Algebra and Discrete Mathematics
Number 1. **(2007).** pp. 24 – 39 RESEARCH ARTICLE

On Frobenius full matrix algebras with structure systems

Hisaaki Fujita, Yosuke Sakai and Daniel Simson

Abstract. Let $n \geq 2$ be an integer. In [5] and [6], an $n \times n$ $\mathbb{A}$-full matrix algebra over a field K is defined to be the set $\mathbb{M}_n(K)$ of all square $n \times n$ matrices with coefficients in K equipped with a multiplication defined by a structure system $\mathbb{A}$, that is, an n-tuple of $n \times n$ matrices with certain properties. In [5] and [6], mainly $\mathbb{A}$-full matrix algebras having $(0,1)$-structure systems are studied, that is, the structure systems $\mathbb{A}$ such that all entries are 0 or 1. In the present paper we study $\mathbb{A}$-full matrix algebras having non $(0,1)$-structure systems. In particular, we study the Frobenius $\mathbb{A}$-full matrix algebras. Several infinite families of such algebras with nice properties are constructed in Section 4.

1. Introduction

Throughout this paper we freely use the rings, modules, and representation theory terminology introduced in [1], [2], [4], [9], [11], and [12]. In particular, given a finite dimensional algebra R over a field K, we denote by $\operatorname{mod} R$ the category of all finite dimensional unitary right R-modules. Given a module M in $\operatorname{mod} R$, we denote by $\operatorname{soc} M$ the socle of M.

Let K be a field and $n \geq 2$ an integer. Let $\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions:

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$, for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$, for all $i, j, k \in \{1, \ldots, n\}$, and

2000 Mathematics Subject Classification: *16G10, 16G30, 16G60.*
Key words and phrases: *Frobenius algebra, quiver, module, socle, tame representation type.*

(A3) $a_{ii}^{(k)} = 0$, for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

We denote by

$$R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} K u_{ij} \tag{1.1}$$

a K-vector space, with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$, equipped with a multiplication (depending on $\mathbb{A}$) defined by the formula

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij}, & \text{if } \ k = l, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $R_{\mathbb{A}}$ is an associative, basic K-algebra $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents of $R_{\mathbb{A}}$ and $1 = u_{11} + \cdots + u_{nn}$ is an identity element of $R_{\mathbb{A}}$, see [5, Proposition 1.1]. We call $R_{\mathbb{A}}$ an $\mathbb{A}$-*full matrix algebra* and $\mathbb{A}$ a *structure system* of $R_{\mathbb{A}}$.

The reader is referred to the recent paper [7] for a degeneration-like approach to the full matrix algebras $R_{\mathbb{A}}$ with structure systems.

Since $u_{ii} R_{\mathbb{A}} u_{jj} \neq 0$, for all $1 \leq i, j \leq n$, then the K-algebra $R_{\mathbb{A}}$ is *connected*, that is, $R_{\mathbb{A}}$ can not be decomposed into a product of two subalgebras. Note also that the Jacobson radical $J(R_{\mathbb{A}})$ of $R_{\mathbb{A}}$ has the form

$$J(R_{\mathbb{A}}) = \bigoplus_{i \neq j} u_{ij} K. \tag{1.2}$$

If V is a simple right $R_{\mathbb{A}}$-module, then $V u_{ii} \neq 0$, for some $1 \leq i \leq n$, and $V \cong u_{ii} R_{\mathbb{A}} / u_{ii} J(R_{\mathbb{A}})$. Therefore the $R_{\mathbb{A}}$-modules

$$u_{11} R_{\mathbb{A}} / u_{11} J(R_{\mathbb{A}}), \ldots, u_{nn} R_{\mathbb{A}} / u_{nn} J(R_{\mathbb{A}})$$

are the representatives of all pairwise non-isomorphic simple right $R_{\mathbb{A}}$-modules. Note that $\dim_K V = 1$, for any simple right $R_{\mathbb{A}}$-module V.

Let $R_{\mathbb{A}}$ be an $\mathbb{A}$-full matrix algebra (1.1) and let M be a right $R_{\mathbb{A}}$-module in $\operatorname{mod} R_{\mathbb{A}}$. The *dimension vector* of M (or the dimension type of M) is defined to be the n-tuple

$$\underline{\dim}\, M = (d_1, \ldots, d_n) \in \mathbb{Z}^n = K_0(R_{\mathbb{A}}) \tag{1.3}$$

of integers $d_i = \dim_K M u_{ii}$, with $1 \leq i \leq n$, see [1] and [5].

2. When $\mathbb{A}$-full matrix algebras are isomorphic?

In this section, we give a criterion for two $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ and $R_{\mathbb{B}}$ to be isomorphic. Moreover, we give a list of the representatives of all non-isomorphic 3×3 $\mathbb{A}$-full matrix algebras.

The isomorphism problem of $\mathbb{A}$-full matrix algebras is also studied in [7] in terms of an action

$$* : \mathbb{G}_n(K) \times \mathbb{ST}_n(K) \ \to \ \mathbb{ST}_n(K)$$

of an algebraic group $\mathbb{G}_n(K)$ (containing the symmetric group S_n) on the algebraic K-variety $\mathbb{ST}_n(K)$ of the structure systems $\mathbb{A}$.

Proposition 2.1. *Let* $R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} Ku_{ij}$ *and* $R_{\mathbb{B}} = \bigoplus_{i,j=1}^{n} Kv_{ij}$ *be full matrix algebras with structure systems*

$$\mathbb{A} = [A_1, \dots, A_n] = [a_{ij}^{(k)}]_{i,j,k} \quad \text{and} \quad \mathbb{B} = [B_1, \dots, B_n] = [b_{ij}^{(k)}]_{i,j,k},$$

respectively. There is a K-algebra isomorphism $R_{\mathbb{A}} \cong R_{\mathbb{B}}$ if and only if there exist a matrix $T = (t_{ij}) \in \mathbb{M}_n(K)$ and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ of the set $\{1, \dots, n\}$ such that

$$t_{ij} \neq 0, \quad t_{ii} = 1, \quad a_{\sigma(i)\sigma(j)}^{(\sigma(k))} t_{ij} = b_{ij}^{(k)} t_{ik} t_{kj}, \quad \textit{for all } 1 \leq i, j, k \leq n.$$

Proof. Suppose that there is a K-algebra isomorphism $f : R_{\mathbb{A}} \to R_{\mathbb{B}}$. Then $f(u_{11}), \dots, f(u_{nn})$ are orthogonal primitive idempotents of $R_{\mathbb{B}}$ such that $1_{R_{\mathbb{B}}} = f(u_{11}) + \cdots + f(u_{nn})$. It follows from [4, Theorem 3.4.1] that there exist a permutation σ of the set $\{1, \dots, n\}$ and an invertible element $b \in R_{\mathbb{B}}$ such that $v_{ii} = bf(u_{\sigma(i)\sigma(i)})b^{-1}$, for all $1 \leq i \leq n$. Hence there is a K-algebra isomorphism $g : R_{\mathbb{A}} \to R_{\mathbb{B}}$ such that $v_{ii} = g(u_{\sigma(i)\sigma(i)})$, for all $1 \leq i \leq n$. Since $g(u_{\sigma(i)\sigma(j)}) = v_{ii} g(u_{\sigma(i)\sigma(j)}) v_{jj}$, then $g(u_{\sigma(i)\sigma(j)}) = t_{ij} v_{ij}$, for some $0 \neq t_{ij} \in K$ $(1 \leq i, j \leq n)$. Clearly, $t_{ii} = 1$, for all $1 \leq i \leq n$. Since

$$g(u_{\sigma(i)\sigma(k)} u_{\sigma(k)\sigma(j)}) = g(u_{\sigma(i)\sigma(k)}) g(u_{\sigma(k)\sigma(j)}),$$

then we have $a_{\sigma(i)\sigma(j)}^{(\sigma(k))} t_{ij} = b_{ij}^{(k)} t_{ik} t_{kj}$, for all $1 \leq i, j, k \leq n$. It follows that $T := (t_{ij}) \in \mathbb{M}_n(K)$ is the desired matrix.

Conversely, suppose that there exist a matrix $T = (t_{ij})$ and a permutation σ of the set $\{1, \dots, n\}$ satisfying the above condition. Then the K-linear map

$$f : R_{\mathbb{A}} \to R_{\mathbb{B}},$$

given by $u_{ij} \mapsto t_{\sigma^{-1}(i)\sigma^{-1}(j)} v_{\sigma^{-1}(i)\sigma^{-1}(j)}$, defines a K-algebra isomorphism. $\square$

As an immedeate consequence of the proposition, we have the following.

Corollary 2.2. *Let* $R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} Ku_{ij}$ *and* $R_{\mathbb{B}} = \bigoplus_{i,j=1}^{n} Kv_{ij}$ *be full matrix algebras with* $(0,1)$*-structure systems*

$$\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k} \quad \text{and} \quad \mathbb{B} = [B_1, \ldots, B_n] = [b_{ij}^{(k)}]_{i,j,k},$$

respectively. Then $R_{\mathbb{A}}$ *is isomorphic to* $R_{\mathbb{B}}$ *as* K*-algebras if and only if there exists a permutation* σ *of the set* $\{1, \ldots, n\}$ *such that* $b_{ij}^{(k)} = a_{\sigma(i)\sigma(j)}^{(\sigma(k))}$ *for all* $1 \leq i, j, k \leq n$.

Lemma 2.3. *Let* $n \geq 3$ *be an integer, and let* $\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$ *be a structure system. Then, for any distinct* $1 \leq i, j, k \leq n$*, the following equalities hold* $a_{ij}^{(k)} a_{ik}^{(j)} = 0$ *and* $a_{kj}^{(i)} a_{ij}^{(k)} = 0$.

Proof. This follows from (A1) and (A3). □

Example 2.4. By applying Lemma 2.3 and Corollary 2.2, one can verify that, for $n = 3$, the following five $(0,1)$-structure systems $\mathbb{A}^{(1)}$, $\mathbb{A}^{(2)}$, $\mathbb{A}^{(3)}$, $\mathbb{A}^{(4)}$, $\mathbb{A}^{(5)}$:

$$\left[\begin{array}{ccc|ccc|ccc} 1&1&1 & 0&1&0 & 0&0&1 \\ 1&0&1 & 1&1&1 & 0&0&1 \\ 1&0&0 & 0&1&0 & 1&1&1 \end{array}\right], \quad \left[\begin{array}{ccc|ccc|ccc} 1&1&1 & 0&1&0 & 0&0&1 \\ 1&0&1 & 1&1&1 & 0&0&1 \\ 1&1&0 & 0&1&0 & 1&1&1 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc|ccc} 1&1&1 & 0&1&1 & 0&0&1 \\ 1&0&0 & 1&1&1 & 0&0&1 \\ 1&1&0 & 0&1&0 & 1&1&1 \end{array}\right], \quad \left[\begin{array}{ccc|ccc|ccc} 1&1&1 & 0&1&0 & 0&1&1 \\ 1&0&1 & 1&1&1 & 0&0&1 \\ 1&0&0 & 1&1&0 & 1&1&1 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc|ccc} 1&1&1 & 0&1&0 & 0&0&1 \\ 1&0&0 & 1&1&1 & 0&0&1 \\ 1&0&0 & 0&1&0 & 1&1&1 \end{array}\right].$$

provide a list of all $(0,1)$-structure systems $\mathbb{A}$ such that every $\mathbb{A}$-full matrix algebra $R_{\mathbb{A}}$ is isomorphic to any of the algebras $R_{\mathbb{A}^{(1)}}$, $R_{\mathbb{A}^{(2)}}$, $R_{\mathbb{A}^{(3)}}$, $R_{\mathbb{A}^{(4)}}$, $R_{\mathbb{A}^{(5)}}$.

Given an arbitrary structure system $\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$, we define a new one $\overline{\mathbb{A}} = [\overline{A}_1, \ldots, \overline{A}_n] = [\overline{a}_{ij}^{(k)}]_{i,j,k}$, where

$$\overline{a}_{ij}^{(k)} := \begin{cases} 1, & \text{if } a_{ij}^{(k)} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\overline{\mathbb{A}}$ is a structure system. Following [7, Definition 3.1], we call the $\overline{\mathbb{A}}$-full matrix algebra $R_{\overline{\mathbb{A}}}$ a $(0,1)$-*limit* of $R_{\mathbb{A}}$.

Theorem 2.5. *For $n = 3$, there are just five 3×3 $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$, up to isomorphism, which are given by the five $(0,1)$-structure systems in Example 2.4.*

Proof. Let $\mathbb{A}$ be a 3×3 $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = [A_1, A_2, A_3] = [a_{ij}^{(k)}]$, and let $R_{\overline{\mathbb{A}}}$ be the $(0,1)$-limit of $R_{\mathbb{A}}$. Then we show that $R_{\mathbb{A}}$ is isomorphic to $R_{\overline{\mathbb{A}}}$, using Proposition 2.1. We put $\sigma = \mathrm{id}$ and $T = (t_{ij}) \in \mathbb{M}_3(K)$, where

$$t_{ij} := \begin{cases} a_{ij}^{(k)}, & \text{if } a_{ij}^{(k)} \neq 0, \text{ for } k \neq i, j, \\ 1, & \text{otherwise,} \end{cases}$$

for distinct $i, j \in \{1, 2, 3\}$, and $t_{ii} := 1$ for $i = 1, 2, 3$. Then using Lemma 2.3, one can check that $\overline{a}_{ij}^{(k)} t_{ij} = a_{ij}^{(k)} t_{ik} t_{kj}$, for all $1 \leq i, j, k \leq n$. This completes the proof. $\square$

3. Frobenius $\mathbb{A}$-full matrix algebras

In this section, we improve the characterization of Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ given by [5, Lemma 4.2], where structure systems are $(0,1)$-matrices.

Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra (1.1) and let M be a right $R_{\mathbb{A}}$-module with $\underline{\dim}\, M = (1, \ldots, 1)$. Then M has a K-basis $\{v_1, \ldots, v_n\}$ such that $v_i u_{ii} = v_i$, for all $1 \leq i \leq n$. Consider the matrix $S = (s_{ij}) \in \mathbb{M}_n(K)$ such that

$$(*) \qquad v_i u_{kj} = \begin{cases} s_{ij} v_j, & \text{if } k = i, \\ 0, & \text{otherwise,} \end{cases}$$

for all $1 \leq i, j, k \leq n$, and that

$$(**) \quad s_{ii} = 1 \quad \text{and} \quad s_{ik} s_{kj} = a_{ij}^{(k)} s_{ij}, \quad \text{for all } \ 1 \leq i, j, k \leq n.$$

We call S a *representation matrix* of M with respect to a K-basis $\{v_1, \ldots, v_n\}$. Conversely, let M be a K-vector space with a K-basis $\{v_1, \ldots, v_n\}$ and $S = (s_{ij}) \in \mathbb{M}_n(K)$ which satisfies the condition $(**)$. Then, by $(*)$, M has a right $R_{\mathbb{A}}$-module structure with $\underline{\dim}\, M = (1, \ldots, 1)$, see [5, Proposition 2.1].

Now we modify [5, Propositions 2.2, 2.3 and Lemma 4.2] to remove the assumption of $(0,1)$-structure systems. We begin with the following lemma.

Lemma 3.1. *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra* (1.1) *and let M, M' be right $R_{\mathbb{A}}$-modules, with $\underline{\dim}\, M = \underline{\dim}\, M' = (1, \ldots, 1)$ and with* *the representation matrices $S = (s_{ij})$ and $S' = (s'_{ij})$, respectively. There exists an isomorphism $M \cong M'$ of right $R_{\mathbb{A}}$-modules if and only if there exist $t_1, \dots, t_n \in K$ such that*

$$t_i \neq 0 \quad \textit{and} \quad s_{ij}t_j = t_i s'_{ij}, \quad \textit{for all } i, j \in \{1, \dots, n\}.$$

Proof. Let $\{v_i \mid 1 \le i \le n\}, \{v'_i \mid 1 \le i \le n\}$ be associated K-bases of M, M' with representation matrices $S = (s_{ij})$ and $S' = (s'_{ij})$, respectively.

First suppose that there is an isomorphism $f : M \to M'$. Since $v'_j = v'_j u_{jj}$, for all $j \in \{1, \dots, n\}$, then there exists $0 \neq t_i \in K$ such that $f(v_i) = f(v_i)u_{ii} = t_i v'_i$, for each $i \in \{1, \dots, n\}$. The equality $f(v_i u_{ij}) = f(v_i)u_{ij}$ yields $s_{ij}t_j = t_i s'_{ij}$, for all $i, j \in \{1, \dots, n\}$.

Conversely, suppose that there exist $t_1, \dots, t_n \in K$ satisfying the above conditions. Since $t_i \neq 0$, for all $i \in \{1, \dots, n\}$, we can define a K-linear isomorphism $f : M \to M'$ by $f(v_i) := t_i v'_i$, for all $i \in \{1, \dots, n\}$. The latter condition implies that f is an $R_{\mathbb{A}}$-module homomorphism, so that $f : M \to M'$ is an isomorphism. □

Indecomposable projective $R_{\mathbb{A}}$-modules are characterized by their representation matrices as follows, see [5, Proposition 2.2].

Lemma 3.2. *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra* (1.1).

(i) *For each indecomposable projective right $R_{\mathbb{A}}$-module $u_{ii}R_{\mathbb{A}}$, we have*

- $\underline{\dim}\, u_{ii}R_{\mathbb{A}} = (1, \dots, 1)$ *and*
- *the representation matrix of the module $u_{ii}R_{\mathbb{A}}$, with respect to the K-basis $\{u_{ij} \mid 1 \le j \le n\}$, is the $n \times n$ matrix $(a^{(k)}_{ij})_{k,j}$, where the (k, j)-entry equals $a^{(k)}_{ij}$.*

(ii) *Let M be a right $R_{\mathbb{A}}$-module with $\underline{\dim} M = (1, \dots, 1)$, and let $S = (s_{ij})$ be a representation matrix of M with respect to a K-basis $\{v_i \mid 1 \le i \le n\}$. Then M is isomorphic to to the projective $R_{\mathbb{A}}$-module $u_{ll}R_{\mathbb{A}}$ if and only if $s_{lk} \neq 0$, for all $k \in \{1, \dots, n\}$.*

Proof. (i) This follows from the definition of the multiplication of $R_{\mathbb{A}}$, that is, $u_{ik}u_{kj} = a^{(k)}_{ij} u_{ij}$, for all $i, j, k \in \{1, \dots, n\}$. Note that (A2) implies $\underline{\dim}\, u_{ii}R_{\mathbb{A}} = (1, \dots, 1)$.

(ii) First suppose that M is isomorphic to $u_{ll}R_{\mathbb{A}}$. Then it follows from Lemma 3.1 that there exist $t_1, \dots, t_n \in K$ such that $t_i \neq 0$ and $s_{ij}t_j = t_i a^{(i)}_{lj}$, for all $i, j \in \{1, \dots, n\}$. Hence $s_{lj}t_j = t_i a^{(l)}_{lj} = t_i \neq 0$, so that $s_{lj} \neq 0$, for all $j \in \{1, \dots, n\}$.

Conversely, suppose that $s_{lj} \neq 0$, for all $j \in \{1, \ldots, n\}$. Since $a_{lj}^{(i)} s_{lj} = s_{li} s_{ij}$, for all $i, j \in \{1, \ldots, n\}$, then there is an $R_{\mathbb{A}}$-module isomorphism $f : u_{ll} R_{\mathbb{A}} \to M,\ u_{lj} \mapsto s_{lj} v_j\ (1 \leq j \leq n)$. □

We denote the standard duality functor $\mathrm{Hom}_K(-, K) : \mathrm{mod}\, R_{\mathbb{A}} \to \mathrm{mod}\, R_{\mathbb{A}}^{op}$ by $(-)^*$. As a dual of Lemma 3.2, we obtain the following, see [5, Proposition 2.3].

Lemma 3.3. *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra (1.1).*

(i) *For each indecomposable injective right $R_{\mathbb{A}}$-module $(R_{\mathbb{A}} u_{jj})^*$, we have*

- $\underline{\dim}\,(R_{\mathbb{A}} u_{jj})^* = (1, \ldots, 1)$ *and*
- *the representation matrix of the module $(R_{\mathbb{A}} u_{jj})^*$, with respect to the dual K-basis $\{u_{ij}^* \mid 1 \leq i \leq n\}$, is the $n \times n$ matrix $(a_{ij}^{(k)})_{i,k}$, where the (i,k)-entry equals $a_{ij}^{(k)}$.*

(ii) *Let M be a right $R_{\mathbb{A}}$-module with $\underline{\dim}\, M = (1, \ldots, 1)$, and let $S = (s_{ij})$ be a representation matrix of M with respect to a K-basis $\{v_i \mid 1 \leq i \leq n\}$. Then M is isomorphic to the injective $R_{\mathbb{A}}$-module $(R_{\mathbb{A}} u_{ll})^*$ if and only if $s_{kl} \neq 0$, for all $k \in \{1, \ldots, n\}$.*

Proposition 3.4. *Let $R_{\mathbb{A}}$ be an $\mathbb{A}$-full $n \times n$ matrix algebra, where $\mathbb{A} = [A_1, \ldots, A_n]$ is the structure system and $A_k = (a_{ij}^{(k)})$ $(1 \leq k \leq n)$. The following two conditions are equivalent.*

(i) *$R_{\mathbb{A}}$ is a Frobenius algebra with Nakayama permutation σ.*

(ii) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$, for all $i \in \{1, \ldots, n\}$, and that $a_{i\sigma(i)}^{(k)} \neq 0$, for all $i, k \in \{1, \ldots, n\}$.*

Proof. (i)⇒(ii) It follows from (i) that $u_{ii} R_{\mathbb{A}} \cong (R_{\mathbb{A}} u_{\sigma(i)\sigma(i)})^*$, for all $i \in \{1, \ldots, n\}$. Since $u_{ii} R_{\mathbb{A}}$ has a representation matrix $(a_{ij}^{(k)})_{k,j}$ with respect to a K-basis $\{u_{i1}, \ldots, u_{in}\}$ then Lemma 3.3 yields $a_{i\sigma(i)}^{(k)} \neq 0$, for all $i, k \in \{1, \ldots, n\}$. Since $\underline{\dim}\, u_{ii} R_{\mathbb{A}} = (1, \ldots, 1)$ then $\mathrm{soc}(u_{ii} R_{\mathbb{A}}) \not\cong u_{ii} R_{\mathbb{A}} / u_{ii} J(R_{\mathbb{A}})$, so that $\sigma(i) \neq i$, for all $i \in \{1, \ldots, n\}$.

(ii)⇒(i) Lemmas 3.2 and 3.3 yield the isomorphism $u_{ii} R_{\mathbb{A}} \cong (R_{\mathbb{A}} u_{\sigma(i)\sigma(i)})^*$ of right $R_{\mathbb{A}}$-modules, for all $1 \leq i \leq n$. Hence (i) follows. □

4. Infinite families of $\mathbb{A}$-full matrix algebras

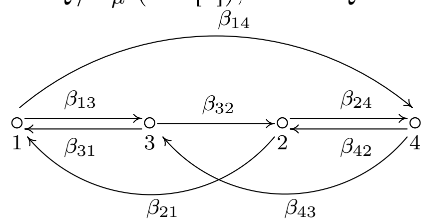
In this section, for $n = 4, 5$ and $n = 6$, we construct several interesting infinite families of $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ that are of infinite representation type. We also determine their representation type (tame or wild), by applying the well-known representation theory diagrammatic criteria, see [1], [11] and [12]. We end the section by presenting an idea of a construction of a large class of Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ such that $\dim_K R_{\mathbb{A}} = n^2$, $n \geq 4$, $\operatorname{soc} R_{\mathbb{A}} = J(R_{\mathbb{A}})^{n-2}$ and $J(R_{\mathbb{A}})^{n-1} = 0$. A characterization of all Frobenius algebras $R_{\mathbb{A}}$ with the above properties remains an open problem.

Example 4.1. Assume that $n = 4$ and K is a field. Consider the one-parameter family of $\mathbb{A}_\mu$-full matrix algebras $C_\mu = R_{\mathbb{A}_\mu}$, where $\mu \in K^* = K \setminus \{0\}$ and $\mathbb{A}_\mu$ is the following structure system

$$\mathbb{A}_\mu = \begin{bmatrix} 1111 & 0100 & 0110 & 0101 \\ 10\mu0 & 1111 & 0010 & 0011 \\ 1001 & 0101 & 1111 & 0001 \\ 1000 & 1100 & 1010 & 1111 \end{bmatrix}.$$

A simple calculation shows that, given $\mu \in K^*$, the matrix satisfies the conditions (A1)–(A3). We show that the algebra C_μ is isomorphic to the bound quiver K-algebra KQ/Ω_μ (see [1]), where Q is the quiver

$Q:$



and Ω_μ is the two-sided ideal of the path K-algebra KQ of Q generated by the following relations:

- $\beta_{21}\beta_{13} - \mu \cdot \beta_{24}\beta_{43}$,
- $\beta_{13}\beta_{32} - \beta_{14}\beta_{42}$,
- $\beta_{32}\beta_{24} - \beta_{31}\beta_{14}$,
- $\beta_{43}\beta_{31} - \beta_{42}\beta_{21}$,
- $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{24}\beta_{42}$, $\beta_{42}\beta_{24}$,
- $\beta_{21}\beta_{14}$, $\beta_{43}\beta_{32}$, $\beta_{32}\beta_{21}$, $\beta_{14}\beta_{43}$.

It is easy to check that the correspondences $\varepsilon_j \mapsto u_{jj}$ and $\beta_{ij} \mapsto u_{ij}$ define a K-algebra homomorphism $h : KQ/\Omega_\mu \to C_\mu$, where ε_j is the primitive idempotent of the path algebra KQ defined by the stationary path at the vertex j, for every $j \in Q_0$. Note that $\dim_K KQ/\Omega_\mu = 16$ and the cosets of the idempotents $\overline{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}$, the eigth arrows $\beta_{ij} \in Q_1$, together with the four cosets $\overline{\beta_{21}\beta_{13}}$, $\overline{\beta_{13}\beta_{32}}$, $\overline{\beta_{32}\beta_{24}}$, and $\overline{\beta_{43}\beta_{31}}$ form a K-basis of the quotient K-algebra KQ/Ω_μ.

Since $e_{23} = h(\overline{\beta_{21}\beta_{13}})$, $e_{12} = h(\overline{\beta_{13}\beta_{32}})$, $e_{34} = h(\overline{\beta_{32}\beta_{24}})$, and $e_{41} = h(\overline{\beta_{43}\beta_{31}})$ then the map h is surjective. Finally, since $\dim_K KQ/\Omega_\mu = \dim_K C_\mu = 16$, the surjection is an isomorphism of K-algebras.

It follows from the shape of Q and Ω_μ that, for each $\mu \in K^*$, $KQ/\Omega_\mu \cong C_\mu$ is a special biserial algebra [13], and therefore it is representation-tame, see [3, 5.2]. Since there is a cyclic walk

$$1 \xrightarrow{\beta_{13}} 3 \xleftarrow{\beta_{43}} 4 \xrightarrow{\beta_{42}} 2 \xleftarrow{\beta_{32}} 3 \xrightarrow{\beta_{31}} 1 \xleftarrow{\beta_{21}} 2 \xrightarrow{\beta_{24}} 4 \xleftarrow{\beta_{14}} 1$$

of the quiver Q then, according to the finite representation type criterion in [13] (see see also [10, Proposition 3.7]), the algebra C_μ is of infinite representation type. Note also that, for each $\mu \in K^*$, C_μ is self-injective, $J(C_\mu)^3 = 0$ and

$$J(C_\mu)^2 = \mathrm{soc}(C_\mu) = K\overline{\beta_{21}\beta_{13}} \oplus K\overline{\beta_{13}\beta_{32}} \oplus K\overline{\beta_{32}\beta_{24}} \oplus K\overline{\beta_{43}\beta_{31}},$$

see also [7, Section 5]. Consequently, the quotient algebras

$$\overline{C}_\mu = C_\mu/\mathrm{soc}\, C_\mu \quad \text{and} \quad \overline{C}_\gamma = C_\gamma/\mathrm{soc}\, C_\gamma$$

are isomorphic, for each pair $\mu, \gamma \in K^*$. In particular, it follows that the numbers of the indecomposable $\overline{C}_\mu$-modules and $\overline{C}_\gamma$-modules are equal and the stable Auslander-Reiten quivers of $\overline{C}_\mu$ and of $\overline{C}_\gamma$ are isomorphic.

Example 4.2. Assume that $n = 6$. Consider the one-parameter family of $\mathbb{A}_\mu$-full matrix algebras $H_\mu = R_{\mathbb{A}_\mu}$, where $\mu \in K$ and

$$\mathbb{A}_\mu = \begin{bmatrix} 111111 & 010000 & 011000 & 010100 & 011110 & 011101 \\ 100000 & 111111 & 001000 & 000100 & 001110 & 001101 \\ 100111 & 010111 & 111111 & 000100 & 000110 & 000101 \\ 101011 & 011011 & 001000 & 111111 & 001010 & 001001 \\ 100000 & 010000 & \mu 11000 & 110100 & 111111 & 000001 \\ 100010 & 010010 & 111010 & 110110 & 000010 & 111111 \end{bmatrix}.$$

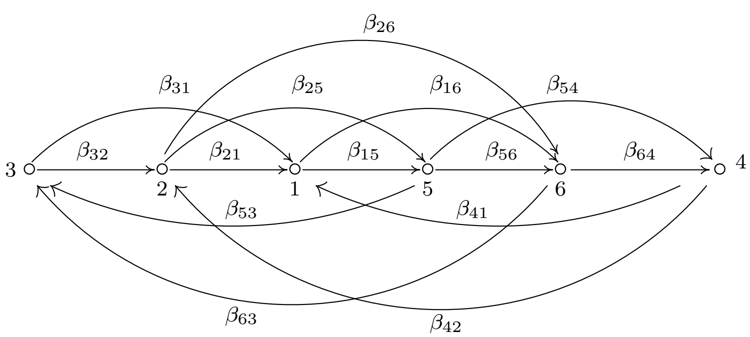
First we observe that:

(a) if K is infinite, then the family $\{H_\mu\}_{\mu \in K \setminus \{0,1\}}$ is infinite, because $H_\mu \cong H_\gamma$ if and only if $\mu = \gamma$, for $\mu, \gamma \in K \setminus \{0,1\}$ (apply Corollary 2.2),

(b) for each $\mu \in K \setminus \{0,1\}$, the algebra H_μ is not self-injective (the right ideals $u_{22}H_\mu$ and $u_{55}H_\mu$ are not injective, by Lemma 3.3, and

(c) for each $\mu \in K \setminus \{0,1\}$, the Gabriel quiver $\mathcal{Q}(H_\mu)$ of the algebra H_μ is the following quiver Q (apply [5, Proposition 1.2]).

$Q :$



Now we show that, for each $\mu \in K^*$, the algebra H_μ is of wild representation type, see [9, Section 14.2] and [12, Chapter XIX]. To see this, we note that $H_\mu/J(H_\mu)^2 \cong H_\nu/J(H_\nu)^2$, for all $\mu, \nu \in K^*$, and the algebra $B := H_\mu/J(H_\mu)^2$ has $J(B)^2 = 0$. It follows that $\mathcal{Q}(B) = \mathcal{Q}(H_{q_\mu})$. Since the separated quiver $\mathcal{Q}^s(B)$ of B (see [2, Section X.2]) contains a wild subquiver of the form

$$\begin{tikzcd}
& 5 \arrow[dl] \arrow[d] \arrow[dr] & 6 \arrow[dl] \arrow[d] \\
6' & 3' & 4'
\end{tikzcd}$$

then, by [9, Theorems 14.14 and 14.15] and [12, Chapter XIX], the algebra B is representation-wild and hence also H_μ is representation-wild, for each $\mu \in K^*$, because there is a fully faithful exact embedding $\operatorname{mod} B \hookrightarrow \operatorname{mod} H_\mu$.

Example 4.3. Assume that $n = 4$, K is a field and $\mathbb{A}$ is a structure system such that $R_{\mathbb{A}}$ is a Frobenius algebra and the Nakayama permutation of $R_{\mathbb{A}}$ is the cyclic permutation $\sigma = (1, 2, 3, 4)$, see [5, Theorem 3.4] and [7, Theorem 5.5]. The structure system $\mathbb{A}$ and the associated $(0, 1)$-structure system $\overline{\mathbb{A}}$ have the following forms

$$\mathbb{A} = \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \mu_6 & 1 & 0 & 0 & \mu_7 & 0 & 1 \\
1 & 0 & \mu_1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \mu_8 & 1 \\
1 & 0 & 0 & \mu_2 & 0 & 1 & 0 & \mu_4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & \mu_3 & 1 & 0 & 0 & \mu_5 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right].$$

$$\overline{\mathbb{A}} = \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right],$$

where $\mu_1, \ldots, \mu_8$ are arbitrary scalars in $K \setminus \{0\}$. By Proposition 3.4 (see also [5, Theorem 3.4] and [7, Theorem 5.5]), each of the algebras $R_{\mathbb{A}}$ in the defined eight parameter family is Frobenius and $\operatorname{soc} R_{\mathbb{A}} = J(R_{\mathbb{A}})^2$. One shows that there is a K-algebra isomorphism $R_{\overline{\mathbb{A}}} \cong KQ/\Omega_\sigma$, where Q is the quiver

$$Q: \quad \begin{tikzcd}
1 \arrow[rr, bend left, "\beta_{13}"] \arrow[rrr, bend left=60, "\beta_{14}"] & 2 \arrow[l, "\beta_{21}"'] \arrow[rr, bend left, "\beta_{24}"] & 3 \arrow[l, "\beta_{32}"'] \arrow[ll, bend left, "\beta_{31}"] & 4 \arrow[l, "\beta_{43}"'] \arrow[ll, bend left, "\beta_{42}"]
\end{tikzcd}$$

and Ω_σ is the two-sided ideal of the path K-algebra KQ of Q generated by the following elements:

1° $\beta_{21}\beta_{13}-\beta_{24}\beta_{43}$, $\beta_{13}\beta_{32}-\beta_{14}\beta_{42}$, $\beta_{32}\beta_{24}-\beta_{31}\beta_{14}$, $\beta_{43}\beta_{31}-\beta_{42}\beta_{21}$,
2° $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{24}\beta_{42}$, $\beta_{42}\beta_{24}$, $\beta_{21}\beta_{14}$, $\beta_{14}\beta_{43}$, $\beta_{32}\beta_{21}$, and $\beta_{43}\beta_{32}$.
It follows that

- the zero relation $\alpha_1\alpha_2\alpha_3$ belongs to Ω_σ, for each path $\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet$ in Q,
- there is an algebra isomorphism $\overline{R}_{\mathbb{A}} \cong KQ/\Omega_\sigma$, and
- $J(\overline{R}_{\mathbb{A}})^3 = 0$ and $J(\overline{R}_{\mathbb{A}})^2 = \mathrm{soc}(\overline{R}_{\mathbb{A}})$.

Now it is easy to see that $\overline{R}_{\mathbb{A}}$ is a special biserial algebra, and therefore it is representation-tame, see [3, 5.2]. Since there is a cyclic walk

$$\begin{tikzcd}[column sep=small]
1 \arrow[dr] & & 2 \arrow[dl] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \\
& 4 & & 1 & & 2 & & 3 &
\end{tikzcd}$$

in Q then, according to [13], the Frobenius algebra $\overline{R}_{\mathbb{A}}$ is of infinite representation type, see also [10, Proposition 3.7].

We end this section by presenting an idea of a construction, for $n = 5$, of tame Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ of infinite representation type such that $J(R_{\mathbb{A}})^4 = 0$ and $J(R_{\mathbb{A}})^3 = \mathrm{soc}\, R_{\mathbb{A}}$.

Example 4.4. Assume that $n = 5$ and K is a field. We construct a set of structure systems $q = [q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}, q^{(5)}]$ such that R_q is a Frobenius algebra, $J(R_q)^4 = 0$, $J(R_q)^3 = \mathrm{soc}\,(R_q)$, and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$ is the Nakayama permutation of R_q.

Suppose that $q = [q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}, q^{(5)}] = [q_{ij}^{(k)}]_{i,j,k}$ is such a structure system and let

$$R_q = \bigoplus_{i,j=1}^{5} K e_{ij}$$

be the corresponding q-full matrix K-algebra with the basis $\{e_{ij} \mid 1 \le i, j \le 5\}$. We recall that the elements $e_1 = e_{11}, \ldots, e_5 = e_{55}$ form a complete set of pairwise orthogonal primitive idempotents of the algebra R_q and $1 = e_1 + e_2 + e_3 + e_4 + e_5$ is the identity of R_q. We denote by $\cdot_q$ the multiplication in R_q.

One shows that $\mathrm{soc}(e_j R_q) = K e_{j\,\sigma(j)}$ (see [7, Theorem 5.3]) and therefore $e_{j\,\sigma(j)} \cdot_q J(R_q) = 0$ and $J(R_q) \cdot_q e_{j\,\sigma(j)} = 0$, for $j = 1, \ldots, 5$. Hence we get the equalities $q_{jr}^{(\sigma(j))} = 0$, for all $r \neq \sigma(j)$, and $q_{s\,\sigma(j)}^{(j)} = 0$, for all $s \neq j$, that is,

- $q^{(1)}_{32} = q^{(1)}_{34} = q^{(1)}_{35} = 0$ and $q^{(1)}_{34} = q^{(1)}_{24} = q^{(1)}_{54} = 0$,
- $q^{(2)}_{41} = q^{(2)}_{43} = q^{(2)}_{45} = 0$ and $q^{(2)}_{45} = q^{(2)}_{35} = q^{(2)}_{15} = 0$,
- $q^{(3)}_{51} = q^{(3)}_{52} = q^{(3)}_{54} = 0$ and $q^{(3)}_{51} = q^{(3)}_{41} = q^{(3)}_{21} = 0$,
- $q^{(4)}_{12} = q^{(4)}_{13} = q^{(4)}_{15} = 0$ and $q^{(4)}_{12} = q^{(4)}_{52} = q^{(4)}_{32} = 0$,
- $q^{(5)}_{23} = q^{(5)}_{24} = q^{(5)}_{21} = 0$ and $q^{(5)}_{23} = q^{(5)}_{43} = q^{(5)}_{13} = 0$.

Consequently, the block matrix q has the form

$$q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & 0 & 1 & * & * & 0 & 0 & * & 1 & * & * & 0 & 0 & 0 & 1 & 0 & 0 & * & 0 & * & 1 \\ 1 & 0 & * & 0 & * & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & * & * & * & 0 & * & 1 & * & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & * & 0 & 1 & 1 & 1 & 1 & 1 & * & 0 & 0 & 1 & * & * & * & 0 & * & 1 \\ 1 & * & * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & * & 1 & 1 & 1 & 1 & 1 & * & * & 0 & 0 & 1 \\ 1 & * & * & 0 & 0 & * & 1 & * & * & 0 & 0 & 0 & 1 & 0 & 0 & * & 0 & * & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

Since we assume that $\operatorname{soc}(e_j R_q) = K e_{j\,\sigma(j)} \subseteq e_j J(R_q)^3$, for $j = 1, \ldots, 5$, then $K e_{j\,\sigma(j)} = K(e_{j\,j_1} \cdot_q e_{j_1\,j_2} \cdot_q e_{j_2\,\sigma(j)})$, where $j_1 \neq j_2$ and $j_1, j_2 \notin \{j, \sigma(j)\}$.

Assume, for simplicity, that there exist non-zero scalars $\lambda_{14}, \lambda_{25}, \lambda_{31}, \lambda_{42}, \lambda_{53} \in K$ such that

$\lambda_{14} e_{14} = e_{12} \cdot_q e_{23} \cdot_q e_{34}$,
$\lambda_{25} e_{25} = e_{23} \cdot_q e_{34} \cdot_q e_{45}$,
$\lambda_{31} e_{31} = e_{34} \cdot_q e_{45} \cdot_q e_{51}$,
$\lambda_{42} e_{42} = e_{45} \cdot_q e_{51} \cdot_q e_{12}$,
$\lambda_{53} e_{53} = e_{51} \cdot_q e_{12} \cdot_q e_{23}$.

Hence we conclude that

$q^{(3)}_{12} = q^{(5)}_{12} = 0$, $q^{(2)}_{34} = q^{(5)}_{34} = 0$, $q^{(1)}_{23} = q^{(4)}_{23} = 0$, $q^{(1)}_{45} = q^{(3)}_{45} = 0$, $q^{(2)}_{51} = q^{(4)}_{51} = 0$.

Indeed, if we assume to the contrary that $q^{(3)}_{12} \neq 0$ then $e_{13} \cdot_q e_{32} = q^{(3)}_{12} e_{12}$ and then the non-zero element $\lambda_{14} q^{(3)}_{12} e_{14} = e_{13} \cdot_q e_{32} \cdot_q e_{23} \cdot_q e_{34}$ belongs to $J(R_q)^4 = 0$, and we get a contradiction. The remaining equalities follow in a similar way.

Moreover, since the elements $\lambda_{14}, \lambda_{25}, \lambda_{31}, \lambda_{42}, \lambda_{53} \in K$ are non-zero then, by the associativity of $\cdot_q$, the equalities above yields

$$q^{(2)}_{13} q^{(3)}_{14} = q^{(2)}_{14} q^{(3)}_{24} \neq 0, \quad q^{(3)}_{24} q^{(4)}_{25} = q^{(3)}_{25} q^{(4)}_{35} \neq 0, \quad q^{(4)}_{35} q^{(5)}_{31} = q^{(4)}_{31} q^{(5)}_{41} \neq 0,$$
$$q^{(5)}_{41} q^{(1)}_{42} = q^{(5)}_{42} q^{(1)}_{52} \neq 0, \quad q^{(1)}_{52} q^{(2)}_{53} = q^{(1)}_{53} q^{(2)}_{13} \neq 0.$$

Equivalently, we get the equalities

$q^{(3)}_{24} = \frac{q^{(2)}_{13} q^{(3)}_{14}}{q^{(2)}_{14}}$,
$q^{(4)}_{35} = \frac{q^{(3)}_{24} q^{(4)}_{25}}{q^{(3)}_{25}} = \frac{q^{(2)}_{13} q^{(3)}_{14} q^{(4)}_{25}}{q^{(2)}_{14} q^{(3)}_{25}}$,
$q^{(5)}_{41} = \frac{q^{(4)}_{35} q^{(5)}_{31}}{q^{(4)}_{31}} = \frac{q^{(2)}_{13} q^{(3)}_{14} q^{(4)}_{25} q^{(5)}_{31}}{q^{(2)}_{14} q^{(3)}_{25} q^{(4)}_{31}}$,
$q^{(1)}_{52} = \frac{q^{(5)}_{41} q^{(1)}_{42}}{q^{(5)}_{42}} = \frac{q^{(2)}_{13} q^{(3)}_{14} q^{(4)}_{25} q^{(5)}_{31} q^{(1)}_{42}}{q^{(2)}_{14} q^{(3)}_{25} q^{(4)}_{31} q^{(5)}_{42}}$,

$$q_{13}^{(2)} = \frac{q_{52}^{(1)} q_{53}^{(2)}}{q_{53}^{(1)}} = \frac{q_{13}^{(2)} q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)} q_{42}^{(1)} q_{53}^{(2)}}{q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)} q_{42}^{(5)} q_{53}^{(1)}}.$$

It follows that if $q_{13}^{(2)} \in K^*$ is arbitrary, then the remaining non-zero scalars $q_{ij}^{(s)}$ that appear in the equalities above satisfy the condition

$$(*) \qquad q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)} q_{42}^{(1)} q_{53}^{(2)} = q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)} q_{42}^{(5)} q_{53}^{(1)}$$

Now we show that $q_{43}^{(1)} = 0$, $q_{54}^{(2)} = 0$, $q_{15}^{(3)} = 0$, $q_{21}^{(4)} = 0$ and $q_{32}^{(5)} = 0$. To see this, assume to the contrary that $q_{43}^{(1)} \neq 0$. Then $0 \neq q_{43}^{(1)} e_{43} = e_{41} \cdot_q e_{13}$. It follows that the non-zero element $e_{45} \cdot_q e_{51} \cdot_q e_{12} \cdot_q e_{23} = q_{41}^{(5)} q_{13}^{(2)} e_{41} \cdot_q e_{13} = q_{41}^{(5)} q_{13}^{(2)} q_{43}^{(1)} e_{43}$ belongs to $J(R_q)^4 = 0$, and we get a contradiction. The equalities $q_{54}^{(2)} = 0$, $q_{15}^{(3)} = 0$, $q_{21}^{(4)} = 0$, $q_{32}^{(5)} = 0$ follow in a similar way. Consequently, the block matrix q has the form

$$q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 & q_{13}^{(2)} & q_{14}^{(2)} & 0 & 0 & 0 & 1 & q_{14}^{(3)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 \\
1 & 0 & 0 & 0 & * & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & q_{24}^{(3)} & q_{25}^{(3)} & 0 & 0 & 0 & 1 & q_{25}^{(4)} & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{31}^{(4)} & 0 & 0 & 1 & q_{35}^{(4)} & q_{31}^{(5)} & 0 & 0 & 0 & 1 \\
1 & q_{42}^{(1)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{41}^{(5)} & q_{42}^{(5)} & 0 & 0 & 1 \\
1 & q_{52}^{(1)} & q_{53}^{(1)} & 0 & 0 & 0 & 1 & q_{53}^{(2)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right].$$

Now we claim that each of the scalars $q_{25}^{(1)}, q_{31}^{(2)}, q_{42}^{(3)}, q_{53}^{(4)}, q_{14}^{(5)}$ is non-zero. Assume, to the contrary, that some of them is zero, say $q_{14}^{(5)} = 0$. It follows from the shape of q that $q_{1r}^{(5)} = 0$, for all $r \neq 5$, and consequently $e_{15} \cdot_q J(R_q) = 0$. It follows that $S' = e_{15}K \subseteq e_1 R_q$ is a simple submodule of $e_1 R_q$; contrary to the assumption that $\mathrm{soc}(e_1 R_q) = e_{14}K$. This finishes the proof of our claim. Consequently, the block matrix q has the form

$$q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 & q_{13}^{(2)} & q_{14}^{(2)} & 0 & 0 & 0 & 1 & q_{14}^{(3)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{14}^{(5)} & 1 \\
1 & 0 & 0 & 0 & q_{25}^{(1)} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & q_{24}^{(3)} & q_{25}^{(3)} & 0 & 0 & 0 & 1 & q_{25}^{(4)} & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & q_{31}^{(2)} & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{31}^{(4)} & 0 & 0 & 1 & q_{35}^{(4)} & q_{31}^{(5)} & 0 & 0 & 0 & 1 \\
1 & q_{42}^{(1)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{42}^{(3)} & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{41}^{(5)} & q_{42}^{(5)} & 0 & 0 & 1 \\
1 & q_{52}^{(1)} & q_{53}^{(1)} & 0 & 0 & 0 & 1 & q_{53}^{(2)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{53}^{(4)} & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right].$$

where $q_{25}^{(1)}, q_{31}^{(2)}, q_{42}^{(3)}, q_{53}^{(4)}, q_{14}^{(5)}$ and $q_{13}^{(2)}$ are arbitrary non-zero scalars in K, the coefficients

$$q_{14}^{(3)}, q_{25}^{(4)}, q_{31}^{(5)}, q_{42}^{(1)}, q_{53}^{(2)}, q_{14}^{(2)}, q_{25}^{(3)}, q_{31}^{(4)}, q_{42}^{(5)}, q_{53}^{(1)}$$

satisfy the equation $(*)$ and the coefficients $q_{52}^{(1)}, q_{13}^{(2)}, q_{24}^{(3)}, q_{35}^{(4)}, q_{41}^{(5)}$ depend of the remaining ones by the formulas preceding the equation $(*)$.

Conversely, if q is a block matrix of the above form, where $q_{25}^{(1)}$, $q_{31}^{(2)}$, $q_{42}^{(3)}$, $q_{53}^{(4)}$, $q_{14}^{(5)}$ and $q_{13}^{(2)}$ are arbitrary non-zero scalars in K, and the remaining ones satisfy the above conditions then q is a structure system and R_q is a Frobenius algebra such that $J(R_q)^4 = 0$ and $\mathrm{soc}(R_q) = J(R_q)^3$. The associated $(0,1)$-matrix $\overline{q}$ structure system has the following form

$$\overline{q} = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right].$$

It follows that the Gabriel quiver $\mathcal{Q}(R_q)$ of R_q has the form

$$\mathcal{Q}(R_q):\quad \begin{tikzcd} 1 \arrow[r, shift left, "\beta_{12}"] \arrow[rrrr, bend left, "\beta_{15}"] & 2 \arrow[l, shift left, "\beta_{21}"] \arrow[r, shift left, "\beta_{23}"] & 3 \arrow[l, shift left, "\beta_{32}"] \arrow[r, shift left, "\beta_{34}"] & 4 \arrow[l, shift left, "\beta_{43}"] \arrow[r, shift left, "\beta_{45}"] & 5. \arrow[l, shift left, "\beta_{54}"] \arrow[llll, bend left, "\beta_{51}"] \end{tikzcd}$$

To view the algebra R_q as a path algebra KQ/Ω_q of a bound quiver, we note that

$$\begin{aligned}
&q_{14}^{(5)} e_{12} \cdot_q e_{23} \cdot_q e_{34} = q_{13}^{(2)} q_{14}^{(3)} e_{15} \cdot_q e_{54},\\
&q_{25}^{(1)} e_{23} \cdot_q e_{34} \cdot_q e_{45} = q_{24}^{(3)} q_{25}^{(4)} e_{21} \cdot_q e_{15},\\
&q_{31}^{(2)} e_{34} \cdot_q e_{45} \cdot_q e_{51} = q_{35}^{(4)} q_{31}^{(5)} e_{32} \cdot_q e_{21},\\
&q_{42}^{(3)} e_{45} \cdot_q e_{51} \cdot_q e_{12} = q_{41}^{(5)} q_{42}^{(1)} e_{43} \cdot_q e_{32},\\
&q_{53}^{(4)} e_{51} \cdot_q e_{12} \cdot_q e_{23} = q_{52}^{(1)} q_{53}^{(2)} e_{54} \cdot_q e_{43}.
\end{aligned}$$

To see the first equality, we note that $e_{15} \cdot_q e_{54} = q_{14}^{(5)} e_{14}$ and $e_{12} \cdot_q e_{23} \cdot_q e_{34} = q_{13}^{(2)} q_{14}^{(3)} e_{14}$. Hence the first equality follows, and the remaining ones follow in a similar way.

Now we prove that there is a K-algebra isomorphism $R_q \cong KQ/\Omega_q$, where $Q = \mathcal{Q}(R_q)$ and Ω_q is the two-sided ideal of the path K-algebra KQ of Q generated by the following relations:

- $\beta_{j+1\,j}\beta_{j\,j+1}$ and $\beta_{j\,j+1}\beta_{j+1\,j}$, for $j = 1, \dots, 5$, where $j+1$ is reduced modulo 5.
- $\beta_1\beta_2\beta_3\beta_4$, if there is a path $\bullet \xrightarrow{\beta_1} \bullet \xrightarrow{\beta_2} \bullet \xrightarrow{\beta_3} \bullet \xrightarrow{\beta_4} \bullet$ in Q.
- $\beta_{21}\beta_{15}\beta_{54}$, $\beta_{32}\beta_{21}\beta_{15}$, $\beta_{43}\beta_{32}\beta_{21}$, $\beta_{54}\beta_{43}\beta_{32}$, $\beta_{15}\beta_{54}\beta_{43}$;
- $q_{14}^{(5)}\beta_{12}\beta_{23}\beta_{34} - q_{13}^{(2)} q_{14}^{(3)}\beta_{15}\beta_{54}$,
- $q_{25}^{(1)}\beta_{23}\beta_{34}\beta_{45} - q_{24}^{(3)} q_{25}^{(4)}\beta_{21}\beta_{15}$,
- $q_{31}^{(2)}\beta_{34}\beta_{45}\beta_{51} - q_{35}^{(4)} q_{31}^{(5)}\beta_{32}\beta_{21}$,
- $q_{42}^{(3)}\beta_{45}\beta_{51}\beta_{12} - q_{41}^{(5)} q_{42}^{(1)}\beta_{43}\beta_{32}$,
- $q_{53}^{(4)}\beta_{51}\beta_{12}\beta_{23} - q_{52}^{(1)} q_{53}^{(2)}\beta_{54}\beta_{43}$.

It is easy to check that the correspondences $\varepsilon_j \mapsto e_j$ and $\beta_{ij} \mapsto e_{ij}$ define a K-algebra homomorphism $h : KQ/\Omega_q \to R_q$, where ε_j is the primitive idempotent of the path algebra KQ defined by the stationary path at the vertex j, for every $j \in Q_0$. Note that the map h is well defined and surjective. Finally, since $\dim_K KQ/\Omega_q = \dim_K R_q = 25$, the surjection h is an isomorphism of K-algebras.

Now it is easy to see that $R_q/\mathrm{soc}(R_q) \cong R_{\overline{q}}/\mathrm{soc}(R_{\overline{q}})$ and the algebra $KQ/\Omega_q \cong R_q$ is special biserial; hence R_q is representation-tame, see [3, 5.2]. Since there is a cyclic walk

$$1 \xrightarrow{\beta_{12}} 2 \xrightarrow{\beta_{23}} 3 \xleftarrow{\beta_{43}} 4 \xrightarrow{\beta_{45}} 5 \xleftarrow{\beta_{51}} 1$$

of the quiver Q and, according to the finite representation type criterion in [13], the algebra R_q is of infinite representation type, see also [10, Proposition 3.7].

Problem 4.5. Give a characterisation of the Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ such that $\dim_K R_{\mathbb{A}} = n^2$, $n \geq 3$, $\mathrm{soc}\, R_{\mathbb{A}} = J(R_{\mathbb{A}})^{n-2}$ and $J(R_{\mathbb{A}})^{n-1} = 0$.

Remark 4.6. In connection with Problem 4.5, we recall that if $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra and $R_{\overline{\mathbb{A}}}$ is the $(0,1)$-limit of $R_{\mathbb{A}}$ then

- $J(R_{\mathbb{A}})^s = J(R_{\overline{\mathbb{A}}})^s$, for each $s \geq 1$ (by [7, Proposition 3.2]),
- $\mathrm{soc}\, R_{\mathbb{A}} = \mathrm{soc}\, R_{\overline{\mathbb{A}}}$ (by [7, Proposition 5.1]), and
- $R_{\mathbb{A}}$ is a Frobenius algebra if and only if the $(0,1)$-limit $R_{\overline{\mathbb{A}}}$ of $R_{\mathbb{A}}$ is a Frobenius algebra (by [7, Theorem 5.3]).

It follows that a solution of the Problem 4.5 for $(0,1)$-structure systems should help to find a solution for arbitrary structure systems $\mathbb{A}$.

We recall from [5] that in case $n = 5$, a list of $(0,1)$-structure systems $\mathbb{A}$ such that $R_{\mathbb{A}}$ is a Frobenius algebra is given in Examples 4.7(4) and 4.7(5) of [5]. It is shown there that, up to isomorphisms of the $\mathbb{A}$-full matrix algebras, there are precisely four Frobenius $(0,1)$-structure systems $\mathbb{A}$. Note that one of them has the property $\mathrm{soc}\, R_{\mathbb{A}} = J(R_{\mathbb{A}})^3$, compare with the $(0,1)$-limit algebra $R_{\overline{q}}$ in Example 4.4.

References

[1] I. Assem, D. Simson and A. Skowroński, *"Elements of the Representation Theory of Associative Algebras"*, Volume 1. Techniques of Representation Theory, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge-New York, 2006.

[2] M. Auslander, I. Reiten and S. Smalø, *"Representation Theory of Artin Algebras"*, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, 1995.

[3] P. Dowbor and A. Skowroński, Galois coverings of representation-infinite algebras, *Comment. Math. Helvetici,* 62(1987), 311–337.

[4] Y. A. Drozd and V. V. Kirichenko, *"Finite Dimensional Algebras",* Springer-Verlag Berlin Heidelberg 1994.

[5] H. Fujita, Full matrix algebras with structure systems, *Colloq. Math.* 98(2003), 249–258.

[6] H. Fujita and Y. Sakai, Frobenius full matrix algebras and Gorenstein tiled orders, *Comm. Algebra*, 34(2006), 1181–1203.

[7] H. Fujita, Y. Sakai and D. Simson, Minor degenerations of the full matrix algebra over a field, *J. Math. Soc. Japan,* 59(2007), 1–33.

[8] K. W. Roggenkamp, V. V. Kirichenko, M. A. Khibina and V. N. Zhuravlev, Gorenstein tiled orders, *Comm. Algebra*, 29(2001), 4231–4247.

[9] D. Simson, *"Linear representations of partially ordered sets and vector space categories"*, Algebra, Logic and Applications, vol. 4, Gordon & Breach Science Publishers, 1992.

[10] D. Simson, On Corner type Endo-Wild algebras, *J. Pure Appl. Algebra*, 202(2005), 118–132.

[11] D. Simson and A. Skowroński, *"Elements of the Representation Theory of Associative Algebras", Volume 2. Tubes and Concealed Algebras of Euclidean Type*, London Math. Soc. Student Texts 71, Cambridge Univ. Press, Cambridge-New York, 2007.

[12] D. Simson and A. Skowroński, *"Elements of the Representation Theory of Associative Algebras", Volume 3. Representation-Infinite Tilted Algebras*, London Math. Soc. Student Texts 72, Cambridge Univ. Press, Cambridge-New York, 2007.

[13] A. Skowroński and J. Waschbüsch, Representation-finite biserial algebras, *J. reine angew. Math.,* 345(1985), 480–500.

Contact information

H. Fujita Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571
E-Mail: `fujita@math.tsukuba.ac.jap`

Y. Sakai Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571
E-Mail: `ysksakai@math.tsukuba.ac.jap`

D. Simson Faculty of Mathematics and Computer Sciences, Nicolaus Copernicus University, 87-100 Toruń, Poland
E-Mail: `simson@mat.uni.torun.pl`

Received by the editors: 29.10.2006
and in final form 28.05.2007.