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On the growth of the identities of algebras

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1. Introduction and Preliminaries

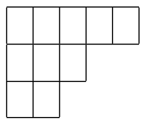
Let F be a field of characteristic zero and let A be an algebra over F. One can associate to the polynomial identities satisfied by the algebra A a numerical sequence $c_n(A), n = 1, 2, \dots$, called the sequence of codimensions of A ([19]). More precisely, if $F\{X\}$ is the free algebra on a countable set $X = \{x_1, x_2, \dots\}$ and P_n is the space of multilinear polynomials in the first n variables, $c_n(A)$ is the dimension of P_n modulo the polynomial identities satisfied by A. The sequence $c_n(A)$ gives in some way a measure of the polynomial relations vanishing in the algebra A and in general, for non-associative algebras, has overexponential growth. Concerning free algebras for instance, if $F\{X\}$ is the free (non-associative) algebra on X, $c_n(F\{X\}) = p_n n!$ where $p_n = \frac{1}{n}\binom{2n-2}{n-1}$ is the n-th Catalan number. For the free associative algebra $F\langle X\rangle$ and the free Lie algebra $L\langle X\rangle$ we have $c_n(F\langle X\rangle) = n!$ and $c_n(L\langle X\rangle) = (n-1)!$.

Since $\operatorname{char} F = 0$, by the well known multilinearization process, every T-ideal is determined by its multilinear polynomials. Hence the T-ideal $Id(A)$ is completely determined by the sequence of spaces $\{P_n \cap Id(A)\}_{n\geq 1}$. Now, the symmetric group S_n acts in a natural way on the space P_n: for $\sigma \in S_n, f(x_1, \dots, x_n) \in P_n$, $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Since for any algebra A, the subspace $P_n \cap Id(A)$ is S_n-invariant, this in turn induces a structure of S_n-module on the space $P_n(A) = \frac{P_n}{P_n \cap Id(A)}$. The S_n-character of $P_n(A)$, denoted $\chi_n(A)$, is called the n-th cocharacter of the algebra A and

$$c_n(A) = \chi_n(A)(1) = \dim_F P_n(A)$$

is the n-th codimension of A.

Since $\operatorname{char} F = 0$, by complete reducibility we can write $\chi_n(A)$ as a sum of irreducible S_n-characters. Now, it is well known that there is a one-to-one correspondence between irreducible S_n-characters and partitions on n (or Young diagrams). Recall that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of n, and we write $\lambda \vdash n$, if $\lambda_1 \geq \cdots \geq \lambda_r > 0$ are integers such that $\sum_{i=1}^{r} \lambda_i = n$. One usually identifies a partition λ with the corresponding Young diagram D_λ whose i-th row has length λ_i. For instance if $\lambda = (5, 3, 2) \vdash 10$, then the corresponding Young diagram is



There is a well-known and useful formula, called the hook formula, for computing the degree of an irreducible S_n-character (see for instance [10]): for $\lambda \vdash n$ a partition of n, let χ_λ denote the corresponding S_n-character. Then

$$\chi_\lambda(1) = \frac{n!}{\prod_{i,j} h_{ij}}$$

where for any box $(i, j) \in D_\lambda$, $h_{ij} = \lambda_i + \lambda_j^{'} - i - j + 1$, with $\lambda^{'} = (\lambda_1^{'}, \ldots, \lambda_s^{'})$ the conjugate partition of λ, is the hook number of the cell (i, j). For instance in the above example $h_{12} = 6$, $h_{21} = 4$.

We define the hook $H(s, t) = \bigcup_{n \geq 1} \{\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{s+1} \leq t\}$ where the integer s is called the hand and t the foot of the hook. If A is an F-algebra, by complete reducibility its n-th cocharacter decomposes as

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{1}$$

where $m_\lambda \geq 0$ is the multiplicity of χ_λ in $\chi_n(A)$. Then we shall write $\chi_n(A) \subseteq H(d, l)$ if $\lambda \in H(d, l)$ for all partitions λ such that $m_\lambda \neq 0$.

The most important feature of an associative algebra A satisfying a polynomial identity (PI-algebra) is that $c_n(A)$ is exponentially bounded ([19]). There is also a wide class of algebras with exponentially bounded codimension growth: for instance, if A is any finite dimensional algebra and $\dim A = d < \infty$, then $c_n(A) \leq d^n$ ([1]). Also, any infinite dimensional simple Lie algebra of Cartan type or any affine Kac-Moody algebra has exponentially bounded codimension growth ([12], [21]).

An interesting remark is that from the hook formula it is easy to show that if the cocharacter of an algebra A lies in a hook $H(s, t)$ then the algebra will have exponentially bounded codimension growth. In case of associative PI-algebras it has been shown that the corresponding cocharacter lies in some hook $H(s, t)$ ([19]). But this is not a general phenomenon. For instance if W_1 is the infinite-dimensional simple algebra of Cartan type (Witt algebra or Virosoro algebra), then it can be shown that $c_n(W_1)$ has exponential growth ([12]) but the corresponding cocharacter is not contained in any hook ([14]).

In case the sequence of codimensions is exponentially bounded, say $c_n(A) \leq d^n$, one can construct the bounded sequence $\sqrt[n]{c_n(A)}$, $n = 1, 2, \ldots$, and it is an open problem if, in this case, $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ exists. Let us define the PI-exponent of the algebra A as $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ in case such limit exists.

Let us recall that the sequence of codimensions $c_n(A)$ of an algebra A has

- *polynomial* growth if there exist constants C and k such that $c_n(A) \leq Cn^k$;

- *overpolynomial* growth if for any constants C and k there exists n such that $c_n(A) > Cn^k$;

- *overexponential* growth if for any constants C and b there exists n such that $c_n(A) > Cb^n$.

If the growth is more than polynomial and less than exponential one says that the sequence $c_n(A)$ has *intermediate* growth.

In the 80's Amitsur conjectured that for any associative PI-algebra A, $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a nonnegative integer. This conjecture was recently confirmed in [7] and [8].

When A is a Lie algebra, the sequence of codimensions has a much more involved behavior. Volichenko in [20] showed that a Lie algebra can have overexponential growth of the codimensions. Starting from this, Petrogradsky in [18] exhibited a whole scale of overexponential functions providing the exponential behavior of the identities of polynilpotent Lie algebras.

Motivated by the good behavior of $c_n(A)$ for associative algebras, one can ask the following question: if a Lie algebra A is such that its sequence of codimensions is exponentially bounded, can we infer that the exponential growth of $c_n(A)$ is integer? In [24] it was shown that the exponential growth of $c_n(A)$ is integer for any finite dimensional Lie algebra A. The exponential growth of $c_n(A)$ is also integer for any infinite dimensional Lie algebra A with a nilpotent commutator subalgebra [17]. Anyway the question was answered in the negative by Mishchenko and Zaicev in [23] by constructing an example of a Lie algebra with exponential growth strictly between 3 and 4. On the other hand no exponential growth between 1 and 2 is allowed ([16]).

In this paper we want to review the results proved in the last years concerning the growth of the codimensions in the general case of non-associative algebras over a field of characteristic zero.

We remark that in the examples constructed below, all algebras are left nilpotent of index 2 i.e., they satisfy the identity $x_1(x_2x_3) \equiv 0$.

In the first part we show that for general algebras whose sequence of codimensions is exponentially bounded, any real number > 1 can actually appear. In fact we construct, for any real number $\alpha > 1$, an algebra A_α whose sequence of codimensions grows exponentially and $\lim_{n\to\infty} \sqrt[n]{c_n(A_\alpha)} = \alpha$. The details of the proof of this result and some more results can be found in the original paper [4].

The second part of the paper concerns intermediate growth. For associative algebras it is known from long time ([11]) that the sequence of codimensions cannot have intermediate growth. A similar result for Lie algebras was proved in [15]. For associative superalgebras, associative algebras with involution and Lie superalgebras it has been shown that the corresponding codimensions cannot have intermediate growth ([3], [2], [22]).

Here we show that in the general case of non-associative algebras, there exist examples of algebras with intermediate growth of the codimensions. To this end for any real number $0 < \beta < 1$ we construct an algebra whose sequence of codimensions grows as n^{n^β}. The details of the proof of this result and some other results can be found in [5]. Finally two more results are presented: for any finite dimensional algebra the sequence of codimensions cannot have intermediate growth, and for any two-dimensional algebra either the exponent is equal to 2 or the growth of the codimensions is polynomially bounded by $n+1$ ([6]).

2. Sturmian or periodic words and real PI-exponent

Given any sequence of integers $K = \{k_i\}_{i\geq 1}$ such that $k_i \geq 2$ for all i, we can define a (non-associative) algebra $A(K)$ in the following way. We let $A(K)$ be the algebra over F with basis

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \dots$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \leq j \leq k_i\}, \quad i = 1, 2, \dots .$$

and with multiplication table given by

$$z_2^{(i)} a = z_3^{(i)}, \ \dots, \ z_{k_i - 1}^{(i)} a = z_{k_i}^{(i)}, \ z_{k_i}^{(i)} a = z_1^{(i)}, \quad i = 1, 2 \dots,$$

$$z_1^{(i)} b = z_2^{(i+1)}, \quad i = 1, 2, \dots$$

and all the remaining products are zero.

Some special types of sequences defined below will be of interest for our purpose. Take $w = w_1 w_2 \dots$ an infinite (associative) word in the alphabet $\{0,1\}$. Given an integer $m \geq 2$, define $K_{m,w} = \{k_i\}_{i\geq 1}$ to be the sequence

$$k_i = \begin{cases} m, & \text{if } w_i = 0 \\ m+1, & \text{if } w_i = 1 \end{cases}$$

and write $A(m,w) = A(K_{m,w})$.

We recall some of the basic definitions concerning infinite words and their complexity. In general, given an infinite word w in a finite alphabet, the complexity Comp_w of w is the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of w of length n.

Recall that an infinite word $w = w_1 w_2 \cdots$ in the alphabet $\{0,1\}$ is periodic with period T if $w_i = w_{i+T}$ for $i = 1, 2, \dots$. It is easy to see that for any such word $\mathrm{Comp}_w(n) \leq T$. Moreover, an infinite word w is called a *Sturmian* word if $\mathrm{Comp}_w(n) = n+1$ for all $n \geq 1$ (see [13]).

For a finite word x, the height $h(x)$ of x is the number of letters 1 appearing in x. Also, if $|x|$ denotes the length of the word x, the *slope* of x is defined as $\pi(x) = \frac{h(x)}{|x|}$. In some cases this definition can be extended to infinite words in the following way. Let w be some infinite word and let $w(1,n)$ denote its prefix subword of length n. If the sequence $\frac{h(w(1,n))}{n}$ converges for $n \to \infty$ and the limit

$$\pi(w) = \lim_{n\to\infty} \frac{h(w(1,n))}{n}$$

exists then $\pi(w)$ it is called the slope of w. It is easy to give examples of infinite words for which the slope is not defined. Nevertheless for periodic words and Sturmian words the slope is well defined. The basic properties of these words are given in the next proposition.

Proposition 1. ([13, Section 2.2]) *Let w be a Sturmian or periodic word. Then there exists a constant C such that*

1) $|h(x) - h(y)| \leq C$, for any finite subwords x, y of w with $|x| = |y|$;

2) the slope $\pi(w)$ of w exists;

3) for any non-empty subword u of w,

$$|\pi(u) - \pi(w)| \leq \frac{C}{|u|};$$

4) for any real number $\alpha \in (0,1)$ there exists a word w with $\pi(w) = \alpha$ and w is Sturmian or periodic according as α is irrational or rational, respectively.

In case w is Sturmian we can take $C = 1$, and if w is periodic of period T, then $\pi(w) = \frac{h(w(1,T))}{T}$.

Recall that, given elements $y_1, y_2, \dots, y_n$ of a non-associative algebra, their left-normed product is defined inductively as $y_1 \cdots y_n = (y_1 \cdots y_{n-1})y_n$. From the definition of the algebra $A(K)$ it easily follows that only left-normed products of the basis elements of $A(K)$ may be non-zero. Moreover the only non-zero products are of the type $z_j^{(i)} f(a,b)$ for some left-normed monomial $f(a,b)$.

Some conclusions can be easily drawn about the cocharacter sequence of $A(K)$. For a partition $\lambda = (\lambda_1, \dots, \lambda_t)$ of n let $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ denote the conjugate partition of λ. Hence $h(\lambda) = \lambda'_1$ is the height of the Young diagram corresponding to λ. Let T_λ be a λ-tableau and let e_{T_λ} be the corresponding essential idempotent of FS_n. For any polynomial $f(x_1, \dots, x_n) \in P_n$, the element $e_{T_\lambda} f(x_1, \dots, x_n)$ is a linear combination of polynomials each alternating on disjoint sets of variables of order $\lambda'_1, \dots, \lambda'_r$, respectively. Since $\mathrm{span}\{z_j^{(i)}\}$ is a two-sided ideal of $A(K)$ with trivial multiplication of codimension 2, it follows that $e_{T_\lambda} f(x_1, \dots, x_n) \equiv 0$ in $A(K)$ as soon as $h(\lambda) > 3$ or $\lambda_3 > 1$. This says that if $\chi_n(A(K)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ is the n-th cocharacter of $A(K)$, then $m_\lambda = 0$ as soon as $h(\lambda) > 3$ or $\lambda_3 > 1$. In other words

$$\chi_n(A(K)) = m_{(n)}\chi_{(n)} + \sum_{\lambda=(\lambda_1,\lambda_2)\vdash n} m_\lambda \chi_\lambda + \sum_{\lambda=(\lambda_1,\lambda_2,1)\vdash n} m_\lambda \chi_\lambda.$$

When studying the cocharacter of the algebra $A(K)$, a special function $\Phi : \mathbb{R} \to \mathbb{R}$ comes into play. This function is defined by

$$\Phi(x) = \frac{1}{x^x (1-x)^{1-x}}.$$

Notice that Φ is continuous in the interval $(0, \frac{1}{2}]$, and $\Phi(a) < \Phi(b)$ whenever $a < b$. Moreover $\lim\limits_{x \to 0} \Phi(x) = 1$ and $\Phi(\frac{1}{2}) = 2$.

At this stage one needs to study the behavior of the n-th cocharacter of $A(m,w)$. Roughly speaking one proves that all characters χ_λ whose diagram λ has long second row, do not participate in $\chi_n(A(m,w))$. This fact is exploited in the next lemmas in order to get an upper bound and a lower bound for $c_n(A(m,w))$.

Lemma 1. *Let w be a Sturmian or periodic word with slope $\pi(w) = \alpha$, let $A = A(m,w)$ and let $\beta = \frac{1}{m+\alpha}$. Then, given any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geq N$, the $(n+1)$-th codimension of A satisfies*

$$c_{n+1}(A) \geq \frac{1}{2^{m+1}\sqrt{\pi n^3}} \Phi(\beta + \varepsilon)^n.$$

Lemma 2. *Let w be a Sturmian or periodic word with slope α and let $A = A(m, w)$. If $\beta = \frac{1}{m+\alpha}$, then $c_{n+1}(A) \leq 3(m+1)(n+2)^5\Phi(\beta)^n$.*

Putting together Lemma 1 and Lemma 2 it is clear that for the algebras $A(m, w)$ the PI-exponent exists and equals $\Phi(\beta)$. We record this in the following.

Proposition 2. *Let w be an infinite Sturmian or periodic word with slope α, $0 < \alpha < 1$. If $m \geq 2$ then for the algebra $A = A(m, w)$ the PI-exponent exists and* $\exp(A) = \Phi(\beta)$ *where $\beta = \frac{1}{m+\alpha}$.*

Recalling that the function Φ is continuous and $\Phi((0, \frac{1}{2})) = (1, 2)$, we immediately obtain.

Corollary 1. *For any real number d, $1 < d < 2$, there exists an algebra A such that* $\exp(A) = d$.

All algebras $A(m, w)$ constructed above are infinite dimensional. In case the word w is periodic we can actually construct a finite dimensional algebra B such that $Id(B) = Id(A(m, w))$ (and $\exp(B) = \exp(A(m, w))$). The construction is the following. Recall that given an infinite word on $\{0, 1\}$ and $m \geq 2$, the sequence $K_{m,w} = \{k_i\}_{i \geq 1}$ is defined by $k_i = \begin{cases} m, & \text{if } w_i = 0 \\ m+1, & \text{if } w_i = 1 \end{cases}$. Then, if w is an infinite periodic word of period T, we define $B(K)$ as the algebra over F with basis

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \cdots \cup Z_T$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \leq j \leq k_i\}, \quad i = 1, 2, \ldots, T,$$

and multiplication table given by

$$z_2^{(i)}a = z_3^{(i)}, \ \ldots, \ z_{k_i-1}^{(i)}a = z_{k_i}^{(i)}, \ z_{k_i}^{(i)}a = z_1^{(i)}, \quad i = 1, 2 \ldots,$$

$$z_1^{(i)}b = z_2^{(i+1)}, \quad i = 1, 2, \ldots, (T-1)$$

and

$$z_1^{(T)}b = z_2^{(1)}.$$

All the remaining products are zero.

The following results are obvious consequences of the definition of the algebra $B(K)$.

Proposition 3. *The algebras $A(K)$ and $B(K)$ satisfy the same identities.*

From this result we have

Proposition 4. *For any rational number $\beta, 0 < \beta \leq \frac{1}{2}$, there exists a finite dimensional algebra B such that* $\exp(B) = \Phi(\beta)$.

We next wish to extend Proposition 2 and Corollary 1 to all real numbers > 1 i.e., we want to construct, for any real number $\alpha > 1$ an algebra A such that $\exp(A) = \alpha$. We can accomplish this by constructing an appropriate algebra B and then by gluing, in an appropriate way, B to one of the algebras $A(m, w)$ constructed above.

Given any positive integer d we define a non-associative algebra $B = B(d)$ as follows: B has basis $\{u_1, \dots u_d,\ s_1, \dots, s_d\}$ with multiplication table given by

$$s_1 u_1 = u_2, \dots s_{d-1} u_{d-1} = u_d, s_d u_d = u_1,$$

and all other products are zero.

Starting with $A(K)$ and B we next define an algebra $A(K, d)$ which will contain both $A(K)$ and B as subalgebras.

Let W be the vector space spanned by the set $\{w_{i,j} \mid 1 \leq i \leq d, j \geq 1\}$ and let $A(K, d)$ be the algebra which is the vector space direct sum of $A(K)$, B and W,

$$A(K, d) = A(K) \oplus B \oplus W.$$

The multiplication in $A(K, d)$ is induced by the multiplication in $A(K), B$ and $u_s z_j^i = w_{si}$, $1 \leq s \leq d$, $1 \leq j \leq k_i$, $i \geq 1$, and all other products are zero.

We start by studying the identities of $B = B(d)$.

Lemma 3. *The algebra B satisfies the right-normed identity*

$$y_1(x_1 \cdots (x_{d-1}(y_2 x_d)) \dots) \equiv y_2(x_1 \cdots (x_{d-1}(y_1 x_d)) \dots)$$

and the left-normed identity $x_1 x_2 x_3 \equiv 0$.

Proposition 5. *Let $m \geq 2$ and let w be a periodic or Sturmian word. Then the PI-exponent of the algebra $A(K_{m,w}, d)$ exists and* $\exp(A(K_{m,w}, d)) = d + \delta$ *where* $\delta = \exp(A(K_{m,w}))$.

At last we formulate the main results about the PI-exponent ([4]).

Theorem 1. *For any real number $t \geq 1$ there exists an algebra A such that*

$$\exp(A) = t.$$

Theorem 2. *For any $1 \leq \alpha < \beta$ there exists a finite dimensional algebra A such that*

$$\alpha < \exp(A) < \beta.$$

3. Constructing intermediate growth

As we mentioned in the introduction, no intermediate growth is allowed for the codimensions of an associative algebra or a Lie algebra. Nevertheless this is not a general phenomenon. In fact we shall next construct non-associative algebras whose codimensions have intermediate growth. Recall that given an algebra A with n-th cocharacter $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$, then $l_n(A) = \sum_{\lambda\vdash n} m_\lambda$ is the n-th colength of A.

Given a sequence $K = \{k_i\}_{i\geq 1}$ we can associate to K a real valued function ρ such that $k_i = \rho(i)$, $i = 1, 2, \ldots$. In this case we also write $A(K) = A(\rho)$.

Let now ρ be a polynomially bounded real function i.e.,

$$\rho(x) \leq cx^\beta$$

for all positive real numbers x, where $\beta > 0$ and $c > 0$ are constants. We next find some condition on such function ρ, so that the algebra $A(\rho)$ has polynomially bounded colength sequence.

Lemma 4. *Let ρ be a polynomially bounded monotone function, $\rho(x) \leq cx^\beta$, for all $x \in \mathbb{R}^+$, and $\beta > 0$, $c > 0$ constants. If $\lim_{x\to\infty} \rho(x) = \infty$ then, for n large enough, the colength sequence of the algebra $A(\rho)$ satisfies*

$$l_n(A(\rho)) \leq (n+1)^3 \left(3n + 3c \left(\frac{2n}{c} \right)^{\frac{\beta+1}{\beta}} \right).$$

We now specialize the polynomially bounded function ρ to a function that behaves like $\gamma x^{\gamma-1}$ for some fixed $\gamma > 1$.

Let β be a real number such that $0 < \beta < 1$ and let $\gamma = \frac{1}{\beta}$. Then we define $A = A(\beta)$ as the algebra $A(K)$ where the sequence $K = \{k_i\}_{i\geq 1}$ is defined by the relation $k_1 + \cdots + k_t = [t^\gamma]$, for all $t \geq 1$, where $[x]$ is the integer part of x. We can show that such algebras have intermediate growth of the codimensions. In fact we have ([5])

Theorem 3. *For any real number β with $0 < \beta < 1$, let $A = A(K)$ where $K = \{k_i\}_{i\geq 1}$ is defined by the relation $k_1 + \cdots + k_t = [t^{\frac{1}{\beta}}]$, for all $t \geq 1$. Then the sequence of codimensions of A satisfies*

$$\lim_{n\to\infty} \log_n \log_n c_n(A) = \beta.$$

Hence, $c_n(A)$ asymptotically equals n^{n^β}.

The above algebras are infinite dimensional. Hence it is natural to ask if one can construct finite dimensional algebras with such property. The answer is negative since, as we shall see, any finite dimensional algebra cannot have intermediate growth of the codimensions.

Concerning the colength sequence of an arbitrary finite dimensional algebra it can be shown that is polynomially bounded and depends only on $\dim A$. In fact the following holds.

Theorem 4. *Let A be a finite dimensional algebra,* $\dim A = d$. *Then*

$$l_n(A) \leq d(n+1)^{d^2+d}.$$

In the next theorem we show that no intermediate growth is allowed for a finite dimensional algebra A and we can exhibit a lower bound for the exponential growth depending only on $\dim A$ ([5]).

Theorem 5. *Let A be a finite dimensional algebra of overpolynomial codimension growth and let* $\dim A = d$. *Then $c_n(A) > \frac{1}{n^2} 2^{\frac{n}{3d^3}}$, for all n large enough.*

More precise information can be obtained in case of an arbitrary two-dimensional algebra. In fact we have the following (see [6]).

Theorem 6. *Let A be a two-dimensional algebra over an algebraically closed field of characteristic zero. Then $c_n(A) \leq n+1$ or* $\exp(A) = 2$.

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