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On minimal ω-composition non-$\mathfrak{H}$-formations

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Abstract. Let $\mathfrak{H}$ be some class of groups. A formation $\mathfrak{F}$ is called a minimal τ-closed ω-composition non-$\mathfrak{H}$-formation [1] if $\mathfrak{F} \nsubseteq \mathfrak{H}$ but $\mathfrak{F}_1 \subseteq \mathfrak{H}$ for all proper τ-closed ω-composition subformations $\mathfrak{F}_1$ of $\mathfrak{F}$. In this paper we describe the minimal τ-closed ω-composition non-$\mathfrak{H}$-formations, where $\mathfrak{H}$ is a 2-multiply local formation and τ is a subgroup functor such that for any group G all subgroups from $\tau(G)$ are subnormal in G.

Introduction

Throughout this paper all groups considered are finite. A non-empty set of formations Θ is called a full lattice of formations [2] if the intersection of any set of formations from Θ again belongs to Θ and in Θ there is a formation $\mathfrak{F}$ such that $\mathfrak{H} \subseteq \mathfrak{F}$ for all $\mathfrak{H} \in \Theta$. Formations belonging to Θ are called Θ-formations. Let $\mathfrak{H}$ be some class of groups. Recall that a Θ-formation $\mathfrak{F}$ is called a minimal non-$\mathfrak{H}$-Θ-formation (L.A. Shemetkov [1]) or $\mathfrak{H}_\Theta$-critical formation (A.N. Skiba [3]) if $\mathfrak{F} \nsubseteq \mathfrak{H}$ but $\mathfrak{F}_1 \subseteq \mathfrak{H}$ for all proper Θ-subformations $\mathfrak{F}_1$ of $\mathfrak{F}$.

The minimal non-$\mathfrak{H}$-Θ-formations, where Θ is the set of all saturated formations have been described in work [4]. This result have been applied in research of local formations with given subformations (see for more in details Chapter 4 in [5]). In the book [2] analogue of this result in the class of τ-closed saturated formations have been obtained. In the work [6] the minimal ω-saturated non-$\mathfrak{H}$-formations, where $\mathfrak{H}$ is a 2-multiply local formation have been described. In [7] the minimal ω-saturated

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non-$\mathfrak{H}$-formations, where $\mathfrak{H}$ is an any formation of classical type have been described. In the work [9] the structure of the minimal non-$\mathfrak{H}$-Θ-formations, where Θ is a class of all ω-composition formations has been described.

In this paper we describe the minimal τ-closed ω-composition non-$\mathfrak{H}$-formations, where $\mathfrak{H}$ is a 2-multiply local formation and τ is a subgroup functor such that for any group G all subgroups from $\tau(G)$ are subnormal in G.

1. Preliminaries

We use standard terminology [10], [11]. In addition we shall need some definitions and notations from the work of L.A.Shemetkov and A.N. Skiba [8] and the concept of subgroup functor given by A.N.Skiba [2].

Let $\mathfrak{L}$ be an arbitrary non-empty class of abelian simple groups and $\omega = \pi(\mathfrak{L})$. Every function

$$f : \omega \bigcup \{\omega'\} \longrightarrow \{formations\ of\ groups\}$$

is called an ω-composition satellite.

We use $C^p(G)$ to denote the intersection of all centralizers of abelian chief p-factors of the group G (we write $C^p(G) = G$ if G has no such chief factors). Let $R(G)$ denote the radical of G (i.e. $R(G)$ is the largest normal soluble subgroup of G).

Let $\mathfrak{X}$ be a set of groups. We use $\mathrm{Com}(\mathfrak{X})$ to denote the class of all abelian simple groups A such that $A \simeq H/K$ for some composition factor H/K of some group $G \in \mathfrak{X}$. Also, we write $\mathrm{Com}(G)$ for the set $\mathrm{Com}(\{G\})$.

For an arbitrary ω-composition satellite f we put following [8]

$$CF_\omega(f) = \{G \mid G/(R(G) \cap O_\omega(G)) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \text{ for all } p \in \pi(Com(G)) \cap \omega\}.$$

If the formation $\mathfrak{F}$ is such that $\mathfrak{F} = CF_\omega(f)$ for some ω-composition satellite f, then we say that $\mathfrak{F}$ is an ω-composition formation and f is an ω-composition satellite of that formation [8]. A ω-composition satellite f of a ω-composition formation $\mathfrak{F}$ is called an inner ω-composition satellite of $\mathfrak{F}$ if $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$.

Recall that a Skiba subgroup functor τ [2] associates with every group G a system of its subgroups $\tau(G)$ such that the following conditions hold:

1) $G \in \tau(G)$ for any group G;

2) for any epimorphism $\varphi : A \longrightarrow B$ and for any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

We write $\tau_1 \leq \tau_2$ if and only if $\tau_1(G) \subseteq \tau_2(G)$.

If for all groups H and G, where $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$, then they say that τ is a closed subgroup functor.

Let $\overline{\tau}$ be the intersection of all closed functors τ_i such that $\tau \leq \tau_i$. The functor $\overline{\tau}$ is called the closure of τ.

In this paper we consider the only subgroup functors τ such that for any group G the set $\tau(G)$ consists of some subnormal subgroups of G.

A formation $\mathfrak{F}$ is called τ-closed if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$. A satellite f is called τ-valued if all values of f are τ-closed formations.

We denote by $c_\omega^\tau form(\mathfrak{X})$ the intersection of all τ-closed ω-composition formations containing the set of groups $\mathfrak{X}$. Then $c_\omega^\tau form(\mathfrak{X})$ is called the τ-closed ω-composition formation generated by $\mathfrak{X}$. If $\mathfrak{X} = \{G\}$ for some group G, then instead of $c_\omega^\tau form(G)$ we write $c_\omega^\tau formG$. Formations of this kind are called one-generated τ-closed ω-composition formations.

Let $\{f_i \mid i \in I\}$ be the set of ω-composition satellites. Then $\bigcap_{i\in I} f_i$ is a satellite such that $(\bigcap_{i\in I} f_i)(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \omega \cup \{\omega'\}$.

Now let $\{f_i \mid i \in I\}$ be the set of all ω-composition τ-valued satellites of the formation $\mathfrak{F}$. By Lemma 2 [8], $f = \bigcap_{i\in I} f_i$ is a ω-composition satellites of $\mathfrak{F}$. The satellite f is called the minimal ω-composition τ-valued satellite of $\mathfrak{F}$.

Let f be the minimal ω-composition τ-valued satellite of $\mathfrak{F}$. And let F be a satellite such that

$$F(a) = \begin{cases} \mathfrak{N}_p f(p), & \text{if } a = p \in \omega; \\ \mathfrak{F}, & \text{if } a = \omega'. \end{cases}$$

Then F is a ω-composition satellite of the formation $\mathfrak{F}$ [8] and it is called the canonical ω-composition satellite of $\mathfrak{F}$.

Let f and h be two ω-composition satellites of the formation $\mathfrak{F}$. Then we write $f \leq h$ if for all $a \in \omega \cup \{\omega'\}$ we have $f(a) \subseteq h(a)$.

Lemma 1.1. *[8, 1]. Let G be a group, p be a prime. Assume that $N \trianglelefteq G$ and that for every composition factor H/K of the subgroup N we have $p \neq |H/K|$. Then $C^p(G/N) = C^p(G)/N$.*

Lemma 1.2. *[12, 2]. Let p be a prime, $O_p(G) = 1$ and $T = Z_p \wr G = [K]G$, where K is the base group of T. Then $K = C^p(T)$.*

Lemma 1.3. *[8, 4]. Let $\mathfrak{F} = CF_\omega(f)$ and $p \in \omega$. If $G/O_p(G) \in \mathfrak{F} \cap f(p)$, then $G \in \mathfrak{F}$.*

Lemma 1.4. *[8, 5]. Let $\mathfrak{F}$ be an arbitrary non-empty set of groups and $\mathfrak{X} \subseteq \mathfrak{H}$, where $\mathfrak{H}$ is a τ-closed formation. Let $\mathfrak{F} = c_\omega^\tau \mathrm{from}(\mathfrak{X})$ and $\pi =$*

$\pi(\mathrm{Com}(\mathfrak{X}))$. *Then* $\mathfrak{F}$ *has the minimal* τ*-valued* ω*-composition satellite* f *and* f *has the following values:*

(1) $f(\omega') = \tau\mathrm{form}(G/(O_\omega(G) \cap R(G))|G \in \mathfrak{X})$.

(2) $f(p) = \tau\mathrm{form}(G/C^p(G)|G \in \mathfrak{X})$, *for all* $p \in \pi \cap \omega$.

(3) $f(p) = \varnothing$, *for all* $p \in \omega \setminus \pi$.

(4) If $\mathfrak{F} = CF_\omega(h)$ *and* h *be the* τ*-valued satellite, then*

$$f(p) = \tau\mathrm{form}(A \mid A \in h(p) \cap \mathfrak{F},\ O_p(A) = 1)$$

for all $p \in \pi \cap \omega$ *and*

$$f(\omega') = \tau\mathrm{form}(A \mid A \in h(\omega') \cap \mathfrak{F} \text{ and } R(A) \cap O_\omega(A) = 1).$$

Lemma 1.5. *[8, 6]. Let* f_i *be the minimal* ω*-composition satellite of the formation* $\mathfrak{F}_i$, $i = 1, 2$. *Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $f_1 \leq f_2$.

Lemma 1.6. *[2, 2.1.5]. Let* A *be a monolithic group and* $R \nsubseteq \Phi(A)$ *is the socle of* G. *Then the formation* $\mathfrak{F} = \tau formA$ *is a* τ*-irreducible and* $\mathfrak{M} = \tau form(\mathfrak{X} \cup \{A/R\})$ *is the unique maximal* τ*-closed subformation of* $\mathfrak{F}$, *where* $\mathfrak{X}$ *is the set of all proper* τ*-subgroups of* A.

2. Main results

A formation $\mathfrak{F}$ is called a 2-multiply local if it has a local satellite f such that all non-empty values of f are local formations.

Theorem 2.1. *Let* f *be the minimal* τ*-valued* ω*-composition satellite of the formation* $\mathfrak{F}$ *and let* H *be the canonical* ω*-composition satellite of a 2-multiply local formation* $\mathfrak{H}$. *A formation* $\mathfrak{F}$ *is a minimal* τ*-closed* ω*-composition non-*$\mathfrak{H}$*-formation if and only if* $\mathfrak{F} = c^\tau_\omega formG$ *where* G *is a monolithic* $\overline{\tau}$*-minimal non-*$\mathfrak{H}$*-group and* $R = G^{\mathfrak{H}} = Soc(G)$ *is the socle of* G, *where* $R \nsubseteq \Phi(G)$ *and either* $\pi = \pi(Com(R)) \cap \omega = \varnothing$ *or* $\pi \neq \varnothing$ *and* G *satisfies one of the following conditions:*

1) $G = R$ *is a group of prime order* $p \notin \pi(\mathfrak{H})$;

2) $G = [R]M$, *where* $R = O_p(G) = F_p(G)$ *for same* $p \in \pi$ *and* M *is a monolithic* $\overline{\tau}$*-minimal non-*$H(p)$*-group and* $Q = M^{H(p)} = Soc(M)$ *is the socle of* M, *where* $p \notin \pi(Com(Q))$ *and* $Q \nsubseteq \Phi(M)$.

Proof. Necessity. Let G be a group of minimal order in $\mathfrak{F}\backslash\mathfrak{H}$. Then G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{H}$-group and $R = G^{\mathfrak{H}} \neq 1$ is the socle of G. Let $\mathfrak{F} \neq c^\tau_\omega formG$. Then $c^\tau_\omega formG \subseteq \mathfrak{H}$ and so $G \in \mathfrak{H}$. This contradiction shows that $\mathfrak{F} = c^\tau_\omega formG$. Since by hypothesis $\mathfrak{H}$ is a local formation, then by Theorem 4.3 [11], $\mathfrak{H}$ is a saturated formation and so $R \nsubseteq \Phi(G)$.

Let $\pi = \pi(Com(R)) \cap \omega = \varnothing$. In this case the condition of the theorem is carried out.

Let's consider the case $\pi \neq \varnothing$ and $p \in \pi$.

Assume that $G = C_G(R)$. Let $R \neq G$. Assume that $H(p) = \varnothing$. Consequently $p \notin \pi(\mathfrak{H})$ and so $Z_p \notin \mathfrak{H}$. Then $\mathfrak{N}_p \not\subseteq \mathfrak{H}$. Consequently $\mathfrak{N}_p = \mathfrak{F}$ and G is a p-group. So $G/R \in \mathfrak{H}$ and G/R is a p-group. If $G/R \neq 1$, then $p \in \pi(\mathfrak{H})$. This contradiction shows that $G = R = Z_p$ and $\mathfrak{F} \neq c^{\tau}_{\omega} form G = \mathfrak{N}_p$. Thus G satisfies Condition 1). Assume that $H(p) \neq \varnothing$ and so $1 = G/C_G(R) \in H(p)$. Hence $G \in \mathfrak{H}$. A contradiction.

Let $G \neq C_G(R)$, where R is an abelian p-group. Let's consider the group $T = [R]M$, where $M = G/C_G(R)$. Let $C = C_G(R)$. Then $C = C \cap RM = R(C \cap M)$. Evidently that $(C \cap M)$ is a normal subgroup of G. But G is a monolithic group, then $C \cap M = 1$ and so $R = C_G(R) = O_p(G) = F_p(G)$. It is not difficult to see that $R = C^p(G)$ and $O_p(G/C^p(G)) = O_p(G/O_p(G)) = O_p(M) = 1$ and so by Lemma 1.2, $C^p(T) = R$. Consequently by Lemma 1.3, s$T \in \mathfrak{F}$. Evidently that $|T| \leq |G|$. Now we suppose that $|T| < |G|$. Then $T \in \mathfrak{H}$ and so

$$T/C^p(T) \simeq T/R \simeq M \simeq G/C_G(R) \simeq G/R \in H(p).$$

But $G/O_p(G) \simeq G/R \in \mathfrak{H}$ and so $G \in \mathfrak{H}$ by Lemma 1.3. This contradiction shows that $T \notin \mathfrak{H}$. Consequently $T \in \mathfrak{F} \backslash \mathfrak{H}$. Thus in view of the choice of G we have $|T| = |G|$ and $\mathfrak{F} = c^{\tau}_{\omega} form T$. It is clear that $R = T^{\mathfrak{H}}$. By Lemma 1.4,

$$f(p) = \tau form(T/C^p(T)) = \tau form(T/R) = \tau form(G/C_G(R)) = \tau form(G/R) = \tau form M.$$

Let $M \in H(p)$. Consequently $G \in \mathfrak{N}_p H(p) = H(p)$. A contradiction. Hence $M \notin H(p)$ and so $\tau form M \not\subseteq H(p)$.

Let $\mathfrak{M}$ be a proper τ-closed subformation of $f(p)$. Assume that $\mathfrak{M} \not\subseteq H(p)$ and A be a group of minimal order in $\mathfrak{M} \backslash H(p)$. Since $H(p) = \mathfrak{N}_p H(p)$, then $O_p(A) = 1$. By Lemma 18.8 [5], exists a simple and faithful $F_p[A]$-module P over F_p. Let $F = [P]A$. Then $P = C_F(P) = O_p(F) = C^p(F)$ and so

$$F/O_p(F) \simeq F/P \simeq A \in \mathfrak{M} \subset f(p) \subseteq f(p) \cap \mathfrak{F}.$$

By Lemma 1.3, $F \in \mathfrak{F}$. Hence $c^{\tau}_{\omega} form F \subseteq \mathfrak{F}$. If $c^{\tau}_{\omega} form F = \mathfrak{F}$, then by Lemma 1.4,

$$f(p) = \tau form(F/C^p(F)) = \tau form(F/P) = \tau form(A) \subseteq \mathfrak{M} \subset f(p).$$

This contradiction shows that $c_\omega^\tau form F \subset \mathfrak{F}$. Then $c_\omega^\tau form F \subseteq \mathfrak{H}$ and so $F \in \mathfrak{H}$. Hence $F/C^p(F) \simeq A \in H(p)$. A contradiction. Hence $\mathfrak{M} \subseteq H(p)$. Thus $f(p)$ is a minimal τ-closed non-$H(p)$-formation.

Let M_1 be a group of minimal order in $\tau form M \backslash H(p)$. Then M_1 is a monolithic $\overline{\tau}$-minimal non-$H(p)$-group with the socle $Q = M_1^{H(p)}$ and $\tau form M = \tau form M_1$.

Assume that $Q \subseteq \Phi(M_1)$. Let t be the minimal 1-multiply local satellite of $\mathfrak{H}$. By Theorem 8.3 [5], t is an inner satellite of $\mathfrak{H}$. Therefore $t(p) \subseteq H(p)$. Applying Theorem 8.3 [5] again and Consequence 8.6 [5] we see that $H(p) = \mathfrak{N}_p t(p)$ is a local formation, as it is the product of two local formations $\mathfrak{N}_p$ and $t(p)$ (see Consequence 7.14 [5]). By Theorem 4.3 [11], $\mathfrak{H}$ is a saturated formation. Since $M_1/Q \in H(p)$, then $M_1/\Phi(M_1) \in H(p)$. Consequently $M_1 \in H(p)$. This contradiction shows that $Q \nsubseteq \Phi(M_1)$.

Assume that $p \in \pi(Com(Q))$. Since $M_1/Q \in H(p)$, then $M_1 \in \mathfrak{N}_p H(p) = H(p)$. This contradiction shows that Q is not a p-group. Hence $O_p(M_1) = 1$. Thus there exists a simple and faithful $F_p[M_1]$-module R_1 over F_p. Let $G_1 = [R_1]M_1$. Hence $R_1 = C_{G_1}(R_1) = O_p(G_1) = C^p(G_1) = F_p(G_1)$ is a minimal normal p-subgroup of G_1 and so

$$G_1/O_p(G_1) \simeq G_1/R_1 \simeq M_1 \in \tau form M = f(p) \subseteq f(p) \cap \mathfrak{F}.$$

By Lemma 1.3, $G_1 \in \mathfrak{F}$.

Let $\mathfrak{H}_1 = c_\omega^\tau form G_1$ and h_1 be the minimal τ-valued ω-composition satellite of $\mathfrak{H}_1$. By Lemma 1.4,

$$h_1(p) = \tau form(G_1/C^p(G_1)) = \tau form(G_1/R_1) = \tau form(M_1).$$

If $\mathfrak{H}_1 \subset \mathfrak{F}$, then $\mathfrak{H}_1 \subseteq \mathfrak{H}$. Therefore by Lemma 1.5, $h_1 \leq H$, consequently,

$$M_1 \simeq G_1/R_1 \in H(p).$$

This contradiction shows that $\mathfrak{H}_1 = \mathfrak{F}$. Thus

$$\mathfrak{F} = c_\omega^\tau form G_1 = c_\omega^\tau form G.$$

Now we shall show that G_1 satisfies the hypothesis of the theorem. In fact we have only to prove that $R_1 = G_1^{\mathfrak{H}}$.

Indeed, if $M_1 \in \mathfrak{H}$, then $G_1/R_1 = G_1/C^p(G_1) \simeq M_1 \in H(p)$. This contradiction shows that $G_1 \notin \mathfrak{H}$. Consequently $G_1^{\mathfrak{H}} = R_1$.

Let $M_1 \notin \mathfrak{H}$. Consequently $c_\omega^\tau form M_1 = \mathfrak{F}$. By Lemma 1.4,

$$f(p) = \tau form(M_1/C^p(M_1)) = \tau form(M_1).$$

But Q is not a p-group, so $Q \subseteq C^p(M_1)$. So $\tau form(M_1/C^p(M_1)) \subseteq \tau form(M_1/Q)$. Therefore $\tau form M_1 \subseteq \tau form(M_1/Q)$. By Lemma 1.6, $\mathfrak{M} = \tau form(\mathfrak{X} \cup \{M_1/Q\})$ is the unique maximal τ-closed subformation of $\tau form M_1$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of M_1. Hence $\mathfrak{M} \subset \tau form M_1$. This contradiction shows that $M_1 \notin \mathfrak{H}$. Therefore $R_1 = G_1^{\mathfrak{H}}$.

Sufficiency. Let G be a group from the theorem. It is clear that $\mathfrak{F} \not\subseteq \mathfrak{H}$.

Let $\pi = \varnothing$. In this case $O_\omega(G) \cap R(G) = 1$. By Lemma 1.4,

$$f(\omega') = \tau form(G/(O_\omega \cap R(G)) = \tau form(G).$$

Since $G \notin \mathfrak{H}$, then

$$f(\omega') = \tau form(G) \not\subseteq \mathfrak{H} = H(\omega').$$

By Lemma 1.6, $\tau form(\mathfrak{X} \cup \{G/R\})$ is the unique maximal τ-closed subformation of $f(\omega') = \tau form G$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of the group G. Since by hypothesis, all proper τ-subgroups of G are contained in $\mathfrak{H}$, then

$$\tau form(\mathfrak{X} \cup \{G/R\}) \subseteq \mathfrak{H} = H(\omega').$$

Hence all proper τ-closed subformations of $f(\omega')$ are contained in $H(\omega')$.

So $f(\omega')$ is a minimal τ-closed non-$H(\omega')$-formation.

Let $\mathfrak{M}$ be a proper τ-closed ω-composition subformation of $\mathfrak{F}$ and m be the minimal τ-valued ω-composition satellite of $\mathfrak{M}$. By Lemma 1.5, $m \leqslant f$. We shall show that $m \leqslant H$.

Since

$$f(\omega') = \tau form(G) \not\subseteq m(\omega') = \mathfrak{M},$$

consequently $m(\omega') \subset f(\omega')$. Hence $m(\omega') \subseteq H(\omega')$. Besides, since $G/R \in \mathfrak{H}$, then $G/R/C^q(G/R) \in H(q)$ for all $q \in \omega \cup \pi(Com(G/R))$. By Lemma 1.1, $C^q(G)/R = C^q(G/R)$ for all $q \in \omega$. Consequently $G/C^q(G) \in H(q)$. Hence

$$m(q) \subseteq f(q) = \tau form(G/C^q(G)) \subseteq H(q).$$

Consequently $m \leqslant H$ and so by Lemma 1.5, $\mathfrak{M} \subseteq \mathfrak{H}$. Thus $\mathfrak{F}$ is a minimal τ-closed ω-composition non-$\mathfrak{H}$-formation.

Let $\pi \neq \varnothing$ and $p \in \pi$.

If the group G satisfies Condition 1), then obviously, $\mathfrak{F}$ is a minimal τ-closed ω-composition non-$\mathfrak{H}$-formation.

Let G satisfies Condition 2). By Lemma 1.4,

$$f(p) = \tau form(G/C^p(G)) = \tau form(G/R) = \tau form(M).$$

But M is a monolithic $\overline{\tau}$-minimal non-$H(p)$-group, then $M \notin H(p)$ and so $f(p) \nsubseteq H(p)$.

Let $\mathfrak{X}$ be the set of all proper $\overline{\tau}$-subgroups of M. Therefore $\mathfrak{X} \subseteq H(p)$. But $M/Q = M/M^{H(p)} \in H(p)$. Hence

$$\tau form(\mathfrak{X} \cup \{M/Q\}) \subseteq H(p).$$

By Lemma 1.6, $\tau form(\mathfrak{X} \cup \{M/Q\})$ is the unique maximal τ-closed subformation of $f(p) = \tau form(M)$. Therefore all proper τ-closed subformations of $f(p)$ are contained in $H(p)$.

Consequently $f(p)$ is a minimal τ-closed non-$H(p)$-formation, where $p \in \pi$. We shall show that in this case the formation $\mathfrak{F}$ is a minimal τ-closed ω-composition non-$\mathfrak{H}$-formation.

Let $\mathfrak{M}$ be a proper τ-closed ω-composition subformation of $\mathfrak{F}$ and m be the minimal τ-valued ω-composition satellite of $\mathfrak{M}$. By Lemma 1.5, $m \leqslant f$. We shall show that $m \leqslant H$. Assume that $m(p) = f(p)$. Then $G/C^p(G) = G/R = G/O_p(G) \in m(p)$. Using now Lemma 1.3 we see that $G \in \mathfrak{M}$ and so

$$\mathfrak{F} = c_\omega^\tau form G \subseteq \mathfrak{M} \subset \mathfrak{F}.$$

This contradiction shows that $m(p) \subset f(p)$ and so from above we know that $m(p) \subseteq H(p)$. By Lemma 1.1, $C^q(G)/R = C^q(G/R)$ for all prime $q \neq p$ and $(R(G) \cap O_\omega(G))/R = R(G/R) \cap O_\omega(G/R)$. And since $G/R \in \mathfrak{H}$, then $f(\omega') \subseteq H(\omega')$ and $f(q) \subseteq H(q)$ for all $q \in \omega \backslash \{p\}$. But $m \leqslant f$ and hence $m(p) \subseteq H(p)$ for all $p \in \{\omega'\} \cup \omega$. By Lemma 1.5, $m \leqslant H$. Consequently $\mathfrak{M} \subseteq \mathfrak{H}$. Thus $\mathfrak{F}$ is a minimal τ-closed ω-composition non-$\mathfrak{H}$-formation. □

Remark 1. If in Theorem 2.1 the formation $\mathfrak{H}$ is those, that $\mathfrak{N} \subseteq \mathfrak{H}$, then G cannot be a group of prime order.

Remark 2. If $\mathfrak{H}$ is a τ-closed formation, then every minimal non-$\mathfrak{H}$-group is a $\overline{\tau}$-minimal non-$\mathfrak{H}$-group.

Let's note that in the case when τ is a trivial subgroup functor (i.e. $\tau(G) = G$ for any group G) we obtain the following corollary:

Corollary 1. *Let f be the minimal ω-composition satellite of the formation $\mathfrak{F}$ and let H be the canonical ω-composition satellite of a 2-multiply* *local formation $\mathfrak{H}$. A formation $\mathfrak{F}$ is a minimal ω-composition non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = c_\omega formG$, where G is a monolithic group and $R = G^{\mathfrak{H}} = Soc(G)$ is the socle of G, where $R \nsubseteq \Phi(G)$ and either $\pi = \pi(Com(R)) \cap \omega = \varnothing$ or $\pi \neq \varnothing$ and G satisfies one of the following conditions:*

1) $G = R$ is a group of prime order $p \notin \pi(\mathfrak{H})$;

2) $G = [R]M$, where $R = O_p(G) = F_p(G)$ for same $p \in \pi$ and M is a monolithic group and $Q = M^{H(p)} = Soc(M)$ is the socle of M, where $p \notin \pi(Com(Q))$ and $Q \nsubseteq \Phi(M)$.

In the case when for all groups G the set $\tau(G)$ is the set of all subnormal subgroups of the group G instead of τ they write s_{sn}.

Corollary 2. *Let f be the minimal s_{sn}-valued ω-composition satellite of the formation $\mathfrak{F}$ and let H be the canonical ω-composition satellite of a 2-multiply local formation $\mathfrak{H}$. A formation $\mathfrak{F}$ is a minimal s_{sn}-closed ω-composition non-$\mathfrak{H}$-formation if and only if $\mathfrak{F} = c_\omega^{s_{sn}} formG$, where G is a monolithic non-$\mathfrak{H}$-group and $R = G^{\mathfrak{H}} = Soc(G)$ is the socle of G, where $R \nsubseteq \Phi(G)$ such that every popper subnormal subgroup of G belongs to $\mathfrak{H}$ and either $\pi = \pi(Com(R)) \cap \omega = \varnothing$ or $\pi \neq \varnothing$ and G satisfies one of the following conditions:*

1) $G = R$ is a group of prime order $p \notin \pi(\mathfrak{H})$;

2) $G = [R]M$, where $R = O_p(G) = F_p(G)$ for same $p \in \pi$ and M is a monolithic non-$H(p)$-group and $Q = M^{H(p)} = Soc(M)$ is the socle of M, where $p \notin \pi(Com(Q))$ and $Q \nsubseteq \Phi(M)$ such that every popper subnormal subgroup of M belongs to $H(p)$.

Corollary 3. *Let $\mathfrak{S}$ be the formation of all soluble groups. A formation $\mathfrak{F}$ is a minimal τ-closed ω-composition non-soluble formation if and only if $\mathfrak{F} = c_\omega^\tau formG$, where G is a monolithic τ-minimal non-soluble group and $R = G^{\mathfrak{S}} = Soc(G)$ is the non-abelian socle of G.*

Proof. Let H be the canonical ω-composition satellite of the formation $\mathfrak{S}$. Hence $H(a) = \mathfrak{S}$ for all $a \in \omega \cup \{\omega'\}$.

Necessity. By Theorem 2.1 and Remark 1, $\mathfrak{F} = c_\omega^\tau formG$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{S}$-group and $R = G^{\mathfrak{S}} \nsubseteq \Phi(G)$ is the socle of G and either $\pi = \pi(Com(R)) \cap \omega = \varnothing$ or $\pi \neq \varnothing$ and $G = [R]M$, where $R = O_p(G) = F_p(G)$ for same $p \in \pi$ and M is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{S}$-group and $Q = M^{\mathfrak{S}}$ is the socle of M, where $Q \nsubseteq \Phi(M)$.

Let's $\pi \neq \varnothing$. In this case R is an abelian p-group. But $G/R \in \mathfrak{S}$ is a soluble group and so G is a soluble group. Then $R = G^{\mathfrak{S}} = 1$. A contradiction. Therefore R is a non-abelian group.

Sufficiency follows from Theorem 2.1. □

Corollary 4. *Let $\mathfrak{N}$ be the formation of all nilpotent groups. A formation $\mathfrak{F}$ is a minimal τ-closed ω-composition non-$\mathfrak{N}$-formation if and only if $\mathfrak{F} = c_\omega^\tau formG$, where G is a monolithic τ-minimal non-$\mathfrak{N}$-group and $R = G^{\mathfrak{N}} = Soc(G)$ is the socle of G and either $\pi = \pi(Com(R)) \cap \omega = \varnothing$ or $\pi \neq \varnothing$ and G is a Schmidt group.*

Proof. Let H be the canonical ω-composition satellite of the formation $\mathfrak{H}$. Hence

$$H(a) = \begin{cases} \mathfrak{N}_p, & \text{if } a = p \in \omega; \\ \mathfrak{N}, & \text{if } a = \omega'. \end{cases}$$

Necessity. By Theorem 2.1 and Remark 1, $\mathfrak{F} = c_\omega^\tau formG$, where G is a monolithic $\overline{\tau}$-minimal non-$\mathfrak{N}$-group and $R = G^{\mathfrak{N}} \not\subseteq \Phi(G)$ is the socle of G and either $\pi = \pi(Com(R)) \cap \omega = \varnothing$ or $\pi \neq \varnothing$ and $G = [R]M$, where $R = O_p(G) = F_p(G)$ for same $p \in \pi$ and M is a monolithic $\overline{\tau}$-minimal non-$H(p)$-group and $Q = M^{H(p)}$ is the socle of M, where $p \notin \pi(Com(Q))$ and $Q \not\subseteq \Phi(M)$.

By Lemma 1.4,

$$f(p) = \tau form(G/C^p(G)) = \tau form(G/R) = \tau formM.$$

It means that $\tau formM$ is a minimal τ-closed non-$\mathfrak{N}_p$-formation. Since $G/R \simeq M \in \mathfrak{N}$ and $\mathfrak{N}$ is hereditary, $\tau formM \subseteq \mathfrak{N}$. Thus by Theorem 2.4 [11], $\tau formM = formM = sformM$. Let H be a group of minimal order in $sformM \backslash \mathfrak{N}_p$. If $sformH \subset sformM$, then $sformH \subseteq \mathfrak{N}_p$. A contradiction. Therefore $sformH = sformM$. By the choice of the group H, it is a minimal non-$\mathfrak{N}_p$-group. Thus all its Sylow subgroups are p-groups. It means that H is p-group. A contradiction. Therefore H is a group of prime order q, where $q \neq p$. Thus $sformH = sformZ_q$ is a hereditary formation generated by the group of prime order q. Since $M \in sformZ_q$, M is a group of exponent q. Since $G = [R]M$ and $R = C_G(R)$, M is a irreducible abelian group of automorphisms for R. Therefore M is a cyclic group. But the order and the exponent of the cyclic group M are the same. Thus we have $|M| = q$. So G is a group Schmidt.

Sufficiency. Let condition of the corollary be satisfied and R be an abelian p-group. Hence G be a Schmidt group. From the description of the Schmidt groups it follows that $G = [R]M$, where $R = C_G(R)$ is a minimal normal p-subgroup of G and $|M| = q$, where q is a prime. It means that M is a minimal non-$\mathfrak{N}_p$-group and $Q = M$ is the socle of M. In this case $\Phi(M) = 1$. Thus by Theorem 2.1, the corollary is proved. □

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