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RESEARCH ARTICLE

On sequences of Mealy automata and their limits

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Abstract. We introduce the notions of n-state Mealy automaton sequence and limit of this sequence. These notions are illustrated by the 2-state Mealy automaton sequences that have the set of finite limit automata.

Introduction

One of the main problem that appears in investigations of the set of discrete objects — how the properties of certain objects are interrelated or how to get the properties of an object through the research of close objects. This problem have been solved by such methods as using operations under these objects that keep the desired properties, selection of special subsets of objects with similar properties, or analysis of object sequences and their limits.

The investigators of Mealy automaton growth deal with countable infinite set of discrete objects. The paper is devoted to consideration the ideas of Mealy automaton sequences and limit automata of these sequences. The limit automaton is constructed over the infinite alphabet, but there can be found automata over finite alphabets with the same properties as the limit automaton. Therefore the using of Mealy automaton sequences have two benefits: it allows to join the automata with similar properties, and to interlink automata-members of a sequence and automata that have the same properties as the limit automaton.

The paper has the following structure. The notions of Mealy automaton sequence and limit are introduced in Section 1. As special case,

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the expanding automaton sequences are considered, and the limits of the growth functions and the semigroups are directly constructed. Section 2 describes two sets of automata of exponential growth. We construct the semigroup, defined by these automata, and calculate their growth functions. These automata are used in Subsection 3.1 for illustrating the properties of introduced in Section 1 notions. Some remarks that concern limit of Mealy automaton sequences are listed in Subsection 3.2.

1. Mealy automaton sequences and limits

1.1. Sequences of Mealy automata

We will use the definitions from [1]. Let us denote a m-symbol alphabet by the symbol X_m, $X_m = \{0, 1, \ldots, m-1\}$, and the infinite alphabet $\{0, 1, \ldots\}$ by X. We denote the set of all infinite to right words over X_m by the symbol X_m^ω. We write a transformation ψ over X_m as

$$(\psi(0), \psi(1), \ldots, \psi(m-1)).$$

Definition 1. *Let us fix the positive integers $n \geq 2$ and $k \geq 2$. We call the sequence $\{A_m, m \geq k\}$ of Mealy automata such that the automaton A_m is n-state Mealy automaton over the alphabet X_m by the n-state Mealy automaton sequence.*

Let us introduce the special type of Mealy automaton sequences, where the next automaton extends the previous one.

Definition 2 ([2]). *Let $A = (X_m, Q, \pi, \lambda)$ be an arbitrary n-state automaton over a m-symbol alphabet. The n-state automaton $A^{(X)} = (X_{m+1}, Q, \pi_1, \lambda_1)$ over the $(m+1)$-symbol alphabet such that the following equalities:*

$$\pi_1(\mathsf{x}, \mathsf{q}) = \pi(\mathsf{x}, \mathsf{q}) \qquad \text{and} \qquad \lambda_1(\mathsf{x}, \mathsf{q}) = \lambda(\mathsf{x}, \mathsf{q})$$

hold for all $\mathsf{x} \in X_m$, $\mathsf{q} \in Q$, is called the extension of A.

The relation of Mealy automaton extension induces the partial order on the set of all n-state Mealy automata, where $A_1 < A_2$ if and only if A_2 is an extension of A_1.

Definition 3. *The n-state Mealy automaton sequence is expanding if the automaton A_{m+1} is an extension of the automaton A_m for all $m \geq k$.*

Let us note that for all $\mathsf{q} \in Q$ and $u \in X_m^\omega$ the equality holds

$$f_{\mathsf{q}, A^{(X)}}(u) = f_{\mathsf{q}, A}(u).$$

Therefore the following proposition holds.

Proposition 1. *Let A be an arbitrary Mealy automaton, and $A^{(X)}$ be an extension of A. Let us denote the semigroups and the growth functions, defined by A and $A^{(X)}$, by the symbols S_A, γ_A and $S_{A^{(X)}}$, $\gamma_{A^{(X)}}$ respectively.*

1. *The semigroup S_A is a factor-semigroup of the semigroup $S_{A^{(X)}}$.*
2. *There are two possible cases: either*

$$\gamma_A(n) = \gamma_{A^{(X)}}(n), \qquad n \geq 1,$$

or there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} \gamma_A(n) &= \gamma_{A^{(X)}}(n), & 1 \leq n < N, \\ \gamma_A(n) &< \gamma_{A^{(X)}}(n), & n \geq N. \end{aligned}$$

Proof. Let some nontrivial relation holds in the semigroup $S_{A^{(X)}}$:

$$f_{q_{i_1}} f_{q_{i_2}} \ldots f_{q_{i_k}} = f_{q_{j_1}} f_{q_{j_2}} \ldots f_{q_{j_l}}, \tag{1.1}$$

where $k, l \geq 1$, $i_p, j_t \in \{0, 1, \ldots, n-1\}$, $1 \leq p \leq k$, $1 \leq t \leq l$. It means that the following equality holds

$$f_{q_{i_1}} f_{q_{i_2}} \ldots f_{q_{i_k}}(u) = f_{q_{j_1}} f_{q_{j_2}} \ldots f_{q_{j_l}}(u) \tag{1.2}$$

for any word $u \in X_{m+1}^{\omega}$. As the equality (1.2) is true for all words $u \in X_m^{\omega}$, then the relation (1.1) holds in S_A. Therefore the set of relations of S_A includes all defining relations of $S_{A^{(X)}}$, and the semigroup S_A is a factor-semigroup of the semigroup $S_{A^{(X)}}$.

It follows from the previous item that there exist the defining relation sets R_A and $R_{A^{(X)}}$ of S_A and $S_{A^{(X)}}$ respectively such that $R_A \supseteq R_{A^{(X)}}$. Here and in the sequel text, the inequality $R_1 \supset R_2$ denotes that each relation of R_2 follows from the relations of R_1. If the equality $R_A = R_{A^{(X)}}$ holds, then S_A and $S_{A^{(X)}}$ are isomorphic semigroups, and the growth functions γ_A and $\gamma_{A^{(X)}}$ coincide for all $n \geq 1$.

Otherwise, let $r : \mathsf{s}_1 = \mathsf{s}_2$ be the relation from $R_A \backslash R_{A^{(X)}}$ with the minimal length of left-side word. Assign $N = \ell(\mathsf{s}_1)$. Then the growth functions γ_A and $\gamma_{A^{(X)}}$ coincide for all $1 \leq n < N$ because the sets R_A and $R_{A^{(X)}}$ include the same subset of relations that can be applied to semigroup words of length less than N. Clearly the inequality $\gamma_A(N) < \gamma_{A^{(X)}}(N)$ holds. Due to this, the inequality $\gamma_A(n) < \gamma_{A^{(X)}}(n)$ holds for all $n > N$.

The Proposition is completely proved. □

1.2. Limit of Mealy automaton sequences

Let us introduce the notion of limit of Mealy automaton sequence. The transition and output functions of A_m are discrete functions, that can be defined by two sets of (nm) values. Therefore the pointwise limits (in discrete Hausdorff metrics) of these functions' values for fixed arguments as m tends to infinity can be considered. Namely

Definition 4. *Let* $\mathfrak{A} = \{A_m = (X_m, Q, \pi_m, \lambda_m)\,,\ m \geq k\}$, $k \geq 2$, *be an arbitrary* n*-state Mealy automaton sequence. The automaton* $A_\infty = (X, Q, \pi, \lambda)$ *is called the limit automaton of the sequence* $\mathfrak{A}$*, if for any state* $q \in Q$ *and any symbol* $x \in X$ *there exists the number* $M \geq k$ *such that the equalities*

$$\pi_m(q, x) = \pi(q, x) \qquad \textit{and} \qquad \lambda_m(q, x) = \lambda(q, x)$$

hold for all $m \geq M$.

It follows from Definition 4 that limit automaton is a unique if it exists. The limit automaton of Mealy automaton sequence have common "limit properties". For example, the pointwise limit of product or sum of automata equals product or sum of the limit automata of these sequences respectively. But the pointwise limit does not preserve some "automaton properties" such as equivalence of states. Let us consider the 2-state automaton sequence $\{A_m,\, m \geq 2\}$ such that A_m have the following automaton transformations:

$$f_0 = (f_0, f_0, \ldots, f_0)\,(0, 1, \ldots, m-2, m-1)\,,$$
$$f_1 = (f_1, f_1, \ldots, f_1)\,(0, 1, \ldots, m-2, 0)\,.$$

Obviously, these transformations are different and A_m has two inequivalent states. The automaton transformations of the limit automaton A_∞ are defined by the equalities

$$f_0 = (f_0, f_0, \ldots, f_0, \ldots)\,(0, 1, \ldots, m-1, \ldots)\,,$$
$$f_1 = (f_1, f_1, \ldots, f_1, \ldots)\,(0, 1, \ldots, m-1, \ldots)\,,$$

whence $f_0 = f_1$ and A_∞ contains one state.

Let $\mathfrak{A}$ be an arbitrary automaton sequence. Each automaton A_m unambiguously defines the automaton transformation semigroup S_{A_m} and the growth function γ_{A_m}. Therefore the automaton sequence $\mathfrak{A}$ defines the sequence of the semigroups $\mathfrak{S} = \{S_{A_m},\, m \geq k\}$ (where the natural set of generators is fixed in each S_{A_m}) and the sequence of the growth functions $\mathfrak{O} = \{\gamma_{A_m},\, m \geq k\}$. Let us define the limit $\gamma_{\mathfrak{A}}$ of the growth function sequence as the pointwise limit of γ_{A_m} as $m \to \infty$, if it exists. Otherwise, we say that the limit of $\mathfrak{O}$ doesn't exist.

Let us define the limit of the semigroup sequence $\mathfrak{S}$, if semigroups compose increasing (S_{A_m} is a factor-semigroup of $S_{A_{m+1}}$) or decreasing ($S_{A_{m+1}}$ is a factor-semigroup of S_{A_m}) sequence. Let R_i be the set of relations of the semigroup S_{A_i}, $i \geq k$. Then the following relations

$$R_k \supset R_{k+1} \supset \ldots \supset R_m \supset \ldots$$

or

$$R_k \subset R_{k+1} \subset \ldots \subset R_m \subset \ldots$$

hold, and the semigroup $S_{\mathfrak{A}}$ is defined as the semigroup with the set of defining relations equals the join or the intersection of semigroups from $\mathfrak{S}$ respectively.

It follows from Definition 4 that the limit automaton is considered over the infinite alphabet. The investigations of automata over the infinite alphabet are more complicated against automata over finite alphabets. Therefore, let us introduce the notion of the finite limit automaton.

Definition 5. *Let $\mathfrak{A} = \{A_m,\, m \geq k\}$, $k \geq 2$, be an arbitrary n-state Mealy automaton sequence. We say that the n-state automaton B over the finite alphabet is the finite limit automaton of the sequence $\mathfrak{A}$, if the equalities $\gamma_B = \gamma_{\mathfrak{A}}$ and $S_B \cong S_{\mathfrak{A}}$ hold.*

Let us note that the finite automaton for certain automaton sequence is not a unique if it exists. There are many unclear aspects, and some remarks are listed in Section 3.

Let's consider the case of expanding automaton sequence. Then there exist the limits of sequences of semigroups and growth functions. Let

$$\mathfrak{A} = \{A_m = (X_m, Q, \pi_m, \lambda_m)\,,\, m \geq k\}\,,$$

where $k \geq 2$, be an arbitrary expanding Mealy automaton sequence. There are two possible cases. Let us assume that there exists the number M such that all growth functions $\{\gamma_{A_m},\, m \geq M\}$ coincide for $n \geq 1$. Then the limit $\gamma_{\mathfrak{A}}$ is the function γ_{A_M}, and each automaton A_m, $m \geq M$, can be considered as the finite limit of $\mathfrak{A}$. Obviously, all semigroups from the set $\{S_{A_m},\, m \geq M\}$ are the same.

Otherwise, it follows from Proposition 1 that there exists a unique infinite sequence

$$k = m_1 < m_2 < m_3 < \ldots$$

such that the automata A_{m_i} and $A_{m_{i+1}}$ have different growth functions, and all automata A_m for $m_i \leq m < m_{i+1}$, have the same growth function,

$i \geq 1$. For each $i \geq 1$ there exists the number $N_{m_i} \in \mathbb{N}$ such that the relations

$$\begin{aligned} \gamma_{A_{m_i}}(n) &= \gamma_{A_{m_{i+1}}}(n), & 1 \leq n < N_{m_i}, \\ \gamma_{A_{m_i}}(n) &< \gamma_{A_{m_{i+1}}}(n), & n \geq N_{m_i}. \end{aligned}$$

hold. It follows from the choice of m_i that the following inequalities

$$1 \leq N_{m_1} < N_{m_2} < N_{m_3} < \dots$$

hold, whence the pointwise limit of the growth functions $\{\gamma_{A_m},\, m \geq k\}$ is defined by the equality

$$\gamma_{\mathfrak{A}}(n) = \begin{cases} \gamma_{A_{m_1}}(n), & \text{if } 1 \leq n < N_{m_1}; \\ \gamma_{A_{m_2}}(n), & \text{if } N_{m_1} \leq n < N_{m_2}; \\ \gamma_{A_{m_3}}(n), & \text{if } N_{m_2} \leq n < N_{m_3}; \\ \dots, & \dots \end{cases}$$

In addition, it follows from Proposition 1 that the defining relations in each semigroup $S_{A_{m_i}}$ can be choose such that

$$R_1 \supset R_2 \supset R_3 \supset \dots,$$

where R_i is the set of defining relations of the semigroup $S_{A_{m_i}}$, $i \geq 1$. Then the defining relation set of the semigroup $S_{\mathfrak{A}}$ is defined by the equality

$$R_{S_{\mathfrak{A}}} = \bigcap_{i \geq 1} R_i.$$

Let us note, that the growth order $[\gamma_{\mathfrak{A}}]$ can be not equal to pointwise limit of growth orders $[\gamma_{A_m}]$ as m tends to ∞. For example, the 2-state Mealy automaton sequence $\{A_m,\, m \geq 3\}$ is considered in [1] such that $[A_m] = \left[n^{m-1}\right]$, but the equality $[\gamma_{\mathfrak{A}}] = [2^n]$ holds.

2. Sets of Mealy automata $\mathrm{A_m}$ and B

2.1. Definitions

Let $m \geq 3$ and $g_i, h_i, y_i \in \{0,1\}$, $i = 2,3,\dots,m-1$, be arbitrary numbers. Consider the following transformations of X_m:

$$\begin{aligned} \alpha &= (0,\dots,0,0), & \alpha_p &= (0,\dots,0,1,2,\dots,m-1-p), p \geq 1; \\ \beta &= (0,\dots,0,m-1), & \beta_p &= (0,\dots,0,1,2,\dots,m-2-p,m-1), p \geq 1. \end{aligned}$$

Clearly the equalities

$$\alpha_1^p = \begin{cases} \alpha_p, & \text{if } 1 \le p \le m-2, \\ \alpha, & \text{if } p \ge m-1; \end{cases} \qquad \beta_1^p = \begin{cases} \beta_p, & \text{if } 1 \le p \le m-3, \\ \beta, & \text{if } p \ge m-2. \end{cases}$$

hold.

Let A_m be an arbitrary 2-state Mealy automaton over X_m such that its automaton transformations f_0 and f_1 are defined by the following equalities:

$$\begin{aligned} f_0 &= \left(f_0, f_1, f_{g_2}, f_{g_3}, \dots, f_{g_{m-1}}\right)(0, 0, 1, 2, \dots, m-3, m-2), \\ f_1 &= \left(f_0, f_1, f_{h_2}, f_{h_3}, \dots, f_{h_{m-1}}\right)(1, 0, y_2, y_3, \dots, y_{m-2}, y_{m-1}). \end{aligned} \tag{2.3}$$

Let us define the set of all automata A_m where g_i, h_i, y_i vary over $\{0,1\}$ by the symbol $\mathbf{A_m}$.

Similarly, let B_m be an arbitrary 2-state Mealy automaton over X_m such that its automaton transformations f_0 and f_1 allow the following decompositions

$$\begin{aligned} f_0 &= \left(f_0, f_1, f_{g_2}, f_{g_3}, \dots, f_{g_{m-1}}\right)(0, 0, 1, 2, \dots, m-3, m-1), \\ f_1 &= \left(f_0, f_1, f_{h_2}, f_{h_3}, \dots, f_{h_{m-1}}\right)(1, 0, y_2, y_3, \dots, y_{m-2}, 0). \end{aligned} \tag{2.4}$$

Let us define the set of all automata B_m where g_i, h_i, y_i vary over $\{0,1\}$ and m varies over $3, 4, \dots$ by the symbol $\mathbf{B}$.

Let Φ_n denote the Fibonacci numbers, defined by $\Phi_n = \Phi_{n-1} + \Phi_{n-2}$, $\Phi_1 = \Phi_2 = 1$. The element Φ_n is defined by the equality [3]:

$$\Phi_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right), \; n \ge 1.$$

For any $n \ge m \ge 0$ the following equality holds

$$\sum_{i=m}^{n} \Phi_i = \Phi_{n+2} - \Phi_{m+1}.$$

Let us note that Φ_n, $n \ge 2$, equals the count of words of length $(n-2)$ over $\{a, b\}$ which don't include subwords bb.

2.2. Properties of $\mathbf{A_m}$ and $\mathbf{B}$

Let us consider the semigroup relations

$$r_p : f_1^2 f_0^p f_1 = f_0 f_1 f_0^p f_1,$$

where $p \ge 1$. The properties of automata from $\mathbf{A_m}$ and $\mathbf{B}$ are described by the following theorems.

Theorem 1. *Let A_m be an arbitrary automaton from $\mathbf{A_m}$.*

1. *A_m defines the automatic transformation semigroup*

$$S_m = \left\langle\ f_0, f_1\ \middle|\ r_p,\ 1 \le p \le m-2;\ f_1^2 f_0^{m-1} = f_0 f_1 f_0^{m-1}\ \right\rangle.$$

2. *The growth function γ_m of A_m is defined by the following equality*

$$\gamma_m(n) = \Phi_{n+4} - \begin{cases} (n+2), & \textit{if } 1 \le n \le m; \\ \Phi_{n+4-m} + (m-1), & \textit{if } n > m. \end{cases}$$

3. *The growth function γ_{S_m} of S_m is defined by the following equality*

$$\gamma_{S_m}(n) = \Phi_{n+6} - \begin{cases} \frac{n^2+5n+16}{2}, & \textit{if } 1 \le n \le m; \\ \Phi_{n+6-m} + \frac{2n(m-1)+7m-m^2}{2}, & \textit{if } n > m. \end{cases}$$

Theorem 2. *Let B be an arbitrary automaton from $\mathbf{B}$.*

1. *The automaton B defines the semigroup*

$$S = \left\langle\ f_0, f_1\ \middle|\ r_p,\ p \ge 1\ \right\rangle.$$

2. *The growth function γ of B is defined by the following equality*

$$\gamma(n) = \Phi_{n+4} - (n+2), n \ge 1.$$

3. *The growth function γ_S of S is defined by the following equality*

$$\gamma_S(n) = \Phi_{n+6} - \frac{n^2+5n+16}{2}, n \ge 1.$$

It follows from Theorems 1 and 2 that the sequences of functions γ_m and γ_{S_m} have the pointwise limits. Namely,

Corollary 1. *The functions γ and γ_S are the pointwise limits of the functions γ_m and γ_{S_m} as m tends to infinity respectively. That is for each $n \ge 1$ the equalities hold:*

$$\gamma_m(n) \xrightarrow[m\to\infty]{} \gamma(n), \qquad \gamma_{S_m}(n) \xrightarrow[m\to\infty]{} \gamma_S(n).$$

2.3. Properties of automaton transformations

Let us fix the numbers $m \geq 3$ and $g_i, h_i, y_i \in \{0,1\}$, $i = 2,3,\ldots,m-1$. Let A_m and B_m be the automata from $\mathbf{A_m}$ and $\mathbf{B}$ respectively, defined by the numbers g_i, h_i, f_i.

Let $[r]$ denotes integral part of rational number r, and $[\![p]\!]$ denotes the parity of nonnegative integer p, $p \geq 0$, $[\![p]\!] \in \{0,1\}$.

Proposition 2. *The relations r_p, $p \geq 1$, hold in the semigroups S_m and S. In addition, in S_m the relation holds*

$$f_1^2 f_0^{m-1} = f_0 f_1 f_0^{m-1}. \tag{2.5}$$

Let us note, that the relations r_p, $p \geq m-1$, follow from the relation (2.5):

$$f_1^2 f_0^p f_1 = f_1^2 f_0^{m-1} \cdot f_0^{p-m+1} f_1 = f_0 f_1 f_0^{m-1} \cdot f_0^{p-m+1} f_1 = f_0 f_1 f_0^p f_1.$$

Proof. Let us consider the automaton A_m. It follows from (2.3) that the following equalities hold:

$$\begin{aligned} f_0^p = \Big(& f_0^p, f_0^{p-1} f_1, f_0^{p-2} f_1 f_{g_2}, \ldots, f_0 f_1 f_{g_2} f_{g_3} \cdots f_{g_{p-1}}, \\ & f_1 f_{g_2} f_{g_3} \cdots f_{g_p}, f_{g_2} f_{g_3} \cdots f_{g_{p+1}}, \ldots, f_{g_{m-p}} f_{g_{m-p+1}} \cdots f_{g_{m-1}} \Big) \alpha_p, \end{aligned} \tag{2.6a}$$

where $0 \leq p \leq m-2$, and

$$\begin{aligned} f_1^2 &= \left(f_1 f_0, f_0 f_1, f_{y_2} f_{h_2}, \ldots, f_{y_{m-1}} f_{h_{m-1}} \right) (0, 1, 1-y_2, \ldots, 1-y_{m-1}), \\ f_0^p f_1 &= \Big(f_0^{p-1} f_1 f_0, f_0^p f_1, f_0^{p-1} f_{y_2} f_{h_2}, f_0^{p-1} f_{y_3} f_{h_3}, \ldots, f_0^{p-1} f_{y_{m-1}} f_{h_{m-1}} \Big) \alpha. \end{aligned} \tag{2.6b}$$

Let us write the unrolled forms of left and right part of r_p, $p \geq 1$:

$$\begin{aligned} f_1^2 f_0^p f_1 = \Big(& f_1 f_0^p f_1 f_0, f_0^{p+1} f_1, f_1 f_0^p f_{y_2} f_{h_2}, \\ & f_1 f_0^p f_{y_3} f_{h_3}, \ldots, f_1 f_0^p f_{y_{m-1}} f_{h_{m-1}} \big) \alpha, \\ f_0 f_1 f_0^p f_1 = \Big(& f_1 f_0^p f_1 f_0, f_0^{p+1} f_1, f_1 f_0^p f_{y_2} f_{h_2}, \\ & f_1 f_0^p f_{y_3} f_{h_3}, \ldots, f_1 f_0^p f_{y_{m-1}} h_{m-1} \big) \alpha, \end{aligned}$$

whence the relations r_p hold in S_m. Let us check the relations (2.5). It follows from (2.6) that

$$\begin{aligned} f_1^2 f_0^{m-1} = \big(& f_1 f_0^m, f_1 f_0^{m-1} f_1, f_1 f_0^{m-2} f_1 f_{g_2}, \ldots, \\ & f_1 f_0^2 f_1 f_{g_2} f_{g_3} \cdots f_{g_{m-2}}, f_1 f_0 f_1 f_{g_2} f_{g_3} \cdots f_{g_{m-1}} \big) \alpha, \\ f_0 f_1 f_0^{m-1} = \big(& f_1 f_0^m, f_1 f_0^{m-1} f_1, f_1 f_0^{m-2} f_1 f_{g_2}, \ldots, \\ & f_1 f_0^2 f_1 f_{g_2} f_{g_3} \cdots f_{g_{m-2}}, f_1 f_0 f_1 f_{g_2} f_{g_3} \cdots f_{g_{m-1}} \big) \alpha, \end{aligned}$$

and therefore (2.5) hold in S_m.

Let us consider B_m. Similarly, it follows from (2.4) that the following equalities hold:

$$f_0^p = \Big(f_0^p, f_0^{p-1} f_1, f_0^{p-2} f_1 f_{g_2}, f_0^{p-3} f_1 f_{g_2} f_{g_3}, \dots, f_0 f_1 f_{g_2} f_{g_3} \dots f_{g_{p-1}},$$
$$f_1 f_{g_2} \dots f_{g_p}, f_{g_2} f_{g_3} \dots f_{g_{p+1}}, \dots, f_{g_{m-p-1}} f_{g_{m-p}} \dots f_{g_{m-2}}, f_{g_{m-1}}^p\Big)\, \beta_p,$$

if $1 \le p \le m-3$, and

$$f_0^p = \Big(f_0^p, f_0^{p-1} f_1, f_0^{p-2} f_1 f_{g_2}, f_0^{p-3} f_1 f_{g_2} f_{g_3}, \dots,$$
$$f_0^{p-m+2} f_1 f_{g_2} f_{g_3} \dots f_{g_{m-2}}, f_{g_{m-1}}^p\Big)\, \beta,$$

otherwise. Hence for any $p \ge 1$ one has

$$f_1^2 = (f_1 f_0, f_0 f_1, f_{y_2} f_{h_2}, f_{y_3} f_{h_3}, \dots,$$
$$f_{y_{m-2}} f_{h_{m-2}}, f_0 f_{h_{m-1}})\,(0, 1, 1-y_2, 1-y_3, \dots, 1-y_{m-2}, 1)\,,$$
$$f_0^p f_1 = \Big(f_0^{p-1} f_1 f_0, f_0^p f_1, f_0^{p-1} f_{y_2} f_{h_2}, \dots, f_0^{p-1} f_{y_{m-2}} f_{h_{m-2}}, f_0^p f_{h_{m-1}}\Big)\, \alpha,$$

and it yields the following unrolled forms

$$f_1^2 f_0^p f_1 = \Big(f_1 f_0^p f_1 f_0, f_0^{p+1} f_1, f_1 f_0^p f_{y_2} f_{h_2}, \dots,$$
$$f_1 f_0^p f_{y_{m-2}} f_{h_{m-2}}, f_1 f_0^{p+1} f_{h_{m-1}}\Big)\, \alpha,$$
$$f_0 f_1 f_0^p f_1 = \Big(f_1 f_0^p f_1 f_0, f_0^{p+1} f_1, f_1 f_0^p f_{y_2} f_{h_2}, \dots,$$
$$f_1 f_0^p f_{y_{m-2}} f_{h_{m-2}}, f_1 f_0^{p+1} f_{h_{m-1}}\Big)\, \alpha.$$

Thus the relations r_p hold in S. Proposition is completely proved. ☐

Proposition 3. *The following equality holds for both A_m and B_m:*

$$f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 f_0^{p_k}(0^*) = 0^{p_1} 1 0^{p_2} 1 \dots 0^{p_{k-2}} 1 0^{p_{k-1}} 1 0^{p_k} \cdot 0^*,$$

where $k \ge 1$, $p_1, p_{k+1} \ge 0$, $p_i > 0$, $1 < i < k$.

Proof. As the restriction of the automata A_m and B_m on a 2-symbol alphabet coincide, then it's enough to consider the case of A_m. It follows from (2.6b) that for any $k \ge 2$ and arbitrary $p_i > 0$, $1 \le i \le k-1$, the following equalities hold:

$$f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 = \Big(f_0^{p_1-1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 f_0,$$
$$f_0^{p_1-1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}+1} f_1, \dots\Big)\,(0, 0, \dots, 0)\,;$$
$$f_1 f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 = \big(f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 f_0,$$
$$f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}+1} f_1, \dots\Big)\,(1, 1, \dots, 1)\,.$$

Using the equality $f_0^p(0^*) = 0^*$, $p > 0$, one has

$$\begin{aligned} &f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 f_0^{p_k}(0^*) = \\ &\quad = f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1(0^*) = 0^{p_1} 1 0^{p_2} 1 \ldots 0^{p_{k-2}} 1 0^{p_{k-1}} 1 0^*, \end{aligned}$$

that was required to prove. ☐

Proposition 4. *Let $s \in S$ be an arbitrary element. It admits a unique presentation as the word*

$$\mathsf{s} = f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}}, \tag{2.7}$$

where $k = 0$, $p_1 > 0$, or $k \geq 2$, $p_1, p_{k+1} \geq 0$, $p_i > 0$, $2 \leq i \leq k$.

Proof. Let s is written as the word

$$\mathsf{s} = f_0^{r_1} f_1^{r_2} f_0^{r_3} f_1^{r_4} \ldots f_0^{r_{2l-1}} f_1^{r_{2l}} f_0^{r_{2l+1}}, \tag{2.8}$$

where $l \geq 0$, $r_1, r_{2l+1} \geq 0$, $r_i > 0$, $2 \leq i \leq 2l$, and $\ell(s) = \sum\limits_{i=1}^{2l+1} r_i > 0$. If s is written in the form (2.7), then Proposition 4 is true.

Otherwise, it follows from r_p that for any $p_1, p_2 \geq 0$ the equalities

$$f_1^{p_1} f_0^{p_2} f_1 = f_1^{p_1-2} f_0 f_1 f_0^{p_2} f_1 = \ldots = f_1^{[\![p_1]\!]} (f_0 f_1)^{\left[\frac{p_1}{2}\right]} f_0^{p_2} f_1$$

hold. Applying the equality at the line above to the word (2.8) from right to left, s can be written as the following word

$$\begin{aligned} \mathsf{s} = f_0^{r_1} f_1^{[\![r_2]\!]} (f_0 f_1)^{\left[\frac{r_2}{2}\right]} f_0^{r_3} f_1^{[\![r_4]\!]} (f_0 f_1)^{\left[\frac{r_4}{2}\right]} \ldots \\ f_0^{r_{2l-3}} f_1^{[\![r_{2l-2}]\!]} (f_0 f_1)^{\left[\frac{r_{2l-2}}{2}\right]} f_0^{r_{2l-1}} f_1 \cdot f_1^{r_{2l}-1} f_0^{r_{2l+1}}, \end{aligned}$$

and it doesn't include subwords f_1^2, may be excepting the last occurrence — the subword $f_1^{r_{2l}}$. The right-side word has the form (2.7), and the proof is completed. ☐

Proposition 5. *An arbitrary element $\mathsf{s} \in S_m$ admits a unique presentation as the word*

$$\mathsf{s} = f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}}, \tag{2.9}$$

where $k = 0$, $p_1 > 0$, or $k \geq 2$, $p_1, p_{k+1} \geq 0$, $p_i > 0$, $2 \leq i \leq k$, and $p_k = 1$ if $p_{k+1} \geq m - 1$.

Proof. It follows from Propositions 2 and 4 that each element $\mathsf{s} \in S_m$ can be written in the form (2.7). In the case $p_{2k+1} \geq m-1$ the relation (2.5) can be applied to s, and the end $f_1^{r_{2l}} f_0^{r_{2l+1}}$ may be replaced by $f_1^{[\![r_{2l}]\!]}(f_0 f_1)^{\left[\frac{r_{2l}}{2}\right]} f_0^{r_{2l+1}}$. After it, the word s has the form (2.9). □

Proposition 6. *The count of words of length n, $n \geq 1$, written in the form* (2.7)*, equals*

$$w(n) = \Phi_{n+4} - (n+2),\ n \geq 1.$$

Proof. Let us separate the words written in the form (2.7) to two types: the words such that $k = 0$ or $p_k = 1$, and the words such that $k \geq 2$ and $p_k > 1$. The count of words of length n of first type is equal to Φ_{n+2}. An arbitrary word s of second type can be written as

$$\mathsf{s} = f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 \cdot f_1^{p_k - 1} f_0^{p_{k+1}}. \tag{2.10}$$

Each word (2.10) unambiguously corresponds to the word of length $(n+1)$:

$$\mathsf{s}' = f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1 \cdot f_0^{p_k - 1} f_1 f_0^{p_{k+1}}.$$

Hence, the count of the words (2.10) of length n equals the count of words of length $(n+1)$ over $\{f_0, f_1\}$ such that they don't include subwords f_1^2 and contain at least two symbols f_1. This count is equal to

$$\Phi_{n+3} - \left(\begin{array}{c} n+1 \\ 0 \end{array} \right) - \left(\begin{array}{c} n+1 \\ 1 \end{array} \right) = \Phi_{n+3} - (n+2).$$

Finally,

$$w(n) = \Phi_{n+2} + \Phi_{n+3} - (n+2) = \Phi_{n+4} - (n+2),$$

for all $n \geq 1$. □

Proposition 7. *The count of words of length n, $n \geq 1$, written in the form* (2.9) *equals*

$$w_m(n) = \Phi_{n+4} - \begin{cases} (n+2), & \text{if } 1 \leq n \leq m; \\ \Phi_{n+4-m} + (m-1), & \text{if } n > m. \end{cases}$$

Proof. For $1 \leq n \leq m$ the value $w_m(n)$ equals $w(n)$, because the relation (2.5) can not be applied. For $n > m$ the value $w_m(n)$ equals $w(n)$ minus the count of words written in the form (2.7) such that $k \geq 2$, $p_{k+1} \geq m-1$ and $p_k > 1$. This count equals the count of words (2.7) of length $(n-(m-1))$ such that $p_k > 1$, that is

$$\Phi_{n-(m-1)+3} - (n-(m-1)+2).$$

Therefore, for $1 \le n \le m$ we have

$$w_m(n) = \Phi_{n+4} - (n+2),$$

and for $n > m$ the following equality holds:

$$w_m(n) = (\Phi_{n+4} - (n+2)) - \big(\Phi_{n-(m-1)+3} - (n-(m-1)+2)\big) = \\ = \Phi_{n+4} - \Phi_{n+4-m} - (m-1). \square$$

2.4. Proofs of Theorems 1 and 2

Proposition 8. *Let* $\mathsf{s}_1, \mathsf{s}_2 \in S_m$ *be arbitrary elements, written in the form* (2.9)*:*

$$\mathsf{s}_1 = f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}},$$
$$\mathsf{s}_2 = f_0^{t_1} f_1 f_0^{t_2} f_1 \dots f_0^{t_{l-2}} f_1 f_0^{t_{l-1}} f_1^{t_l} f_0^{t_{l+1}},$$

They define the same transformation over the set X_m^ω *if and only if they coincide graphically, that is*

$$k = l; p_i = t_i, 1 \le i \le k.$$

Proof. Clearly s_1 and s_2 define the same transformation over X_m^ω, if they coincide graphically. Therefore it's enough to prove that if s_1 and s_2 have different forms (2.9), then they define different transformations over X_m^ω. Let us assume that elements s_1 and s_2 define the same transformation over X_m^ω, but written in the different form (2.9). Then for any $u \in X_m^\omega$ the following equality holds

$$\mathsf{s}_1(u) = \mathsf{s}_2(u). \tag{2.11}$$

It follows from (2.11) that the equality holds

$$f_0 \mathsf{s}_1 f_0 f_1(0^*) = f_0 \mathsf{s}_2 f_0 f_1(0^*). \tag{2.12}$$

Using Proposition 3, let us rewrite the left- and right-side words of the equality at the line above:

$$f_0 \mathsf{s}_1 f_0 f_1(0^*) = f_0^{p_1+1} f_1 f_0^{p_2} \dots f_1 f_0^{p_{k-1}} f_1^{[\![p_k]\!]} (f_0 f_1)^{\left[\frac{p_k}{2}\right]} f_0^{p_{k+1}+1} f_1(0^*) = \\ = 0^{p_1+1} 1 0^{p_2} 1 \dots 0^{p_{k-2}} 1 0^{p_{k-1}} 1^{[\![p_k]\!]} (01)^{\left[\frac{p_k}{2}\right]} 0^{p_{k+1}+1} 1 0^*,$$

$$f_0 \mathsf{s}_2 f_0 f_1(0^*) = f_0^{t_1+1} f_1 f_0^{t_2} \dots f_1 f_0^{t_{l-1}} f_1^{[\![t_l]\!]} (f_0 f_1)^{\left[\frac{t_l}{2}\right]} f_0^{t_{l+1}+1} f_1(0^*) = \\ = 0^{t_1+1} 1 0^{t_2} 1 \dots 0^{t_{l-2}} 1 0^{t_{l-1}} 1^{[\![t_l]\!]} (01)^{\left[\frac{t_l}{2}\right]} 0^{t_{l+1}+1} 1 0^*.$$

As s_1 and s_2 have the different forms (2.9) then the equality (2.12) can be true if and only if one of the element, say s_1, have the form

$$\mathsf{s}_1 = f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}},$$

and the other, s_2, is written as

$$\mathsf{s}_2 = f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{\llbracket p_k \rrbracket} (f_0 f_1)^{r_1} f_0 f_1^{2r_2+1} f_0^{p_{k+1}},$$

where $r_1 \geq 0$, $r_2 \geq 0$, $r_1 + 1 + r_2 = \left[\frac{p_k}{2}\right]$, $p_{k+1} \leq m-2$, $p_k \geq 2$.

It follows from (2.6) and (2.3), that for arbitrary $p_k \geq 1$, $1 \leq p_{k+1} \leq m-2$, and for all $u \in X_m^\omega$ the equalities hold:

$$\begin{aligned} f_0^{p_{k+1}}(p_{k+1} \cdot u) &= 1 \cdot f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}(u), \\ f_1^{p_k} f_0^{p_{k+1}}(p_{k+1} \cdot u) &= (1 - \llbracket p_k \rrbracket) \cdot f_1^{\llbracket p_k \rrbracket} (f_0 f_1)^{\left[\frac{p_k}{2}\right]} f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}(u), \\ f_0 f_1^{p_k} f_0^{p_{k+1}}(p_{k+1} \cdot u) &= 0 \cdot f_0^{\llbracket p_k \rrbracket} f_1 (f_0 f_1)^{\left[\frac{p_k}{2}\right]} f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}(u). \end{aligned}$$

Without restricting the generality, let us assume that $p_{k+1} \geq 1$. Otherwise, we can repeat sequel speculations for the elements $\mathsf{s}_1 f_0$ and $\mathsf{s}_2 f_0$. It follows from the equalities at the line above that

$$f_0 \mathsf{s}_1(p_{k+1} \cdot u) = 0 \cdot \mathsf{s}_1'(u) \qquad \text{and} \qquad f_0 \mathsf{s}_2(p_{k+1} \cdot u) = 0 \cdot \mathsf{s}_2'(u),$$

where

$$\begin{aligned} \mathsf{s}_1' &= f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} \cdot f_0^{\llbracket p_k \rrbracket} f_1 (f_0 f_1)^{\left[\frac{p_k}{2}\right]} f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}, \\ \mathsf{s}_2' &= f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{\llbracket p_k \rrbracket} (f_0 f_1)^{r_1 - 1} f_0 \cdot \\ &\qquad\qquad\qquad f_0 f_1 (f_0 f_1)^{r_2} f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}. \end{aligned}$$

For any word $u \in X_m^\omega$ the equality $\mathsf{s}_1'(u) = \mathsf{s}_2'(u)$ holds due by the assumption (2.11). Therefore the following equality hold:

$$f_0 \mathsf{s}_1' f_0 f_1 (0^*) = f_0 \mathsf{s}_2' f_0 f_1 (0^*).$$

Obviously, right multiplying by $f_0 f_1$ may change only the ends of semigroup words s_i', $i = 1, 2$, and affects on the subword $f_1 f_{g_2} f_{g_3} \dots f_{g_{p_{k+1}+1}}$. Hence it follows from Proposition 3 that the elements s_1' and s_2' define different transformations over X_m^ω. We get the contradiction with the assumption (2.11). □

Proof of Theorem 1. It follows from Proposition 2 that the relations r_p, $p \geq 1$, and (2.5) hold in the semigroup S_m. In Proposition 5 these relations allow to reduce an arbitrary element $s \in S_m$ to the form (2.9).

It is shown in Proposition 8 that two semigroup elements define the same transformation over X_m^ω if and only if they have the same form (2.9). Therefore, the set of relations

$$f_1^2 f_0^{m-1} = f_0 f_1 f_0^{m-1}, \qquad f_1^2 f_0^p f_1 = f_0 f_1 f_0^p f_1,\ 1 \le p \le m-2;$$

is the set of defining relations. In addition, the relations do not depends on the numbers g_i, h_i, y_i, and all automata from the set $\mathbf{A_m}$ define the same semigroup, whence Item 1 of Theorem 1 is true.

Let us calculate the growth functions of A_m and S_m. As the defining relations of S_m does not change the length of semigroup words then the equalities hold

$$\gamma_m(n) = w_m(n), \qquad \gamma_{S_m}(n) = \sum_{i=1}^{n} w_m(i).$$

Using Proposition 7, we have

$$\gamma_m(n) = \Phi_{n+4} - \begin{cases} (n+2), & \text{if } 1 \le n \le m, \\ \Phi_{n+4-m} + (m-1), & \text{if } n > m; \end{cases}$$

and Item 2 holds.

Let $n \le m$. Then

$$\gamma_{S_m}(n) = \sum_{i=1}^{n} (\Phi_{i+4} - (i+2)) = \Phi_{n+6} - \Phi_6 - 2n - \frac{n^2+n}{2} = \\ = \Phi_{n+6} - \frac{n^2+5n+16}{2}.$$

Similarly for $n > m$ the following equalities hold

$$\gamma_{S_m}(n) = \gamma_{S_m}(m) + \sum_{i=m+1}^{n} (\Phi_{i+4} - \Phi_{i+4-m} - (m-1)) = \\ = \Phi_{m+6} - \frac{m^2+5m+16}{2} + \Phi_{n+6} - \Phi_{m+6} - \Phi_{n+6-m} + \Phi_{m+6-m} - \\ - (n-m)(m-1) = \Phi_{n+6} - \Phi_{n+6-m} - \frac{2n(m-1)+7m-m^2}{2},$$

that coincides with the formulae in Item 3. □

Proposition 9. *Let* $\mathsf{s}_1, \mathsf{s}_2 \in S$ *be arbitrary elements, written in the form* (2.7)*:*

$$\mathsf{s}_1 = f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}},$$
$$\mathsf{s}_2 = f_0^{t_1} f_1 f_0^{t_2} f_1 \ldots f_0^{t_{l-2}} f_1 f_0^{t_{l-1}} f_1^{t_l} f_0^{t_{l+1}},$$

They define the same transformation over the set X_m^ω if and only if they coincide graphically, that is

$$k = l; p_i = t_i, 1 \leq i \leq k.$$

Proof. Proposition 9 can be proved in the same way as Proposition 8. It follows from (2.3) and (2.4) that the actions of A_m and B_m differ only at the symbol $(m-1)$. Therefore it's enough to consider the case when the elements s_1 and s_2 are written in the following way:

$$\begin{aligned}
\mathsf{s}_1 &= f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{p_k} f_0^{p_{k+1}},\\
\mathsf{s}_2 &= f_0^{p_1} f_1 f_0^{p_2} f_1 \dots f_0^{p_{k-2}} f_1 f_0^{p_{k-1}} f_1^{\llbracket p_k \rrbracket} (f_0 f_1)^{r_1} f_0 f_1^{2r_2+1} f_0^{p_{k+1}},
\end{aligned}$$

where $r_1 \geq 0$, $r_2 \geq 0$, $r_1 + 1 + r_2 = \left[\frac{p_k}{2}\right]$, $p_k \geq 2$. Using the unrolled forms (2.4) the equalities hold for arbitrary $p_k, p_{k+1} \geq 1$, and any $u \in X_m^\omega$:

$$\begin{aligned}
f_0^{p_{k+1}}((m-1)\cdot u) &= (m-1)\cdot f_{g_{m-1}}^{p_{k+1}}(u),\\
f_1^{p_k} f_0^{p_{k+1}}((m-1)\cdot u) &= (1 - \llbracket p_k \rrbracket)\cdot f_0^{1-\llbracket p_k \rrbracket}(f_1 f_0)^{\left[\frac{p_k-1}{2}\right]} f_{h_{m-1}} f_{g_{m-1}}^{p_{k+1}}(u),\\
f_0 f_1^{p_k} f_0^{p_{k+1}}((m-1)\cdot u) &= 0 \cdot f_1^{1-\llbracket p_k \rrbracket} f_0 (f_1 f_0)^{\left[\frac{p_k-1}{2}\right]} f_{h_{m-1}} f_{g_{m-1}}^{p_{k+1}}(u).
\end{aligned}$$

The sequel speculations are carried out similarly to the case of A_m. □

Proof of Theorem 2. The relations r_p, $p \geq 1$, hold in S, and an arbitrary element is reduced to the form (2.7) by applying these relations in Proposition 4. It is proved in Proposition 9 that two semigroup elements written in different form (2.7) define the different transformations over X_m^ω. Therefore, the presentation of the semigroup, defined by B_m, does not depend on m and the numbers g_i, h_i, y_i, and S have the presentation, stated in Theorem 2.

The count of words of length n, that written in the form (2.7), is calculated in Proposition 6, and this count doesn't depend on m. Using the equalities

$$\gamma(n) = w(n), \qquad \gamma_S(n) = \sum_{i=1}^{n} w(i),$$

and the calculations in the proof of Theorem 1, the growth functions of each automaton from the set $\mathbf{B_m}$ and the semigroup S are defined by the equalities:

$$\gamma(n) = \Phi_{n+4} - (n+2), \qquad \gamma_S(n) = \Phi_{n+6} - \frac{n^2+5n+16}{2},$$

for all $n \geq 1$. □

Proof of Corollary 1. Let us fix $n \geq 1$. It follows from Theorem 1 and 2 that for all $m \geq M = n + 1$ the equalities hold:

$$\gamma_m(n) = \gamma(n), \qquad \gamma_{S_m}(n) = \gamma_S(n).$$

Hence the functions γ and γ_S are the pointwise limits of the sequences $\{\gamma_m,\, m \geq 3\}$ and $\{\gamma_{S_m},\, m \geq 3\}$ as m tends to the infinity respectively. □

3. Conclusions

3.1. Examples of expanding sequences

Let us construct examples of expanding sequences, using the automata from the sets $\mathbf{A_m}$ and $\mathbf{B}$. Let $\{g_2, g_3, \ldots\}$, $\{h_2, h_3, \ldots\}$, and $\{y_2, y_3, \ldots\}$ be the infinite sequences such that $g_i, h_i, y_i \in \{0, 1\}$, $i \geq 2$. Let us consider the 2-state Mealy automaton sequence $\mathfrak{A} = \{A_m,\, m \geq 3\}$ such that A_m belongs to $\mathbf{A_m}$ and its automaton transformations are defined by the numbers $\{g_2, g_3, \ldots, g_{m-1}\}$, $\{h_2, h_3, \ldots, h_{m-1}\}$, and $\{y_2, y_3, \ldots, y_{m-1}\}$. Obviously, the automaton A_{m+1} is an expansion of A_m for all $m \geq 3$. Therefore $\mathfrak{A}$ has the limit automaton A_∞ such that its automaton transformations are defined by the following equalities:

$$\begin{aligned} f_0 &= \left(f_0, f_1, f_{g_2}, f_{g_3}, \ldots, f_{g_{m-1}}, \ldots\right)(0, 0, 1, 2, \ldots, m-2, \ldots), \\ f_1 &= \left(f_0, f_1, f_{h_2}, f_{h_3}, \ldots, f_{h_{m-1}}, \ldots\right)(1, 0, y_2, y_3, \ldots, y_{m-1}, \ldots). \end{aligned}$$

Let us note, that all growth functions $\{\gamma_m,\, m \geq 3\}$ are different, and let $m_i = i + 2$, $i \geq 1$. It follows from Theorem 1 that the equalities hold

$$\begin{aligned} \gamma_{A_m}(m+1) &= \Phi_{m+5} - \Phi_5 - (m-1) = \Phi_{m+5} - (m+4), \\ \gamma_{A_m}(m+1) &= \Phi_{m+5} - (m+1+2) = \Phi_{m+5} - (m+3), \end{aligned}$$

whence $N_{m_i} = m_i$, $i \geq 1$, (in definitions of Section 1) and we have

$$\begin{aligned} \gamma_{A_m}(n) &= \gamma_{A_{m+1}}(n), & 1 \leq n \leq m, \\ \gamma_{A_m}(n) &< \gamma_{A_{m+1}}(n), & n > m. \end{aligned}$$

Therefore, the limit growth function is defined by the equalities

$$\gamma_{\mathfrak{A}}(n) = \begin{cases} \gamma_{A_3}(n), & \text{if } 1 \leq n < 3, \\ \gamma_{A_{n+1}}(n), & \text{if } n \geq 3; \end{cases}$$

whence

$$\gamma_{\mathfrak{A}}(n) = \Phi_{n+4} - (n+2),\ n \geq 1.$$

As the relations r_p for $p \geq m$ follow from the relation (2.5) then it is enough to choose in S_m the set of relations

$$R_i = \left\{ r_p,\, 1 \leq p \leq m-2;\; f_1^2 f_0^{m-1} = f_0 f_1 f_0^{m-1} \right\},$$

and these sets form the sequence

$$R_1 \supset R_2 \supset R_3 \supset \dots$$

The set of defining relations of $S_{\mathfrak{A}}$ is defined by the equality

$$R_{\mathfrak{A}} = \bigcap_{i \geq 1} R_i = \{r_p, p \geq 1\}.$$

It follows from Theorem 2 that any automaton $B \in \mathfrak{B}$ can be considered as the finite limit of the sequence $\mathfrak{A}$. In this case, the finite limit automaton exists.

3.2. Some remarks

The ideas and notions introduced in Section 1 require more attention. There are appear many questions, that can be separated into the following groups:

1. the development of constructing methods of Mealy automaton sequences such that the limit automaton and the finite limit automaton exist;
2. the research of sequences of automatic transformation semigroup, defined by Mealy automaton sequences, and research of correlation between the semigroup defined by the limit automaton and the limit of semigroup sequence;
3. the investigations of interrelation between sequences and finite limit automata, the existence of these automata, and constructing methods of the finite limit automata.

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