposition.

2.1. *Formulation of the problem.* In the book of Mitropolsky Yu. A. and Lopatin A. K. [31], a new method for investigating systems of differential equations with a small parameter have been developed. It was a further development of N. N. Bogolyubov's averaging method called by the authors the method of asymptotic decomposition. The idea of the new approach originates from N. N. Bogolyubov's averaging method but its realization needed the use of essentially new apparatus — the theory of continuous transformation groups. Also the Campbell – Hausdorff formula was used.

Let us explain the idea of the new approach. As is known, the starting point of investigation by the averaging method is a system in a standard form

$$\frac{dx}{dt} = \varepsilon X(x, t, \varepsilon), \tag{1}$$

where $x = \text{col} \| x_1, x_2, \dots, x_n \|$, $X(x, t, \varepsilon)$ is an n-dimensional vector.

After the averaging*

$$X_{01}(\xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(\xi, t) dt$$

and a special change of variables, system (1) is reduced to the averaged system

$$\frac{d\bar{x}}{dt} = \varepsilon X_0^{(1)}(\bar{x}) + \varepsilon^2 X_0^{(2)}(\bar{x}) + \dots , \tag{2}$$

which does not explicitly contain the variable t. Let us rewrite the initial system (1) in the equivalent form

$$\frac{dx}{dt} = \varepsilon X(x, y, \varepsilon), \quad \frac{dy}{dt} = 1 \tag{3}$$

and the averaged system (2) correspondingly in the form

$$\frac{d\bar{x}}{dt} = \varepsilon X_0(\bar{x}), \quad \frac{d\bar{y}}{dt} = 1, \tag{4}$$

where $X_0(\bar{x}) = X_0^{(1)}(\bar{x}) + \varepsilon X_0^{(2)}(\bar{x}) + \dots$. Integration of system (4) is simpler than that of system (3), since variables are separated: the system for slow variables $\bar{x}$ does not contain the fast variable $\bar{y}$ and is integrated independently.

Everything stated above allows us to interpret the averaging method in the following way: *The averaging method transforms system* (3) *with nonseparated variables into system* (4) *with separated fast and slow variables.*

The described property of separation of variables with the help of the averaging method has group–theoretical characteristics. Really, let us put $\varepsilon = 0$ in systems (3) and (4) and write initial unperturbed systems (systems of zero approximation) in the form

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 1 \tag{5}$$

and correspondingly

$$\frac{d\bar{x}}{dt} = 0, \quad \frac{d\bar{y}}{dt} = 1. \tag{6}$$

Systems (5) and (6) coincide to within the notation. Let the vectors X and X_0 in systems (3) and (4) have components

$$X = \text{col} \| X_1, \dots, X_n \|, \quad X_0 = \text{col} \| X_{10}, \dots, X_{n0} \|.$$

Put the first-order partial linear differential operator in accordance with system (3):

$$W_0 = W + \varepsilon \tilde{W} , \tag{7}$$

where

$$W = \frac{\partial}{\partial y}, \quad \tilde{W} = X_1 \frac{\partial}{\partial x_1} + \dots + X_n \frac{\partial}{\partial x_n},$$

and correspondingly the operator

$$U_0 = U + \varepsilon \tilde{U} , \tag{8}$$

where

$$U = \frac{\partial}{\partial \bar{y}}, \quad \tilde{U} = X_{01} \frac{\partial}{\partial \bar{x}_1} + \dots + X_{0n} \frac{\partial}{\partial \bar{x}_n}.$$

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* To ensure the existence of the average we impose special conditions on the functions $X_j(x, t, \varepsilon)$, $j = \overline{1, n}$. We omit the explicit form of these conditions.

in accordance with system (4). Operators (7) and (8) are called associated with systems (3) and (4), respectively. If we put $\varepsilon = 0$ in formulas (7) and (8), then operator (7) turns into the operator

$$W_0' = W \equiv \frac{\partial}{\partial y} \tag{9}$$

associated with the system of zero approximation (5) and operator (8) turns into the operator

$$U_0' \equiv U \equiv \frac{\partial}{\partial y}, \tag{10}$$

associated with the system of zero approximation (6). It is easy to show that the Poisson bracket of the operators U and $\tilde{U}$ is identically equal to zero,

$$[U, \tilde{U}] = U\tilde{U} - \tilde{U}U \equiv 0. \tag{11}$$

Let us consider a one parameter transformation group determined by the operator U and given by the Lie series

$$\bar{x}_1 = e^{sU(\bar{x}_0, \bar{y}_0)} \bar{x}_{10},$$

$$\dotfill$$

$$\bar{x}_n = e^{sU(\bar{x}_0, \bar{y}_0)} \bar{x}_{n0}, \tag{12}$$

$$\bar{y} = e^{sU(\bar{x}_0, \bar{y}_0)} \bar{y}_0,$$

where $\bar{x}_{10}, \dots, \bar{x}_{n0}, \bar{y}_0$ are new variables; $U(\bar{x}_0, \bar{y}_0) = \partial/\partial\bar{y}_0$; s is a parameter characterizing the group. It is known from the theory of continuous transformation groups that identity (11) means that the system of differential equations (4) is invariant under group (12), i.e., after the change of variables (12), it turns into the system

$$\frac{d\bar{x}_0}{dt} = \varepsilon X_0(\bar{x}_0), \quad \frac{d\bar{y}_0}{dt} = 1$$

which coincides with the original system (4) to within the notation.

In the considered case, the invariance of system (4) with respect to transformations (12) can be easily established by the immediate check, since relations (12) are defined in the finite form by the formulas

$$\bar{x}_1 = \bar{x}_{10}, \dots, \bar{x}_n = \bar{x}_{n0}, \bar{y} = \bar{y}_0 + s.$$

At the same time, by the immediate check one can easily ascertain that, in general case, the identity similar to (11) does not hold for the operators W, $\tilde{W}$ of the perturbed system: $[W, \tilde{W}] = \tilde{W}W - W\tilde{W} \not\equiv 0$. This implies that system (3) is not invariant with respect to the one-parameter group

$$x_1 = e^{sW(x_0, y_0)} x_{10},$$

$$\dotfill$$

$$x_n = e^{sW(x_0, y_0)} x_{n0}, \tag{13}$$

$$y = e^{sW(x_0, y_0)} y_0,$$

where $x_{10}, \dots, x_{n0}, y_0$ are new variables; $W(x_0, y_0) = \partial/\partial y_0$; s is a parameter characterizing the group generated by the operator W, which is associated with the system of zero approximation.

Indeed, relations (13) can be easily represented in the finite form

$$x_1 = x_{10}, \ \dots \ , \ x_n = x_{n0}, \quad y = y_0 + s. \tag{14}$$

Under the action of transformation (14), system (3) turns into the system

$$\frac{dx_0}{dt} = \varepsilon X(x_0, y_0 + s, \varepsilon), \quad \frac{dy_0}{dt} = 1$$

which does not coincide with the original system to within the notation.

The above considerations allow us to give the following group–theoretical interpretation of the averaging method: *the averaging method transforms system* (3), *which is not invariant with respect to the one-parameter transformation group generated by the operator* **W** (9), *associated with the system of zero approximation* (5), *into averaged system* (7), *which is invariant with respect to the one-parameter transformation group generated by the operator* **U** (10) *associated with the system of zero approximation* (6).

2.2. *Main algorithm.* Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x), \quad x(t_0) = x_0, \tag{15}$$

where $x = \text{col} \| x_1, x_2, \dots , x_n \|$, $\omega = \text{col} \| \omega_1(x), \dots , \omega_n(x) \|$, $\omega_i(x) \in \mathcal{D}(G)$, $i = \overline{1,n}$, $G_0 = I \times G \in R^{n+1}$, $G \in R^n$, $t \in I$, is the domain of existence and uniqueness of the solution of Cauchy problem of the system (15); $\mathcal{D}(G)$ is the manifold of analytic functions defined in G.

Let $\mathcal{D}^1(G)$ denote the set of linear differential operators in partial derivatives of the first order (further, they are called the operators) with the coefficients from $\mathcal{D}(G)$.

Put the structural properties characterized by some invariance group as a basis for studying system (15) acted by small perturbations. System (15) is invariant under the local one-parameter transformations Lie group, which has the form of the series

$$x = \exp(\mu Z(\bar{x})) \bar{x} \tag{16}$$

(μ is a parameter characterizing the group; Z is some operator of $\mathcal{D}^1(G)$), if, under the action of this transformation, this system turns into the system

$$\frac{d\bar{x}}{dt} = \omega(\bar{x})$$

coinciding with the initial system to within the notation. As is known, for system (15) to be invariant under the transformation of the form (16), it is necessary and sufficient that the operator Z be a solution of the equation (see [31, p. 35])

$$[U, S] = 0, \tag{17}$$

where $U = \omega_1(x)\partial/\partial x_1 + \dots + \omega_n(x)\partial/\partial x_n$ is the linear differential operator associated with system (15) and $[\ ,\ ]$ is the Poisson bracket.

If Z_1, Z_2 are some solutions of equation (17), then the Poisson bracket $[Z_1, Z_2]$ is also a solution as the Jacobian identity implies

$$[U, [Z_1, Z_2]] + [Z_2, [U, Z_1]] + [Z_1, [Z_2, U]] \equiv 0.$$

The set of solutions of equation (17) yields a Lie algebra $\mathcal{B}_0$, which characterizes completely the initial system (15). Note that the algebra $\mathcal{B}_0$ is not empty since it contains the element $S = U$. It is clear that transformation (16) preserves the invariance of system (15) for any element $Z \in \mathcal{B}_0$.

The totality of transformations (16), where $Z \in \mathcal{B}_0$, generates a group $G(\mathcal{B}_0)$. Further, it will be sufficient to consider only the algebra $\mathcal{B}_0$ generating this group. The Lie algebra $\mathcal{B}_0$ will be called the algebra of the centralizer of the element U (further, this item will be considered in detail).

Under small perturbations $\varepsilon\tilde{\omega}(x')$, system (15), which will be called the system of zero approximation, is transformed into:

$$\frac{dx'}{dt} = \omega(x') + \varepsilon\tilde{\omega}(x'), \quad x'(t) = x_0, \tag{18}$$

where $\tilde{\omega}(x') = \mathrm{col} \, \| \, \tilde{\omega}_1(x'), \dots, \tilde{\omega}_n(x') \, \|$, $\tilde{\omega}_i(x') \in \mathcal{D}(G)$, $i = \overline{1, n}$, ε is a small positive parameter. Denote by $G_{0\varepsilon} = J \times J_\varepsilon \times G \in R^{n+2}$, $J_\varepsilon = [0, 1]$, the domain of existence and uniqueness of a solution of the Cauchy problem for system (18), which will be called the perturbed system. Following the general idea described in the introduction, we compare system (18) with some standard system. For this purpose, we make the change of variables as a Lie transformation in system (18)

$$x'_j = \exp(\varepsilon S) x_j, \quad j = \overline{1, n}, \tag{19}$$

where

$$\exp(\varepsilon S) = 1 + \frac{\varepsilon}{1!} S + \frac{\varepsilon^2}{2!} S + \dots,$$

$$S = S_1 + \varepsilon S_2 + \dots; \quad S_i = \gamma_{i1}(x) \frac{\partial}{\partial x_1} + \dots + \gamma_{in}(x) \frac{\partial}{\partial x_n}, \tag{20}$$

$$S_i \in \mathcal{D}^1(G), \quad i = \overline{1, n}.$$

It is easy to write the transformation $x_j = \exp(-\varepsilon S') x'_j$, which is the inverse to (19). Using a close connection of system (18) and the associated differential operator

$$U'_0 = U' + \varepsilon\tilde{U}', \tag{21}$$

where

$$\tilde{U}' = \tilde{\omega}_1(x') \frac{\partial}{\partial x'_1} + \dots + \tilde{\omega}_n(x') \frac{\partial}{\partial x'_n},$$

we subject this operator to transformation (19) and then consider the transformed system of ordinary differential equations after the application of transformation (19). For simplicity, we use the following notation for the functions $f(x') \equiv f'$ and $f(x) = f$. The form of the operator U'_0 is given by the Campbell–Hausdorff formula

$$U'_0 \rightarrow U_0 - \frac{\varepsilon}{1!} [U_0, S] + \frac{\varepsilon^2}{2!} [[U_0, S], S] - \frac{\varepsilon^3}{3!} [[[U_0, S], S], S] + \dots.$$

Substituting the values of the operators S and U_0, given by relations (20) and (21), after simple calculation, we obtain for the operator U'_0 in new variables

$$U'_0 \rightarrow U_0 = U + \varepsilon(-[U, S_1] + F_1) + \dots + \varepsilon^v(-[U, S_v] + F_v) + \dots, \tag{22}$$

where $F_1 = U$; $F_2 = -[\tilde{U}, S_1] + \frac{1}{2}[[U, S_1], S_1] + \dots$, F_v are known functions of the operators $U, \tilde{U}, S_1, \dots, S_{v-1}$. The explicit form of these functions can be obtained after some calculations.

The form of the transformed operator U_0 and, therefore, the form of the

corresponding system of differential equations depends on the way of choosing the sequences of the operators

$$S_1, S_2, \dots, \tag{23}$$

which have not been defined yet. To find these operators we form the sequence of the operator equations

$$[U, S_j] = F_j, \quad j = 1, 2, \dots. \tag{24}$$

The formed infinite sequence has the recursive character, as it follows from the structure of the right-hand sides $F_1, F_2, \dots$. After solving the first equations of the system, we define the right-hand side F_2 of the second system and so on. Since the equations of system (24) have the same homogeneous part, for studying the problem of solvability of (24), it suffices to consider one equation

$$[U, S] = F, \tag{25}$$

which will be called the equation-representative of system (24). Generally speaking, an arbitrary operator $F \in \mathcal{D}^1(G)$ can be in the right-hand side of the equation (25). The solvability of the operator equation (25) is determined by the structure of the solution of homogeneous equation (17), which, in turn, gives the algebra of the centralizer $\mathcal{B}_0$.

The inhomogeneous equation (25) must have solutions with definite analytic properties. For example, it should not contain secular terms on the trajectories of the system of zero approximation, preserve the stationary point, and so on. This is possible only if the right-hand side of equation (25) does not contain elements of $\mathcal{B}_0$.

Necessary properties of the solution of the operator equations can be realized after replacing system (24) by the system

$$[U, S_j] = F_j - \text{pr} F_j, \quad j = 1, 2, \dots, \tag{26}$$

where $\text{pr} F_j$ denotes the projection of the operator F_j to the algebra $\mathcal{B}_0$ (the exact definition of this notion is given further). Let the sequence of operators (23) be determined from the system of equations (26). Then the transformed operator (22) U_0 will take the form

$$U_0 = U + \varepsilon N_1 + \varepsilon^\nu N_\nu + \dots, \tag{27}$$

where the notation

$$N_\nu \equiv \text{pr} F_\nu \equiv \sum_{j=1}^{n} b_{\nu j 0} \frac{\partial}{\partial x_j}, \quad \nu = 1, 2, \dots, \tag{28}$$

is introduced. We restore the transformed system

$$\frac{dx_j}{dt} = \omega_j(x) + \varepsilon N(x) x_j, \quad j = \overline{1, n}, \tag{29}$$

where $N(x) = N_1(x) + \dots + \varepsilon^{\nu-1} N_\nu(x) + \dots$ by using operator (27). Taking into account the structure of operator (28), this system can be written by using the known coefficients:

$$\frac{dx_j}{dt} = \omega_j(x) + \sum_{j=1}^{\infty} \varepsilon^\nu b_{\nu j 0}(x), \quad j = \overline{1, n}. \tag{30}$$

2.3. *Equivalence between the construction of the standard system and the problem of finding the symmetry algebra.* The algebra of the centralizer $\mathcal{B}_0$ is the generating

Lie algebra of the obtained system (29) (or (30)). To underline the connection between system (29) and the algebra of the centralizer $\mathcal{B}_0$, we shall call system (29) the *centralized system*. The operator form (29) of the centralized system would be preferable in our theoretical reasoning.

Thus, the centralized system (29) is the standard system for the initial perturbed system (18). This system is obtained from the perturbed system with the help of a transformation similar to series (20), where the operators $S_1, S_2, \ldots$ are solutions of the system of equations (26).

The centralized system (29) possesses the following properties: its zero approximation coincides with the system of zero approximation (15) and it is invariant with respect to the one-parameter transformation group:

$$x = \exp(\mu U(\bar{x}))\,\bar{x}, \tag{31}$$

where $\bar{x} = \mathrm{col} \| \bar{x}_1, \ldots, \bar{x}_n \|$ and $\dot{U}$ is the operator associated with system (15) of the zero approximation.

The described algorithm of passing from the perturbed system (18) to the centralized system (29) will be called the algorithm of *asymptotic decomposition*.

The invariance of the centralized system under one-parameter group (31) can be taken as its definition. Then the obtained result can be formulated as follows:

The algorithm of asymptotic decomposition puts into correspondence to a perturbed system (18) a centralized system (29);

and only the zero approximation of the perturbed system is invariant with respect to (31).

The integration of the centralized system (29) is simpler than that of the initial perturbed system (18). Now we introduce more precise definitions and auxiliary notions, which were omitted in the previous discussion of the problem.

We have already mentioned the defining role of solutions of homogeneous equation (17) in the construction of the algorithm of asymptotic decomposition. We shall find these solutions in the generating Lie algebra $\tilde{\mathcal{B}}$ of the perturbed system. This algebra is generated by the operators U, $\tilde{U}$ and it contains the elements U, $\tilde{U}$, $[U, \tilde{U}]$, $[U, [U, \tilde{U}]] \ldots$, obtained by calculations of the Poisson bracket of these operators. It is evident that $\mathcal{B}_0 \subset \tilde{\mathcal{B}} \subseteq \mathcal{D}^1(G)$. Together with equation (17), we can consider the equation

$$[U, [U, Y]] = 0 \tag{32}$$

in the algebra $\tilde{\mathcal{B}}$. It is easy to show that all solutions of equation (32) of $\tilde{\mathcal{B}}$ also give the Lie algebra $\mathcal{B}^{(1)}$ and $\mathcal{B}_0 \subseteq \mathcal{B}^{(1)}$. The algebra $\mathcal{B}^{(1)}$ will be called the algebra of the centralizer of the second degree (by the number of the Poisson bracket in equation (32)). By induction, we can define the algebra of the centralizer of an arbitrary degree k. Further, we restrict ourselves (if it is not mentioned otherwise) to a study of the algebras of the centralizer of the first degree, i.e., the identity $\mathcal{B}^{(1)} \equiv \mathcal{B}_0$ should hold. This case has been of the most practical significance. If we considered only the general situation, we would not have obtained important results but only complicated our calculations.

With $\mathcal{B}_0$, there is another important subalgebra $\mathcal{B}_n \subseteq \tilde{\mathcal{B}}$ admitted by the operator U. Its elements satisfy the relation $[U, \mathcal{B}_{..}] = \mathcal{B}_n$.

Define a linear mapping L_U acting on the algebra $\tilde{\mathcal{B}}$ by the formula $L_U X = [U, X]$, $X \in \tilde{\mathcal{B}}$. Thus, the algebra of the centralizer $\mathcal{B}_0$ is the kernel of the mapping L_U and the algebra $\mathcal{B}_n$ is the image of the mapping L_U.

Returning to the inhomogeneous equation (25), we should note that the choice of

If $a_{ij} = \partial \omega_i / \partial x_j$, $i, j = \overline{1, n}$, then system (35) can be represented as follows:

$$U \gamma_j = a_{j1}(x) \gamma_1 + \ldots + a_{jn}(x) \gamma_n + b_j(x), \quad j = \overline{1, n}.$$

Now consider the matrix $\mathcal{A} = \| a_{ij} \|$, $i, j = \overline{1, n}$, the vectors

$$\gamma = \mathrm{col} \| \gamma_1, \ldots, \gamma_n \|, \quad b(x) = \mathrm{col} \| b_1(x), \ldots, b_n(x) \|,$$

and the operator $W = U \otimes \mathcal{E} - \mathcal{A}$, where

$$U \otimes \mathcal{E} = \mathrm{diag} \underbrace{\| U, \ldots, U \|}_{n},$$

$\mathcal{E}$ is the unit $n \times n$ matrix. Then system (25) can be represented in the compact form $W \gamma = b$. The system

$$U \gamma_1 = \tilde{b}_1(x), \quad \ldots, \quad U \gamma_n = \tilde{b}_n(x) \tag{36}$$

is a particular form of system (35). The system of the form (35) or (36) is called a Jacobian system. Together with the inhomogeneous system (35), we consider the homogeneous system $W \gamma = 0$. It is evident that we obtain the homogeneous equation in the described way by solving the homogeneous operator equation (17). It is easy to see the converse statement is valid too.

2.5. *Various approaches to solution of the Jacobian system give rise to various methods in nonlinear mechanics.* By choosing a special basis, we can simplify the integration of the operator equation (25) and that of the Jacobian system (35) corresponding to this equation. The problem of reducing the system of equations (35) to the system of equations (36) is considered there too.

Let there exist n linearly unconnected solutions $Z_1, \ldots, Z_n$ of the homogeneous operator equation (17) and right-hand sides of equation (25) are expanded in this basis:

$$F = \tilde{b}_1(x) Z_1 + \ldots + \tilde{b}_n(x) Z_n.$$

Now we obtain the solution S of operator equation (25) as the sum

$$S = \gamma_1(x) Z_1 + \ldots + \gamma_n(x) Z_n.$$

Since $[U, Z_j] \equiv 0$, $j = \overline{1, n}$, we obtain the expression for the Poisson bracket $[U, S]$:

$$[U, \gamma_1 Z_1 + \ldots + \gamma_n Z_n] = U \gamma_1 Z_1 + \ldots + U \gamma_n Z_n +$$

$$+ \gamma_1 [U, Z_1] + \ldots + \gamma_n [U, Z_n] \equiv U \gamma_1 Z_1 + \ldots + U \gamma_n Z_n.$$

Finally, we indicate the result:

$$U \gamma_1 Z_1 + \ldots + U \gamma_n Z_n \equiv \tilde{b}_1 Z_1 + \ldots + \tilde{b}_n Z_n.$$

By equating the coefficients of the basis operators, we obtain Jacobi equations in the form (36).

We use the approach to the solution of the operator equation (25) that has been just described in the previous section. It leads to an important particular case of implementation of the asymptotic decomposition algorithm. This implementation admits the following group–theoretical interpretation. The assumption that n linearly unconnected solutions of equation (17) are known is equivalent to the assumption that the symmetry algebra and its corresponding group of symmetry for the system of zero approximation are known. The averaging method (KBM method) connected to passing to a standard form and applying the averaging operation also fits in the described above scheme. From this point of view, it corresponds to passing, in the zero approximation,

to the commutative Lie group generated by n linearly unconnected commuting operators. We consider these questions in detail in section 3.

Here, it can be said that the choice of a method for solving operator equation (25) leads to various implementations of the asymptotic decomposition method.

The main idea of the approach that is developed below is using the group–theoretical properties of the zero approximation system. Under various assumptions about a particular group connected with the zero approximation system, different modifications of existing and new methods are obtained. We consider particular implementations of algorithms for the groups $GL(2)$, $SO(2)$.

3. The asymptotic method of separation of variables by Krylov, Bogolyubov, and Mitropolsky (KBM method) and the asymptotic decomposition method.

3.1. *General remarks.* In the present section, we apply the asymptotic decomposition method to a class of systems that traditionally were studied by the asymptotic method developed by N. M. Krylov, N. N. Bogolyubov, and Yu. A. Mitropolsky (see, for example, Bogolyubov N. N., Mitropolsky Yu. A. [4]). This also enables us to establish explicitly a connection between the asymptotic decomposition method and the above mentioned method. We consider purely algorithmic aspects only, casting aside such questions as justification of algorithms, appearance of resonances, and passage of a system through the resonance.

Below, we will show that the asymptotic decomposition method being applied to the same objects as the classical asymptotic method yields the identical results. However, the algorithm of the asymptotic decomposition method is in essence simpler. The following points of simplification can be pointed out. First, logical clearness of the structure of the changes carried out. This is implied by the general logic of the asymptotic decomposition method. Second, actual simplification of computations, since there are no needs to invert equations of the change and carry out the accompanying symbolic computations. Third, the computations needed for an arbitrary fixed approximation can be carried out according to an explicit recursion formula, which is suitable for computer usage.

The principal conclusion that can be made after comparison of two methods in the present section is the following.

In the asymptotic decomposition method, the operation of averaging, which is used in the KBM method, is a certain way to construct the projection $\mathrm{pr}\,F$ *of the operator* F.

In the asymptotic decomposition method, *the centralized system is a direct analog of the averaged system* of the KBM method.

We refer to the operation of averaging used in the asymptotic decomposition method to construct the projection of an operator onto the algebra of the centralizer as the *Bogolyubov projector.*

It is worthy to note that existence of the Bogolyubov projector is not a sufficient condition for coefficients of the change of variables to be bounded, and ensures only that the order of their growth does not exceed t. To ensure boundedness of these coefficients it is necessary to impose additional restrictions on the properties of coefficients of the system.

3.2. *Systems of the standard form. The Bogolyubov projector.* N. N. Bogolyubov and his disciples studied systems of the form

$$\dot{x}' = \varepsilon X(x', t, \varepsilon), \tag{37}$$

where $x' = \mathrm{col}\,\| x_1', \dots, x_n' \|$, $X' = \mathrm{col}\,\| X_1', \dots, X_n' \|$.

Systems of form (37) are referred to as systems *of the standard* form. Let us introduce an additional variable s' into system (37)

$$\dot{x}' = \varepsilon X(x', s', \varepsilon),$$

$$\dot{s}' = 1 \tag{38}$$

and apply to it the asymptotic decomposition method.

Let us write down the operator U'_0 associated to the system

$$U'_0 = U' + \varepsilon \tilde{U}',$$

where

$$U' = \frac{\partial}{\partial s'}, \quad \tilde{U} = \sum_{j=1}^{n} X'_j \frac{\partial}{\partial x_j}.$$

Operators $U'_1 = \partial/\partial x'_1, \dots, U'_n = \partial/\partial x'_n$ commute with the operator $U' = \partial/\partial s'$ and form a basis of algebra $\mathcal{B}_0$ of centralizer. This Lie algebra is finite-dimensional and commutative. Hence, the case described at the beginning of subsection 2.5 has been realized.

After transformation (20) operator U'_0 turns into the operator

$$U'_0 \rightarrow U_0 = U + \varepsilon(-[U, S_1] + F_1) + \dots + \varepsilon^{\nu}(-[U, S_{\nu}] + F_{\nu}) + \dots .$$

The coefficients of operators of the transformation

$$S_j = \gamma_{j1}(x, s) \frac{\partial}{\partial x_1} + \dots + \gamma_{jn}(x, s) \frac{\partial}{\partial x_n}$$

are found from the operator equation

$$[U, S_j] = F_j - \mathrm{pr} F_j, \quad j = 1, 2, \dots . \tag{39}$$

Let us write the operator F_j of the right-hand side of (39)

$$F_j = f_{j1}(x, s) \frac{\partial}{\partial x_1} + \dots + f_{jn}(x, s) \frac{\partial}{\partial x_n}. \tag{40}$$

Assume that the average values for the coefficients f_{jk} of operator (40) exist

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f_{jk}(x, s) ds = f^0_{jk}(x) < +\infty, \quad k = \overline{1, n}.$$

Let us introduce a particular notation for the average value of a function $f_{jk}(x, s)$

$$\langle f_{jk}(x, s) \rangle \overset{\text{def}}{=} f^0_{jk}(x).$$

Define the projection of an operator $\mathrm{pr} F_j$ by the following

$$\mathrm{pr} F_j = \langle f_{j1}(x, s) \rangle \frac{\partial}{\partial x_1} + \dots + \langle f_{jn}(x, s) \rangle \frac{\partial}{\partial x_n}. \tag{41}$$

In virtue of independence of average values from the variable s, the following identities hold

$$[U, \mathrm{pr} F_j] \equiv 0.$$

Therefore, $\mathrm{pr} F_j$ actually belongs to the algebra of centralizer $\mathcal{B}_0$. Operator (41) will be referred to as *Bogolyubov projection*.

Now let us turn to the problem of determination of the coefficients γ_{jk}, $k = \overline{1, n}$, of the operator of transformation S_j of equation (39).

By subsection 2.4, it can be easily established that equation (39) is reduced to the system of partial differential equations

$$\frac{\partial \gamma_{j1}}{\partial s} = f_{j1}(x, s) - f_{j1}^0(x),$$

....

$$\frac{\partial \gamma_{jn}}{\partial s} = f_{jn}(x, s) - f_{jn}^0(x).$$

These systems are integrated easily,

$$\gamma_{jk}(x, T) = \int_0^T \left(f_{jk}(x, s) - f_{jk}^0 \right) ds + C_k(x), \quad k = \overline{1, n}.$$

Functions $C_1(x), \dots, C_n(x)$ can be regarded as integration constants and require a particular choice. To simplify the calculations we suppose that all of them are equal to zero identically, i.e., $C_j(x) \equiv 0, \; k = \overline{1, n}$.

The following statement is true.

Theorem 1. *The existence of the Bogolyubov projector ensures that the order of growth of the coefficients $\gamma_{jk}(x, T)$ does not exceed T when $T \to \infty$.*

Proof. Let us find the limit

$$\lim_{T \to \infty} \frac{\gamma_{jk}(x, T)}{T} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f_{jk}(x, s) ds - f_{j1}^0(x) \equiv 0.$$

Consequently, $\gamma_{ik}(x, T) \equiv o(T)$.

It follows from the proved theorem that additional assumptions are needed for coefficients $f_{jk}(x, t)$ to be bounded in t. In problems of nonlinear mechanics, it is supposed that functions $f_{ik}(x, t)$ are periodic in t and that ensures boundedness of coefficients $f_{ik}(x, t)$ in the variable t.

Let us illustrate the above considerations by examples.

Example 1. Consider the standard system of the form

$$\dot{x}_1' = \varepsilon \sin x_1' \left(1 + \frac{1}{s' + 1} \right), \quad \dot{s}' = 1. \tag{42}$$

The right-hand sides of system (42) are bounded in s'. Let us write equation (39) for system (42) in the first approximation

$$[U, S_1] = F_1 - \operatorname{pr} F_1.$$

Here,

$$U = \frac{\partial}{\partial s}, \quad F_1 = \bar{U} = \sin x_1 \left(1 + \frac{1}{s + 1} \right) \frac{\partial}{\partial x_1}.$$

The operator S_1 is searched in the form

$$S_1 = \gamma_1(x_1, s) \frac{\partial}{\partial x_1}.$$

For the coefficient of the operator F_1, its average exists

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin x_1 \left(1 + \frac{1}{s + 1} \right) ds = \lim_{T \to \infty} \frac{\sin x_1 (T + \ln(T + 1))}{T} = \sin x_1,$$

therefore, the Bogolyubov projection

$$\text{pr } F_1 = \sin x_1 \frac{\partial}{\partial x_1}$$

can be defined.

The coefficient $\gamma_1(x, s)$ is determined as follows:

$$\gamma_1(x, T) = \int_0^t \left[\sin x_1 \left(1 + \frac{1}{s+1} \right) - \sin x_1 \right] ds = \sin x_1 \ln(T+1).$$

When $T \to \infty$, the function $\gamma_1(x, T)$ grows slower than T since the identity

$$\lim_{T \to \infty} \frac{\ln(T+1)}{T} = 0$$

holds.

Example 2. Consider the second-order system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1 - \varepsilon y_1^3, \tag{43}$$

which is equivalent to the Duffing equation.

After the change of variables

$$y_1 = \rho' \sin \varphi', \quad \rho' = \sqrt{y_1^2 + y_2^2},$$

$$y_2 = \rho' \cos \varphi, \quad \varphi' = \operatorname{arctg} \frac{y_1}{y_2},$$

system (43) turns into the system of standard form

$$\dot{\rho}' = +\varepsilon (\rho')^2 \left(-\frac{1}{8} \sin 4\varphi' + \frac{1}{4} \sin 2\varphi' \right),$$

$$\dot{\varphi}' = 1 + \varepsilon (\rho')^2 \left(\frac{3}{8} + \frac{1}{8} \cos 4\varphi' - \frac{1}{4} \cos 2\varphi' \right). \tag{44}$$

Write down the operator U_0' associated with system (44) $U_0' = U_1' + \varepsilon \tilde{U}'$, where $U_0' = \partial/\partial \varphi'$, $\tilde{U}' = b_1(\rho', \varphi') \partial/\partial \rho' + b_2(\rho', \varphi') \partial/\partial \varphi'$,

$$b_1(\rho', \varphi') = +(\rho')^2 \left(\frac{1}{8} \sin 4\varphi' - \frac{1}{4} \sin 2\varphi' \right),$$

$$b_2(\rho', \varphi') = (\rho')^2 \left(\frac{3}{8} + \frac{1}{8} \cos 4\varphi' - \frac{1}{2} \cos 2\varphi' \right). \tag{45}$$

Let us apply to system (44) the algorithm of asymptotic decomposition. We restrict ourselves by the first approximation and consider the operator equation

$$[U, S_1] = \tilde{U} - \operatorname{pr} \tilde{U}, \tag{46}$$

where $S_1 = \gamma_1(\rho, \varphi) \partial/\partial \rho + \gamma_2(\rho, \varphi) \partial/\partial \varphi$. Compute the average value of coefficients (45)

$$\langle b_1(\rho, \varphi) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T b_1(\rho, \varphi) \, d\varphi \equiv 0,$$

$$\langle b_2(\rho, \varphi) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T b_2(\rho, \varphi) \, d\varphi = \frac{3}{8} \rho^2.$$

According to the general theory,

$$\mathrm{pr}\,\tilde{U} = \frac{3}{8}\rho^2\frac{\partial}{\partial\varphi}.$$

The operator equation (46) is replaced by the system of differential equations

$$\frac{\partial\gamma_j}{\partial\varphi} = b_j(\rho,\varphi) - \langle b_j(\rho,\varphi)\rangle, \quad j = 1, 2.$$

These systems can be easily integrated in terms of trigonometric functions

$$\gamma_1 = \rho^2\left(\frac{1}{8}\cos 2\varphi - \frac{1}{32}\cos 4\varphi\right),$$

$$\gamma_2 = \rho^2\left(\frac{1}{32}\sin 4\varphi - \frac{1}{4}\sin 2\varphi\right).$$

Therefore, the centralized (averaged) system in the first approximation takes the form

$$\dot{\rho} = 0, \quad \dot{\varphi} = 1 + \varepsilon\frac{3}{8}\rho^2.$$

The relation with the original variables is given by the formulas

$$\rho' = \rho + \varepsilon\gamma_1(\rho,\varphi) + \varepsilon^2 + ..., \quad \varphi' = \varphi + \varepsilon\gamma_2(\rho,\varphi) + \varepsilon^2 + ...,$$

where $\rho = \rho_0$, $\varphi = t(1 + \varepsilon 3/8\,\rho_0^2)$.

The system with rotating phase

$$\dot{x} = \varepsilon X(x, y, \varepsilon), \quad \dot{y} = \omega(x) + \varepsilon Y(x, y, \varepsilon) \tag{47}$$

is more general than the standard system (37). Here, x is an n-dimensional vector and y is a scalar.

3.3. *Systems of nonlinear mechanics with several fast variables.* If we suppose that in system (47) the variable y is a vector, then we have a case of several rotating phases (or several fast variables). The application of the asymptotic decomposition method to such systems brings a series of new features into the algorithm comparatively with the standard system (38). The theory of these systems appears to be highly complicated.

In the present section, for the sake of definiteness, we consider a system with two rotating phases.

The motion of a pendulum, which is under the action of an external force, is described by the equation $\ddot{z} + \omega^2 z = \varepsilon f(z, \dot{z}, t)$. By using the change of variables $z = x'\cos y'$, $\dot{z} = -\omega x'\sin y'$, we can reduce the oscillator equation to the following system

$$\dot{x}' = \varepsilon X(x', y', s'), \quad \dot{y}' = \omega + \varepsilon Y(x', y', s'), \quad s' = 1. \tag{48}$$

The operator U_0' associated with system (48), $U_0' = U' + \varepsilon\tilde{U}'$, where

$$U' = \frac{\partial}{\partial s'} + \omega\frac{\partial}{\partial y'}, \quad \tilde{U}' = X(x', y', s')\frac{\partial}{\partial x'} + Y(x', y', s')\frac{\partial}{\partial y'},$$

after the application of the asymptotic decomposition method (in the first approximation), turns into the operator $U_0 = U + \varepsilon\,\mathrm{pr}\,\tilde{U}$.

The coefficients $\gamma_j(x, y, s)$ of the operator of transformation

$$S_1 = \gamma_1(x, y, s)\frac{\partial}{\partial x} + \gamma_2(x, y, s)\frac{\partial}{\partial y}$$

are obtained as a result of solving the operator equation

$$[U, S_1] = \tilde{U} - \text{pr}\,\tilde{U}. \tag{49}$$

We will assume that the average values of the coefficients $X(x, y, s)$, $Y(x, y, s)$ of the operator $\tilde{U}$ with respect to the variables y and s exist, .

$$\lim_{\substack{T_1 \to \infty \\ T_2 \to \infty}} \frac{1}{T_1} \frac{1}{T_2} \int_0^{T_1} \int_0^{T_2} X(x, y, s)\, dy\, ds = X^0(x) < +\infty,$$

$$\lim_{\substack{T_1 \to \infty \\ T_2 \to \infty}} \frac{1}{T_1} \frac{1}{T_2} \int_0^{T_1} \int_0^{T_2} Y(x, y, s)\, dy\, ds = Y^0(x) < +\infty. \tag{50}$$

Let us introduce the notation $\langle X(x, y, s) \rangle$, $\langle Y(x, y, s) \rangle$ for the average values of the functions $X(x, y, s)$, $Y(x, y, s)$. Therefore, the Bogolyubov projector

$$\text{pr}\,\tilde{U} = \langle X(x, y, s) \rangle \frac{\partial}{\partial x} + \langle Y(x, y, s) \rangle \frac{\partial}{\partial y}$$

can be chosen to be $\text{pr}\,\tilde{U}$. It is clear that $[U, \text{pr}\,\tilde{U}] \equiv 0$ and $\text{pr}\,\tilde{U}$ belongs to the algebra of centralizer $\mathcal{B}_0$.

To find coefficients of the transformation S_1, we pass from the operator equation (49) to the system of partial differential equations

$$\frac{\partial \gamma_1}{\partial s} + \omega \frac{\partial \gamma_1}{\partial y} = X(x, y, s) - X^0(x),$$

$$\frac{\partial \gamma_2}{\partial s} + \omega \frac{\partial \gamma_2}{\partial y} = Y(x, y, s) - Y^0(x),$$

$$\frac{\partial \gamma_3}{\partial s} + \omega \frac{\partial \gamma_3}{\partial s} = 0.$$

Take the coefficient $\gamma_3 = 0$. The equations for γ_1 and γ_2 have identical structure and, therefore, we consider in detail the integration of the first equation of the described system.

The problem of integrating a linear inhomogeneous equation is equivalent to the problem of integrating the homogeneous equation

$$\frac{\partial V}{\partial s} + \omega \frac{\partial V}{\partial y} + (X(x, y, s) - X^0(x)) \frac{\partial V}{\partial \gamma_1} = 0. \tag{51}$$

In its turn, equation (51) can be replaced by the system of ordinary differential equations

$$\dot{s} = 1,$$

$$\dot{y} = \omega,$$

$$\dot{\gamma}_1 = X(x, y, s) - X^0(x).$$

Finding the coefficient $\gamma_1(x, y, s)$ is reduced to the simple quadratures

$$\gamma_1(x, \omega t, t) = \int_0^t \left(X(x, \omega t, t) - X^0(x) \right) dt.$$

If we make an additional assumption that function $f(z, \dot{z}, t)$ in the original equation of nonlinear oscillator is periodic in t with the period $T = 2\pi / m$, then

functions X, Y in (48) are periodic in y and s with the periods 2π and $2\pi/m$ correspondingly. In virtue of conditions (50), the Fourier series in ωt, t of the function $X(x, \omega t, t) - X^0(x)$ will not contain a free term. As a result, the function $\gamma_1(x, \omega t, t)$ will be limited by t. This problem is treated from other point of view in [36, p. 201]. The second coefficient $\gamma_2(x, y, s)$ of the transformation S_1 is found similarly.

The solution of the original system (48) (in the first approximation) is determined by the formulas

$$x' = x + \varepsilon \gamma_1(x, y, s) + \varepsilon^2 + ... ,$$

$$\cdot y' = y + \varepsilon \gamma_2(x, y, s) + \varepsilon^2 + ...$$

Here, the variables x, y, s are a solution of the centralized (averaged) system of the first approximation

$$\dot{x} = \varepsilon X^0(x),$$

$$\dot{y} = \omega + \varepsilon Y^0(x),$$

$$\dot{s} = 1.$$

The structure of averaged operators depends in large degree on resonance relations between frequencies determined by the variables y and s. We do not touch this problem here.

1. *Krylov N. M., Bogolyubov N. N.* Applications of the methods of nonlinear mechanics to the theory of stationary oscillations. – Kiev: Izd-vo AN USSR, 1934. – 112 p.
2. *Krylov N. M., Bogolyubov N. N.* Introduction to nonlinear mechanics. – Kiev: Izd-vo AN USSR, 1937. – 363 p.
3. *Bogolyubov N. N.* On some statistic methods in mathematical physics. – Kiev: Izd-vo AN USSR, 1945. – 137 p.
4. *Bogolyubov N. N, Mitropolsky Yu. A.* Asymptotic methods in the theory of nonlinear oscillations. – M.: Nauka, 1974. – 504 p.
5. *Bogolyubov N. N., Mitropolsky Yu. A, Samoilenko A. M.* Method of accelerated convergence in nonlinear mechanics. – Kiev: Nauk. Dumka, 1969. – 140 p.
6. *Mitropolsky Yu. A.* Problems of asymptotic theory of nonstationary oscillations. – M.: Nauka, 1964. – 432 p.
7. *Mitropolsky Yu. A.* Averaging method in nonlinear mechanics. – Kiev: Nauk. Dumka, 1971. – 440 p.
8. *Sanders J. A., Verhulst F.* Averaging methods in nonlinear dynamical systems. – New York: Springer – Verlag, 1985. – 248 p.
9. *Mitropolsky Yu. A.* The ideals of Krylov and Bogolyubov in the theory of differential equations and mathematical physics and their development // Ukr. Mat. Zhurn. – 1990. – **42**, № 3. – P. 291 – 302.
10. *Samoilenko A. M.* N. N. Bogolyubov and nonlinear mechanics // Uspekhi mat. nauk. – 1994 (to appear).
11. *Hori G.* Theory of general perturbations with unspecified canonical variables // J. Japan. Astr. Soc. – 1966. – **18**, № 5. – P. 287 – 296.
12. *Hori G.* Lie transformations in nonhamiltonian systems // Lect. notes. Summer Inst. Orbital Mech. – Austin: Univ. Texas, 1970.
13. *Kamel A.* A perturbations methods in the theory of nonlinear oscillations // Celest. Mech. – 1970. – **3**, № 1. – P. 90 – 106.
14. *Giacaglia G. E. O.* Perturbation methods in nonlinear systems. – New York etc.: Springer – Verlag, 1972. – 320 p.
15. *Nayfeh A. H.* Perturbation methods. – New York etc.: John Wiley & Sons, 1973. – 456 p.
16. *Kirchgraber U., Stiefel E.* Methoden der analytischen storungsrechnung und Anwedungen (in German). – Anwedungen etc.: Teubner, 1978. – 294 p.
17. *Kirchgraber U.* On the Lie-series approach to the method of averaging // IX Mezhdunar. Konf. po nelinejn. koleban. – Kiev: Nauk. dumka, 1984. – P. 173 – 178.
18. *Povzner A.* Linear methods in problems of nonlinear differential equations with a small parameter // Int. J. Non-Linear Mech. – 1979. – **9**, № 4. – 279 – 323.

19. *Bogaevsky V. N., Povzner A. Ya.* Linear methods in nonlinear theory of perturbations of differential equations // IX Mezhdunar. Konf. po nelinejnym koleban. – Kiev: Nauk. dumka, 1984. – P. 90 – 93.

20. *Bogaevsky V. N., Povzner A. Ya.* Linear methods in nonlinear problems with a small parameter // Lect. Notes Math. Asympt. Anal. – 1983. – № 985. – P. 441 – 448.

21. *Bogaevsky V. N., Povzner A. Ya.* Algebraic methods in nonlinear perturbation theory. – M.: Nauka, 1987. – 256 p.

22. *Zhuravlev V. F.* Method of Lie series in the problem of separation of motions in nonlinear mechanics // Prikladn. matematika i mekhanika. – 1983. – **47**, № 4. – P. 559 – 565.

23. *Zhuravlev V. F., Klimov D. N.* Applied methods in the oscillations theory. – M.: Nauka, 1988. – 326 p.

24. *Molchanov A. M.* Separation of motions and asymptotic methods in the theory of nonlinear oscillations // Dokl. AN SSSR. – 1961. – **5**. – P. 1030 – 1033.

25. *Brjuno A. D.* Normal forms and methods of averaging // Dokl. AN SSSR. – 1976. – **240**, № 2. – P. 257 – 260.

26. *Brjuno A. D.* Local method of nonlinear analysis of differential equations. – M.: Nauka, 1979. – 254 p.

27. *Mitropolsky Yu. A., Samoilenko A. M.* To the problem on asymptotic decompositions of nonlinear mechanics // Ukr. Mat. Zhurn. – 1979. – **31**, № 1. – P. 42 – 53.

28. *Lomov S. A.* Introduction to general theory of singular perturbations. – M.: Nauka, 1981. – 400 p.

29. *Gubin Yu. P.* Connection of regularization method with averaging method // Vsesojuz. konf. po asimptoticheskim metodam, pt I. – Alma–Ata: Nauka, 1979. – P. 73 – 75.

30. *Lomov S. A., Safonov V. F.* Regularization method for the system with weak nonlinearity in the case of resonance // Mat. zametki. – 1979. – **25**, № 6. – P. 371 – 889.

31. *Mitropolsky Yu. A., Lopatin A. K.* Group theory, approach in asymptotic methods of nonlinear mechanics. – Kiev: Nauk. Dumka, 1988. – 272 p.

32. *Mitropolsky Yu. A.* Development of the averaging method using group transformation of the Lie theory // Proc. of the Intern. Simp. "Functional Differential Equations". – Kyoto, Japan, World Scientific, 1990. – P. 229 – 258.

33. *Lopatin A. K.* Some aspects of the group-theoretical approach in problems of nonlinear mechanics // Ukr. Mat. Zhurn. – 1992. – **44**, № 1. – P. 17 – 22.

34. *Lopatin A. K.* Normal form procedures on groups in nonlinear mechanics. – Preprint, Inst. Mat. AN Ukrainy. – Kiev, 1994 (to appear).

35. *Lopatin A. K.* Averaging, normal forms and symmetry in nonlinear mechanics. – Preprint, Inst. Mat. AN Ukrainy. – Kiev, 1994 (to appear).

36. *Moiseev N. N.* Asymptotic methods of nonlinear mechanics. –M.: Nauka, 1981. – 400 p.