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Integral group ring of the McLaughlin simple group

V. A. Bovdi, A. B. Konovalov

Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. We consider the Zassenhaus conjecture for the normalized unit group of the integral group ring of the McLaughlin sporadic group `McL`. As a consequence, we confirm for this group the Kimmerle's conjecture on prime graphs.

1. Introduction, conjectures and main results

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. A long-standing conjecture of H. Zassenhaus **(ZC)** says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [26]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [22] to solve it for A_5. Later M. Hertweck in [17] extended the Luthar–Passi method and applied it for the investigation of the Zassenhaus conjecture for $PSL(2,p^n)$. The Luthar–Passi method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [5, 7, 16, 17, 18, 19]. Also, some related properties and some weakened variations of the Zassenhaus conjecture can be found in [1, 23] and [3, 21].

First of all, we need to introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the

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primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G. In [21] W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

In particular, in the same paper W. Kimmerle verified that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 8, 9, 10, 12] **(KC)** was confirmed for the Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24} and the sporadic Janko simple groups J_1, J_2 and J_3.

Here we continue these investigations for the McLaughlin simple group $\mathtt{McL}$. Although using the Luthar–Passi method we cannot prove the rational conjugacy for torsion units of $V(\mathbb{Z}\mathtt{McL})$, our main result gives a lot of information on partial augmentations of these units. In particular, we confirm the Kimmerle's conjecture for this group.

Let $G = \mathtt{McL}$. It is well known (see [15]) that $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ and $exp(G) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. Let

$$\begin{aligned}\mathcal{C} = \{&C_1, C_{2a}, C_{3a}, C_{3b}, C_{4a}, C_{5a}, C_{5b}, C_{6a}, C_{6b}, C_{7a}, C_{7b}, C_{8a}, C_{9a}, C_{9b},\\ &C_{10a}, C_{11a}, C_{11b}, C_{12a}, C_{14a}, C_{14b}, C_{15a}, C_{15b}, C_{30a}, C_{30b}\}\end{aligned}$$

be the collection of all conjugacy classes of $\mathtt{McL}$, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [2] and [25], Ch.5, p.102) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \qquad (1)$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following

Theorem 1. Let G denote the McLaughlin simple group $\mathtt{McL}$. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the tuple

$$\begin{aligned}(&\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{7b}, \nu_{8a}, \nu_{9a}, \nu_{9b},\\ &\nu_{10a}, \nu_{11a}, \nu_{11b}, \nu_{12a}, \nu_{14a}, \nu_{14b}, \nu_{15a}, \nu_{15b}, \nu_{30a}, \nu_{30b}) \in \mathbb{Z}^{23}\end{aligned}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. The following properties hold.

(i) There is no elements of orders 21, 22, 33, 35, 55, 77 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{18, 20, 24, 28, 36, 40, 45, 56, 60, 72, 90, 120, 180, 360\}$, then $|u|$ coincides with the order of some element $g \in G$.

(ii) If $|u| = 2$, then u is rationally conjugate to some $g \in G$.

(iii) If $|u| = 3$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{3a} and ν_{3b}, and the pair (ν_{3a}, ν_{3b}) is one of

$$\{\ (-2,3),\ (-1,2),\ (0,1),\ (1,0)\ \}.$$

(iv) If $|u| = 5$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{5a} and ν_{5b}, and the pair (ν_{5a}, ν_{5b}) is one of

$$\{\ (-4,5),\ (-3,4),\ (-2,3),\ (-1,2),\ (0,1),\ (1,0)\ \}.$$

(v) If $|u| = 7$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{7a} and ν_{7b}, and the pair (ν_{7a}, ν_{7b}) is one of

$$\{\ (\nu_{7a}, \nu_{7b})\ \mid\ -86 \le \nu_{7a} \le 87, \quad \nu_{7a} + \nu_{7b} = 1\ \}.$$

(vi) If $|u| = 11$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{11a} and ν_{11b}, and the pair (ν_{11a}, ν_{11b}) is one of

$$\{\ (\nu_{11a}, \nu_{11b})\ \mid\ -9 \le \nu_{11a} \le 10, \quad \nu_{11a} + \nu_{11b} = 1\ \}.$$

As an immediate consequence of part (i) of the Theorem we obtain

Corollary 1. If $G = \mathtt{McL}$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.

2. Preliminaries

The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1. (see [22] and Theorem 2.5 in [24]) Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.

The next result now yield that several partial augmentations are zero.

Proposition 2. (see [16], Proposition 3.1; [17], Proposition 2.2) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3. (see [22, 17]) Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G, then for every integer l the number

$$\mu_l(u,\chi,p) = \tfrac{1}{k}\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\} \qquad (2)$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u,\chi,*)$ for $\mu_l(u,\chi,0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4. (see [13]) The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.

3. Proof of the Theorem

Throughout this section we denote `McL` by G. The character table of G, as well as the p-Brauer character tables, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2,3,5,7,11\}$, can be found using the computational algebra system GAP [15], which derives these data from [14, 20]. For the characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

First of all we will investigate units of orders 2, 3, 5, 7 and 11, since the group G possesses elements of these orders. After this, by Proposition 4, the order of each torsion unit divides the exponent of G, so to prove the Kimmerle's conjecture, it remains to consider units of orders 21, 22, 33, 35, 55 and 77. We prove that no units of all these orders do appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

• Let u be an involution. Using Proposition 2 we obtain that all partial augmentations except one are zero. Thus by Proposition 1 the proof of part (ii) of Theorem 1 is done.

• Let u be a unit of order 3. By (1) and Proposition 2 we get $\nu_{3a}+\nu_{3b} = 1$. Put $t_1 = 5\nu_{3a} - 4\nu_{3b}$. By (2) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{3}(-2t_1+22) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{3}(t_1+22) \geq 0,$$

from which $t_1 \in \{-22, -19, -16, -13, -10, -7, -4, -1, 2, 5, 8, 11\}$. Now for each possible value of t_1 consider the system of linear equations

$$\nu_{3a} + \nu_{3b} = 1, \qquad 5\nu_{3a} - 4\nu_{3b} = t_1.$$

Since $\left|\begin{smallmatrix} 1 & 1 \\ 5 & -4 \end{smallmatrix}\right| \neq 0$, this system always has the unique solution. First we select only integer solutions, and then using the condition that all $\mu_i(u, \chi_j, *)$ are non-negative integers, we obtain only four pairs (ν_{3a}, ν_{3b}) listed in part (iii) of Theorem 1.

• Let u be a unit of order 5. By (1) and Proposition 2 we get $\nu_{5a} + \nu_{5b} = 1$. Put $t_1 = 3\nu_{5a} - 2\nu_{5b}$. By (2) we obtain the system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{5}(-4t_1 + 22) \geq 0; \qquad \mu_1(u, \chi_2, *) = \tfrac{1}{5}(t_1 + 22) \geq 0,$$

so $t_1 \in \{-22, -17, -12, -7, -2, 3\}$. Using the same arguments as in the previous case, we obtain only six pairs (ν_{5a}, ν_{5b}) listed in part (iv) of Theorem 1.

• Let u be a unit of order 7. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{7b} = 1$. Put $t_1 = 4\nu_{7a} - 3\nu_{7b}$. Using $\mathfrak{BCT}(3)$ and $\mathfrak{BCT}(5)$, by (2) we have

$$\mu_3(u, \chi_7, 3) = \tfrac{1}{7}(t_1 + 605) \geq 0; \quad \mu_1(u, \chi_{12}, 5) = \tfrac{1}{7}(-t_1 + 3245) \geq 0;$$
$$\mu_1(u, \chi_7, 3) = \tfrac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 605) \geq 0.$$

It follows that we have only 174 pairs (ν_{7a}, ν_{7b}), given in part (v) of Theorem 1. Note that using our implementation of the Luthar–Passi method, which we intended to make available in the GAP package LAGUNA [11], we checked that it is not possible to further reduce the number of solutions, and the same remark also applies for the remaining part of the paper.

• Let u be a unit of order 11. By (1) and Proposition 2 we have $\nu_{11a} + \nu_{11b} = 1$. Using $\mathfrak{BCT}(3)$, by (2) we obtain the system of inequalities

$$\mu_1(u, \chi_3, 3) = \tfrac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 104) \geq 0;$$
$$\mu_2(u, \chi_3, 3) = \tfrac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 104) \geq 0,$$

that has only that twenty pairs (ν_{11a}, ν_{11b}) listed in part (vi) of the Theorem 1.

• Let u be a unit of order 21. By (1) and Proposition 2 we have

$$\nu_{3a} + \nu_{3b} + \nu_{7a} + \nu_{7b} = 1.$$

Put $t_1 = 5\nu_{3a} - 4\nu_{3b} - \nu_{7a} - \nu_{7b}$, $t_2 = 5\nu_{3a} + 2\nu_{3b}$ and $t_3 = 3\nu_{7a} - 4\nu_{7b}$. Since $|u^7| = 3$, for any character χ of G we need to consider four cases, defined by part (iii) of the Theorem. Now we consider each case separately:

Case 1. Let $\chi(u^7) = \chi(3a)$. Using (2), we obtain the system of inequalities

$$\mu_3(u,\chi_2,*) = \tfrac{1}{21}(2t_1+11) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{21}(-12t_1+18) \geq 0,$$

which has no integral solution.

Case 2. Let $\chi(u^7) = \chi(3b)$. Again, using (2), we obtain the system of inequalities

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{21}(-12t_1+36) \geq 0; & \mu_7(u,\chi_2,*) &= \tfrac{1}{21}(6t_1+24) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{21}(36t_2+243) \geq 0; & \mu_7(u,\chi_3,*) &= \tfrac{1}{21}(-18t_2+225) \geq 0;\\
\mu_1(u,\chi_{16},*) &= \tfrac{1}{21}(-t_3+8386) \geq 0; & \mu_9(u,\chi_{16},*) &= \tfrac{1}{21}(2t_3+8386) \geq 0;
\end{aligned}$$
$$\mu_1(u,\chi_5,*) = \tfrac{1}{21}(-13\nu_{3a}+5\nu_{3b}+765) \geq 0.$$

This yields $t_1 \in \{-4,3\}$, $t_2 \in \{-5,2,9\}$ and $t_3 \in \{7+21k \mid -200 \leq k \leq 399\}$, but none of possible combinations of t_i's gives us any solution.

Case 3. Let $\chi(u^7) = -2\chi(3a)+3\chi(3b)$. Then using (2), we obtain the system

$$\begin{aligned}
\mu_1(u,\chi_2,*) &= \tfrac{1}{21}(-t_1-1) \geq 0; & \mu_7(u,\chi_2,*) &= \tfrac{1}{21}(6t_1+6) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{21}(36t_2+207) \geq 0; & \mu_7(u,\chi_3,*) &= \tfrac{1}{21}(-18t_2+243) \geq 0;\\
\mu_1(u,\chi_{16},*) &= \tfrac{1}{21}(-t_3+8218) \geq 0; & \mu_9(u,\chi_{16},*) &= \tfrac{1}{21}(2t_3+8218) \geq 0,
\end{aligned}$$

from which $t_1 = -1$, $t_2 \in \{-4,3,10\}$ and $t_3 \in \{7+21k \mid -196 \leq k \leq 391\}$, and again we have no solution for every combination of t_i's.

Case 4. Let $\chi(u^7) = -\chi(3a)+2\chi(3b)$. Using (2), we obtain the system

$$\begin{aligned}
\mu_7(u,\chi_2,*) &= \tfrac{1}{21}(6t_1+15) \geq 0; & \mu_0(u,\chi_2,*) &= \tfrac{1}{21}(-12t_1+54) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{21}(36t_2+225) \geq 0; & \mu_7(u,\chi_3,*) &= \tfrac{1}{21}(-18t_2+234) \geq 0;\\
\mu_9(u,\chi_{16},*) &= \tfrac{1}{21}(2t_3+8015) \geq 0; & \mu_1(u,\chi_{16},*) &= \tfrac{1}{21}(-t_3+8015) \geq 0,
\end{aligned}$$

so $t_1 = 1$, $t_2 \in \{-1,6,13\}$ and $t_3 \in \{14+21k \mid -191 \leq k \leq 381\}$, that also gives us no solutions.

• Let u be a unit of order 22. By (1) and Proposition 2 we have

$$\nu_{2a}+\nu_{11a}+\nu_{11b} = 1.$$

Now by (2) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{22}(60\nu_{2a}+28) \geq 0; \qquad \mu_{11}(u,\chi_2,*) = \tfrac{1}{22}(-60\nu_{2a}+16) \geq 0,$$

which has no integral solution.

• Let u be a unit of order 33. By (1) and Proposition 2 we have

$$\nu_{3a}+\nu_{3b}+\nu_{11a}+\nu_{11b}=1.$$

Put $t_1=5\nu_{3a}-4\nu_{3b}$, $t_2=5\nu_{3a}+2\nu_{3b}$ and $t_3=32\nu_{3a}-4\nu_{3b}-6\nu_{11a}+5\nu_{11b}$. Since $|u^{11}|=3$, for any character χ of G we need to consider four cases, defined by part (iii) of the Theorem.

Case 1. Let $\chi(u^{11})=\chi(3a)$. Then by (2) we obtain the system of inequalities

$$\mu_{11}(u,\chi_2,*)=\tfrac{1}{33}(10t_1+27)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{33}(-20t_1+12)\geq 0,$$

that has no integral solution.

Case 2. Let $\chi(u^{11})=\chi(3b)$. Now (2) gives us the system

$$\mu_{11}(u,\chi_2,*)=\tfrac{1}{33}(10t_1+18)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{33}(-20t_1+30)\geq 0,$$

which also has no integral solution.

Case 3. Let $\chi(u^{11})=-2\chi(3a)+3\chi(3b)$. By (2) we obtain that

$$\begin{aligned}
&\mu_1(u,\chi_2,*)=\tfrac{1}{33}(-5\nu_{3a}+4\nu_{3b})\geq 0; && \mu_{11}(u,\chi_2,*)=\tfrac{1}{33}(50\nu_{3a}-40\nu_{3b})\geq 0;\\
&\mu_0(u,\chi_3,*)=\tfrac{1}{33}(60t_2+207)\geq 0; && \mu_{11}(u,\chi_3,*)=\tfrac{1}{33}(-30t_2+243)\geq 0;\\
&\mu_1(u,\chi_7,*)=\tfrac{1}{33}(t_3+978)\geq 0; && \mu_3(u,\chi_7,*)=\tfrac{1}{33}(-2t_3+750)\geq 0.
\end{aligned}$$

It follows that $t_1=0$, $t_2=7$ and $t_3\in\{12+33k\mid -30\leq k\leq 11\}$, and we have no solutions again.

Case 4. Let $\chi(u^{11})=-\chi(3a)+2\chi(3b)$. By (2) we obtain the system

$$\mu_{11}(u,\chi_2,*)=\tfrac{1}{33}(10t_1+9)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{33}(-20t_1+48)\geq 0,$$

which has no integral solution.

• Let u be a unit of order 35. By (1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{7a}+\nu_{7b}=1.$$

Put $t_1=3\nu_{5a}-2\nu_{5b}-\nu_{7a}-\nu_{7b}$, $t_2=6\nu_{5a}+\nu_{5b}$ and $t_3=6\nu_{5a}+\nu_{5b}+3\nu_{7a}-4\nu_{7b}$. Since $|u^7|=5$, for any character χ of G we need to consider six cases, defined by part (iv) of the Theorem.

Case 1. Let $\chi(u^7)=\chi(5a)$. By (2) we obtain the system

$$\mu_5(u,\chi_2,*)=\tfrac{1}{35}(4t_1+9)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{35}(-24t_1+16)\geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7)=\chi(5b)$. Now the non-compatible inequalities are

$$\mu_7(u,\chi_2,*)=\tfrac{1}{35}(6t_1+26)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{35}(-24t_1+36)\geq 0.$$

Case 3. Let $\chi(u^7) = -2\chi(5a) + 3\chi(5b)$. By (2) we obtain the system

$$\mu_7(u,\chi_2,*) = \tfrac{1}{35}(6t_1 + 16) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{35}(-24t_1 + 76) \geq 0,$$

which has no integral solution.

Case 4. Let $\chi(u^7) = -3\chi(5a) + 4\chi(5b)$. By (2) we obtain the system

$$\begin{aligned} \mu_0(u,\chi_2,*) &= \tfrac{1}{35}(-24t_1 + 96) \geq 0; & \mu_7(u,\chi_2,*) &= \tfrac{1}{35}(6t_1 + 11) \geq 0;\\ \mu_0(u,\chi_3,*) &= \tfrac{1}{35}(24t_2 + 175) \geq 0; & \mu_7(u,\chi_3,*) &= \tfrac{1}{35}(-6t_2 + 245) \geq 0;\\ \mu_{15}(u,\chi_{16},*) &= \tfrac{1}{35}(4t_3 + 8071) \geq 0; & \mu_1(u,\chi_{16},*) &= \tfrac{1}{35}(-t_3 + 8001) \geq 0; \end{aligned}$$
$$\mu_0(u,\chi_2,3) = \tfrac{1}{35}(-96\nu_{5a} + 24\nu_{5b} + 85) \geq 0,$$

so $t_1 = 4$, $t_2 \in \{0, 35\}$ and $t_3 \in \{21 + 35k \mid -58 \leq k \leq 228\}$, and we have no solutions again.

Case 5. Let $\chi(u^7) = -4\chi(5a) + 5\chi(5b)$. Using (2), we obtain

$$\begin{aligned} \mu_7(u,\chi_2,*) &= \tfrac{1}{35}(6t_1 + 6) \geq 0; & \mu_1(u,\chi_2,*) &= \tfrac{1}{35}(-t_1 - 1) \geq 0;\\ \mu_0(u,\chi_3,*) &= \tfrac{1}{35}(24t_2 + 155) \geq 0; & \mu_5(u,\chi_3,*) &= \tfrac{1}{35}(-4t_2 + 155) \geq 0;\\ \mu_{15}(u,\chi_{16},*) &= \tfrac{1}{35}(4t_3 + 8091) \geq 0; & \mu_1(u,\chi_{16},*) &= \tfrac{1}{35}(-t_3 + 7996) \geq 0. \end{aligned}$$

Then $t_1 = 1$, $t_2 \in \{-5, 30\}$ and $t_3 \in \{16 + 35k \mid -58 \leq k \leq 228\}$, and we have no solutions as before.

Case 6. Let $\chi(u^7) = -\chi(5a) + 2\chi(5b)$. By (2) we have incompatible inequalities

$$\mu_7(u,\chi_2,*) = \tfrac{1}{35}(6t_1 + 21) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{35}(-24t_1 + 56) \geq 0.$$

• Let u be a unit of order 55. By (1) and Proposition 2 we have

$$\nu_{5a} + \nu_{5b} + \nu_{11a} + \nu_{11b} = 1.$$

Put $t_1 = 3\nu_{5a} - 2\nu_{5b}$, $t_2 = 6\nu_{5a} + \nu_{5b}$ and $t_3 = 4\nu_{5a} - \nu_{5b} + 6\nu_{11a} - 5\nu_{11b}$. Since $|u^{11}| = 5$, for any character χ of G we need to consider six cases, defined by part (iv) of the Theorem.

Case 1. Let $\chi(u^{11}) = \chi(5a)$. Then by (2) we obtain incompatible inequalities

$$\mu_5(u,\chi_2,*) = \tfrac{1}{55}(4t_1 + 10) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{55}(-40t_1 + 10) \geq 0.$$

Case 2. Let $\chi(u^{11}) = \chi(5b)$. Using (2) we obtain the system

$$\begin{aligned} \mu_{11}(u,\chi_2,*) &= \tfrac{1}{55}(10t_1 + 20) \geq 0; & \mu_0(u,\chi_2,*) &= \tfrac{1}{55}(-40t_1 + 30) \geq 0;\\ \mu_0(u,\chi_3,*) &= \tfrac{1}{55}(40t_2 + 235) \geq 0; & \mu_{11}(u,\chi_3,*) &= \tfrac{1}{55}(-10t_2 + 230) \geq 0;\\ \mu_5(u,\chi_7,*) &= \tfrac{1}{55}(4t_3 + 939) \geq 0; & \mu_1(u,\chi_7,*) &= \tfrac{1}{55}(-t_3 + 934) \geq 0, \end{aligned}$$

so $t_1 = -2$, $t_2 \in \{1, 12, 23\}$ and $t_3 \in \{-1 + 55k \mid -4 \le k \le 17\}$, and in every case we have no integer solution.

Case 3. Let $\chi(u^{11}) = -2\chi(5a) + 3\chi(5b)$. By (2) we obtain that

$$\begin{aligned} \mu_{11}(u, \chi_2, *) &= \tfrac{1}{55}(10t_1 + 10) \ge 0; & \mu_0(u, \chi_2, *) &= \tfrac{1}{55}(-40t_1 + 70) \ge 0; \\ \mu_0(u, \chi_3, *) &= \tfrac{1}{55}(40t_2 + 195) \ge 0; & \mu_{11}(u, \chi_3, *) &= \tfrac{1}{55}(-10t_2 + 240) \ge 0; \\ \mu_5(u, \chi_7, *) &= \tfrac{1}{55}(4t_3 + 946) \ge 0; & \mu_1(u, \chi_7, *) &= \tfrac{1}{55}(-t_3 + 891) \ge 0. \end{aligned}$$

From this follows that $t_1 = -1$, $t_2 \in \{2, 13, 24\}$ and $t_3 \in \{11+55k \mid -4 \le k \le 16\}$, and for every combination of t_i's we have no solution.

Case 4. Let $\chi(u^{11}) = -3\chi(5a) + 4\chi(5b)$. By (2) we obtain the system

$$\mu_{11}(u, \chi_2, *) = \tfrac{1}{55}(10t_1 + 5) \ge 0; \qquad \mu_0(u, \chi_2, *) = \tfrac{1}{55}(-40t_1 + 90) \ge 0,$$

which has no integral solution.

Case 5. Let $\chi(u^{11}) = -4\chi(5a) + 5\chi(5b)$. By (2) we have that

$$\begin{aligned} \mu_{11}(u, \chi_2, *) &= \tfrac{1}{55}(10t_1) \ge 0; & \mu_1(u, \chi_2, *) &= \tfrac{1}{55}(-t_1) \ge 0; \\ \mu_0(u, \chi_3, *) &= \tfrac{1}{55}(40t_2 + 155) \ge 0; & \mu_{11}(u, \chi_3, *) &= \tfrac{1}{55}(-10t_2 + 250) \ge 0; \\ \mu_5(u, \chi_7, *) &= \tfrac{1}{55}(4t_3 + 986) \ge 0; & \mu_1(u, \chi_7, *) &= \tfrac{1}{55}(-t_3 + 881) \ge 0, \end{aligned}$$

so $t_1 = 0$, $t_2 \in \{3, 14, 25\}$ and $t_3 \in \{1 + 55k \mid -4 \le k \le 16\}$, and again we have no solutions in every combination of t_i's.

Case 6. Let $\chi(u^{11}) = -\chi(5a) + 2\chi(5b)$. Using (2) we obtain the system

$$\mu_{11}(u, \chi_2, *) = \tfrac{1}{55}(10t_1 + 15) \ge 0; \qquad \mu_0(u, \chi_2, *) = \tfrac{1}{55}(-40t_1 + 50) \ge 0,$$

which has no integral solution.

• Let u be a unit of order 77. By (1) and Proposition 2 we have

$$\nu_{7a} + \nu_{7b} + \nu_{11a} + \nu_{11b} = 1.$$

Then using (2) we obtain the non-compatible system of inequalities

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \tfrac{1}{77}(60(\nu_{7a} + \nu_{7b}) + 28) \ge 0; \\ \mu_{11}(u, \chi_2, *) &= \tfrac{1}{77}(-10(\nu_{7a} + \nu_{7b}) + 21) \ge 0. \end{aligned}$$

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Contact information

V. A. Bovdi — Institute of Mathematics, University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary Institute of Mathematics and Informatics, College of Nyíregyháza, Sóstói út 31/b, H-4410 Nyíregyháza, Hungary
E-Mail: vbovdi@math.klte.hu

A. B. Konovalov — School of Computer Science, University of St Andrews, Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland
E-Mail: konovalov@member.ams.org