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STRONGLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS. II

СТРОГО НЕЛИНЕЙНЫЕ ВЫРОЖДЕННЫЕ ЭЛЛИПТИЧЕСКИЕ УРАВНЕНИЯ С РАЗРЫВНЫМИ КОЭФФИЦИЕНТАМИ. II

We use energy methods to prove the existence and uniqueness of solutions of the Dirichlet problem for an elliptic nonlinear second-order equation of divergence form with a superlinear term [i.e., $g(x,u) = \nu(x)\,a(x)\,|u|^{p-1}u,\ p>1$] in unbounded domains. Degeneracy in the ellipticity condition is allowed. Coefficients $a_{i,j}(x,r)$ may be discontinuous with respect to the variable r.

Використано енергетичні методи для доведення існування та єдиності розв'язків задачі Діріхле для еліптичного рівняння другого порядку дивергентної форми з суперлінійним членом (тобто $g(x,u) = \nu(x)\,a(x)\,|u|^{p-1}u,\ p>1$) в необмеженій області. В умові еліптичності дозволяється виродженність, коефіцієнти $a_{i,j}(x,r)$ можуть бути розривними відносно r.

1. Introduction. This work is continuation of [1]. Let Ω be an open subset of $\mathbb{R}^m$ $(m \ge 2)$. We consider a strongly nonlinear degenerate elliptic equation of the type

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{m} a_{i,j}(x,u)\frac{\partial u}{\partial x_j}\right) + a(x)\big(\nu(x)|u|^{p-1}u\big) = f, \tag{$*$}$$

where p is a real number greater than 1. The equation is degenerate elliptic for a condition of the following form is fulfilled

$$\sum_{i,j=1}^{m} a_{i,j}(x,r)\,\xi_i\xi_j \ \ge\ \nu(x)\,|\xi|^2, \tag{1}$$

for every $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$, a. e. $(x,r) \in \Omega \times \mathbb{R}$. When Ω is bounded, it is known that there exists a unique weak bounded solution for the Dirichlet problem related to (*) (see [1]). In this paper we extend these results to an unbounded set Ω. Regularity hypotheses on Ω and on other data are minima. Our results are in some respects similar to those of F. Guglielmino, F. Nicolosi [2] and generalize the ones obtained by A. V. Ivanov, P. Z. Mkrtycjan [3]. The main difference with our results comes from the fact that we merely use energy methods which allow us to treat a greater class of functions f. So we do not need, as in [2] and [3], any hypothesys on the growth of the data at infinity. Moreover, we replace the continuity hypothesys of coefficients $a_{i,j}(x,r)$ with respect to the variable r with a weaker one (see [4]). In the non degenerate case, $\nu(x) = 1$, the first result where the existence and uniqueness of a solution of the problem (*) was given without growth at infinity on f is due to Brezis [5]; then, this result has been generalized in [6] and [7]. Our argument has some points of contact with the one introduced by J. Diaz, O. Oleinik in [7]. More precisely, for every $R > 0$ we define

$$B_R \equiv \{x \in \mathbb{R}^m : |x| < R\},$$

$$\Omega_R \equiv \Omega \cap B_R,$$

and we consider the unique solution u_N of problem (*) in Ω_N. Then, taking $u_N \Lambda_R$ as test function in the integrale equality satisfied by u_N $(2R < N)$, we obtain a priori estimate from above for the norm of u_N in $H^1_\nu(\Omega_R) \cap L^{p+1}(\nu(x), \Omega_R)$ (Lemma (5.1)); here $\Lambda_R = \theta^2\left(\frac{|x|}{R}\right)$ where $\theta \in C^\infty(\mathbb{R})$ is a cut-off function such that $\theta'(s) = O((1-s)^{t-1})$ $(t > 0)$. Finally, by diagonal extraction we obtain a weak solution of (*) with Dirichlet data. The uniqueness of solution is obtained in the same manner of [7] assuming that $a_{i,j}$ does not depend on r.

2. Function spaces. Let $\mathbb{R}^m$ be the Euclidean m-space with generic point $x = (x_1, x_2, \ldots, x_m)$.

Hypothesys 2.1. Let $\nu(x)$ be a positive function defined on Ω; there exists a real number $g > m/2$ such that:

$$\nu(x) \in L^s(\Omega_R), \qquad \frac{1}{\nu(x)} \le L^g(\Omega_R)$$

for every $R > 0$; here $s = \dfrac{mg}{2g - m}$.

Let D be a bounded open subset of $\mathbb{R}^m$. We shall denote by $H^1_\nu(D)$ the completion of $C^1(\overline{D})$ with respect to the norm

$$\| u \|_{1,\nu,D} = \left(\int_D \nu(x) \{ |u|^2 + |\nabla u|^2 \} \, dx \right)^{1/2}.$$

$H^{1,0}_\nu(D)$ will be the closure of $C_0^\infty(D)$ in $H^1_\nu(D)$. By Hypothesys 2.1 we get the imbedding

$$H^{1,0}_\nu(D) \hookrightarrow L^{\tilde{2}}(D)$$

where $\tilde{2} = \dfrac{2mg}{mg + m - 2g}$ is greater than 2; moreover, the inequality

$$\left(\int_D \nu(x) |u|^2 \, dx \right)^{1/2} \le c(m, g, \nu) \left(\int_D \nu(x) |\nabla u|^2 \, dx \right)^{1/2}$$

holds for every $u \in H^{1,0}_\nu(D)$. Let $u \in H^1_\nu(D)$, $E \subseteq \partial\Omega$, we will say that $u|_E = 0$ if there exists $\{\varphi_k(x)\} \in C^1(\overline{D})$ such that $\varphi_k(x) = 0$ on E and $\varphi_k(x) \to u(x)$ in $H^1_\nu(D)$. We also consider

$$W_\nu(\Omega_R) \equiv \{ w \in H^1_\nu(\Omega_R) : w|_{\partial\Omega \cap B_R} = 0 \},$$

$$\tilde{W}_\nu(\Omega) \equiv \bigcap_{R>0} W_\nu(\Omega_R).$$

Obviously by $W^*_\nu(\Omega_R)$, $\tilde{W}^*_\nu(\Omega)$ we will denote the dual spaces of $W_\nu(\Omega_R)$ and $\tilde{W}^*_\nu(\Omega)$ respectively. For definitions concerning the spaces $H^1(\nu, D)$, $H^1_0(\nu, D)$ and $L^k(\nu(x), D)$, $k \ge 1$ we refere to [1].

***Remark* 2.2.** If D is a bounded open subset of $\mathbb{R}^m$ satisfying cone-property and Hypothesys 2.1 holds, then

$$H^1_\nu(D) \subset H^1(\nu, D).$$

Moreover, for any bounded open subset D of $\mathbb{R}^m$, we have

$$H_\nu^{1,0}(D) = H_0^1(\nu, D).$$

For more details about these spaces we refere to [8, 9].

3. Hypotheses on coefficients, main result. It is reasonable to postulate the following hypotheses on the coefficients of (*):

Hypothesys 3.1. The functions $a_{i,j}(x, r)$ $(i, j = 1, 2, \dots, m)$ are defined and measurable in $\Omega \times \mathbb{R}$, moreover

$$\frac{a_{i,j}(x, r)}{\nu(x)} \in L^\infty(\Omega_S \times \mathbb{R})$$

for every $S \in \mathbb{R}^+$.

Hypothesys 3.2. Function $a(x)$ is defined and measurable in Ω, moreover $a(x) \in L^{mg/(mg-2g+m)}(\Omega_S)$, for every $S \in \mathbb{R}^+$, $a(x) \geq a_0 > 0$, a.e. $x \in \Omega$ and some constant a_0.

Let us denote by $\tilde{a}_{i,j,s}(x) = a_{i,j}(x, s)$.

Hypothesys 3.3. For every $S \in \mathbb{R}^+$, for every $\varepsilon > 0$ there exists a compact subset $K_{\varepsilon,S} \subset \Omega_S$ with $\operatorname{meas}(\Omega_S \setminus K_{\varepsilon,S}) < \varepsilon$, such that the functions of the family $\{\tilde{a}_{i,j,s}(x),\ s \in \mathbb{R},\ i, j = 1, 2, \dots, m\}$ are equicontinuous on $K_{\varepsilon,S}$.

We refere to [4] for further details concerning the last hypothesys.

Remark **3.4.** It is easy to check that the Hypothesys 3.3 can be formulate with A instead of Ω_S, where A is any bounded open subset of Ω.

Let $f \in \tilde{W}_\nu^*(\Omega)$ and $p > 1$. A function $u(x) \in \tilde{W}_\nu(\Omega)$ is called a weak solution of problem (*) if $a(x)|u|^{p-1}u \in L^1_{\text{loc}}(\nu, \Omega)$ and

$$\int_\Omega \sum_{i,j=1}^m a_{i,j}(x, u(x)) \frac{\partial u}{\partial x_j}(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \; +$$

$$+ \int_\Omega a(x)\, \nu(x)\, |u(x)|^{p-1} u(x)\, \varphi(x)\, dx = (f, \varphi(x)) \tag{2}$$

for every $\varphi \in \tilde{W}_\nu(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, φ with compact support in Ω. Now, we can formulate the main result.

Theorem. *Under the condition* (1) *and hypotheses* 2.1, 3.1 – 3.3 *there exists a weak solution* $u(x)$ *of problem* (*).

4. Preliminary Lemmas. Let us introduce the cut-off function

$$\Lambda_R \equiv \theta^2\left(\frac{|x|}{R}\right) \quad \text{for} \quad R > 0, \tag{3}$$

where $\theta \in C^\infty(\mathbb{R})$ is such that $\theta(s) = 1$ if $|s| \leq \frac{1}{2}$ and $\theta(s) = 0$ if $|s| \geq 1$.

Remark **4.1.** We can choose $\theta(s)$ such that

$$\theta' = O((1-s)^{t-1}), \qquad t > 0. \tag{4}$$

In this way, $\theta = O\left(\dfrac{(1-s)^t}{t}\right)$ for every $s \geq 0$.

Lemma 4.2. *Let D a bounded open subset of $\mathbb{R}^m$ and $u(x) \in H_\nu^1(D)$ with compact support in D, then $u(x) \in H_\nu^{1,0}(D)$.*

Proof. Since $u(x) \in H^1_\nu(D)$, there exists a sequence of functions $\{\varphi_h(x)\}$ such that $\varphi_h(x) \in C^1(\overline{D})$, $h = 1, 2, \ldots$, and $\|\varphi_h(x) - u(x)\|_{1,\nu,D} \to 0$ if $h \to 0$; moreover, there exists an open subset ω of D such that

$$\operatorname{supp} u \subset \omega \subset \overline{\omega} \subset D.$$

Let us fix $\varphi(x)$ in $C_0^\infty(\mathbb{R}^m)$ such that $\varphi(x) = 1$ in $\operatorname{supp} u$ and $\operatorname{supp} \varphi \subset \omega$. If we put $\psi_h(x) = \varphi_h(x)\varphi(x)$, $h = 1, 2, \ldots$, then we get

$$\psi_h(x) \in C_0^1(D), \quad \text{for every} \quad h = 1, 2, \ldots,$$

and

$$\|\psi_h(x) - u(x)\|_{1,\nu,D} \to 0 \quad \text{if} \quad h \to 0.$$

From density of $C_0^\infty(D)$ in $C_0^1(D)$ with respect to $H^1_\nu(D)$, we have the assertion of Lemma.

Lemma 4.3. *Let D be a bounded open subset of $\mathbb{R}^m$ satisfying cone-property. If hypotheses* 2.1, 3.1, 3.3, *with D instead of Ω_S, are satisfied, then the operator*

$$B : H^1(\nu, D) \to (H^1(\nu, D))^*$$

such that

$$(B(u), v) = \int_D \sum_{i,j=1}^m a_{i,j}(x, u) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx, \qquad u, v \in H^1(\nu, D),$$

is sequentially weakly continuous.

Proof. Analogous to [10, p. 57 – 60].

Lemma 4.4. *Let D a bounded open subset of $\mathbb{R}^m$ and $u(x) \in H^{1,0}_\nu(D)$. Then, there exists a sequence of function $h_p(x)$ such that $h_p(x) \in H^{1,0}_\nu(D) \cap \cap L^\infty(D)$, $p = 1, 2, \ldots$, and $h_p(x) \to u(x)$ in $H^{1,0}_\nu(D)$ for $p \to \infty$.*

Proof. For every $p = 1, 2, \ldots$, it will be sufficient to define

$$h_p(x) = \operatorname{sgn} u \min(|u|, p)$$

(see [8, Prop. (2.7), p. 10]).

5. Proof of Theorem. Let us fix $f \in \tilde{W}^*_\nu(\Omega)$ and let us indicate by f_N, $N \in \mathbb{N}$, the extension of f from $\tilde{W}_\nu(\Omega)$ to $W_\nu(\Omega_N)$ such that

$$\|f\|_{\tilde{W}^*_\nu(\Omega)} = \|f_N\|_{W^*_\nu(\Omega_N)}.$$

It is clear that f_N is a continuous linear functional on $H^1_0(\nu, \Omega_N)$. Now, we consider the problem

(P_N)

$$\begin{cases} -\displaystyle\sum_{i=1}^m \frac{\partial}{\partial x_i}\left(\sum_{j=1}^m a_{i,j}(x, u_N) \frac{\partial u_N}{\partial x_j}\right) + a(x)(\nu(x)|u_N|^{p-1} u_N) = f_N & \text{in } \Omega_N, \\ u = 0 & \text{on } \partial\Omega_N. \end{cases}$$

The existence of a solution $u_N \in H^1_0(\nu, \Omega_N)$ satisfying (P_N) in the sense that $a(x)|u_N|^{p-1}u \in L^1(\nu, \Omega_N)$, and the integral identity (2) holds when replacing Ω by Ω_N for every $\varphi \in H^1_0(\nu, \Omega_N) \cap L^\infty(\Omega_N)$ is a consequence of the results of [1], Theorem 3.1.

These results also imply that $a(x)|u_N|^{p+1} \in L^1(\nu, \Omega_N)$ and the integral identity (2) also holds for $\varphi = u_N$.

We proceed to show that

$$\int_{\Omega_R} \sum_{i,j=1}^{m} a_{i,j}(x, u_N) \frac{\partial u_N}{\partial x_j} \frac{\partial (u_N \Lambda_R)}{\partial x_i} dx + \\ + \int_{\Omega_R} a(x)\nu(x)|u_N|^{p+1} \Lambda_R \, dx = (f_N, u_N \Lambda_R), \tag{5}$$

for every $0 < R < N$.

Fix $N \in \mathbb{N}$ and define, for every $p = 1, 2, \ldots$,

$$T_p(u_N) = \operatorname{sgn} u_N \min(|u_N|, p).$$

By definition (3) and Lemma 4.4 we have that

$$T_p(u_N)\Lambda_R \in H_0^1(\nu, \Omega_N) \cap L^\infty(\Omega_N)$$

and

$$T_p(u_N)\Lambda_R \to u_N \Lambda_R \quad \text{in} \quad H_0^1(\nu, \Omega_N) \quad \text{as} \quad p \to \infty.$$

Then, setting $\varphi = T_p(u_N)\Lambda_R$ in the integral identity satisfed by $u_N(x)$ on Ω_N and making $p \to \infty$ we obtain (5).

We extend the function u_N by zero over $\Omega \setminus \Omega_N$ and, for simplicity, we denote again by u_N this extension. The convergence of u_N as $N \to \infty$ will be consequence of the following useful auxiliary result

Lemma 5.1. *Let* $g \in W_\nu^*(\Omega_N)$, $u \in W_\nu(\Omega_N)$ *such that* $a(x)|u|^{p+1} \in L^1(\nu, \Omega_N)$ *and the inequality*

$$\int_{\Omega_R} \sum_{i,j=1}^{m} a_{i,j}(x, u) \frac{\partial u}{\partial x_j} \frac{\partial (u \Lambda_R)}{\partial x_i} dx + \\ + \int_{\Omega_R} a(x)\nu(x)|u|^{p+1} \Lambda_R \, dx \le (g, u\Lambda_R), \tag{6}$$

holds, for every $R \in [0, N]$. *Then, for every* $R \in [1, N]$, *it follows the inequality*

$$\int_{\Omega_{R/2}} \nu(x)|\nabla u|^2 dx + \int_{\Omega_{R/2}} \nu(x)|u(x)|^{p+1} dx \le \\ \le AR^{m-2(p+1)/(p-1)} + BR^{-2(p+1)/(p-1)} \|\nu(x)\|^s_{L^s(\Omega_R)} + CR^2 \|g\|^2_{W_\nu^*(\Omega_N)}; \tag{7}$$

here B *and* C *are constant independent of* Ω, R *and* u, *while the constant* A *independent of* u, *is linearly depend on* $M_R = \operatorname*{ess\,sup}_{\Omega_R \times \mathbb{R}} \left| \frac{a_{i,j}(x, r)}{\nu(x)} \right|$.

Proof. Let $R = 1$. From (6), hypotheses 3.1, 3.2 and assumption (1) we have

$$\int_{\Omega_1} \nu(x)|\nabla u|^2 \theta^2(|x|)dx + a_0 \int_{\Omega_1} \nu(x)|u|^{p+1}\theta^2(|x|)dx \le \\ \le 2C_1 \varepsilon \int_{\Omega_1} \nu(x)|\nabla u|^2 \theta^2(|x|)dx + C_2 \delta \int_{\Omega_1} \nu(x)|u|^{p+1}\theta(|x|)dx +$$

$$+ \; C_3(M_1, \varepsilon, \delta) \int_{\Omega_1} \nu(x)\,|\theta(|x|)|^{-4/(p-1)}|\theta'(|x|)|^{2(p+1)/(p-1)}\,dx \; +$$

$$+ \; \varepsilon \int_{\Omega_1} \nu(x) \left\{ \sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i}(u\theta^2(|x|)) \right|^2 + |u\theta^2(|x|)|^2 \right\} dx \; + \; \frac{1}{\varepsilon}\|g\|^2_{W_\nu^*(\Omega_N)}, \tag{8}$$

for any ε and δ positive numbers, where we have used the Hölder and the Young inequalities; here and in the sequel we denote by C_i a constant independent of u. Choosing $\theta(s)$ such that (4) holds we have

$$P \; = \; \int_{\Omega_1} \nu(x)\,|\theta(|x|)|^{-4/(p-1)}|\theta'(|x|)|^{2(p+1)/(p-1)}\,dx \; \le$$

$$\le \; \frac{C_4}{t} \int_{\Omega_1} \nu(x)\,(1-|x|)^{-4t/(p-1)}(1-|x|)^{2(p+1)(t-1)/(p-1)}\,dx \; \le$$

$$\le \; \frac{C_4}{t} \left(\int_{\Omega_1} \nu(x)^s\,dx \right)^{1/s} \left(\int_{B_1} (1-|x|)^{(2t(p-1)-2(p+1))s'/(p-1)}\,dx \right)^{1/s'},$$

where $\dfrac{1}{s} + \dfrac{1}{s'} = 1$.

Therefore, setting $t > \dfrac{p+1}{p-1}$, we get

$$P \; \le \; C_5\,\|v(x)\|_{L^s(\Omega_1)}. \tag{9}$$

Analogously, we obtain

$$\int_{\Omega_1} \nu(x) \sum_{i=1}^{m} \left| \frac{\partial}{\partial x_i}(u\theta^2(|x|)) \right|^2 dx \; \le \; C_6 \int_{\Omega_1} \nu(x)\,|\nabla u|^2\,\theta^2(|x|)\,dx \; +$$

$$+ \; C_7\,\frac{\delta}{\varepsilon} \int_{\Omega_1} \nu(x)\,|u|^{p+1}\,\theta^2(|x|)\,dx \; + \; C_8\left(\frac{\varepsilon}{\delta}\right)^{2/(p-1)} \|v(x)\|_{L^s(\Omega_1)}\,; \tag{10}$$

$$\int_{\Omega_1} \nu(x)\,|u\theta^2(|x|)|^2\,dx \; \le$$

$$\le \; \frac{\delta}{\varepsilon} \int_{\Omega_1} \nu(x)\,|u|^{p+1}\,\theta^2(|x|)\,dx \; + \; C_9\left(\frac{\varepsilon}{\delta}\right)^{2/(p-1)} \|v(x)\|_{L^s(\Omega_1)}\,\tilde{P}, \tag{11}$$

where $\tilde{P} \; = \; \left(\int_{B_1} (1-|x|)^{4prs'/(p-1)}\,dx \right)^{1/s'}$.

From (8) by (9), (10) and (11) we have

$$\int_{\Omega_1} \nu(x)\,|\nabla u|^2\,\theta^2(|x|)\,dx \; + \; a_0 \int_{\Omega_1} \nu(x)\,|u|^{p+1}\,\theta^2(|x|)\,dx \; \le$$

$$\le \; C_{10}\,\varepsilon \int_{\Omega_1} \nu(x)\,|\nabla u|^2\,\theta^2(|x|)\,dx \; + \; C_{11}\,\delta \int_{\Omega_1} \nu(x)\,|u|^{p+1}\,\theta^2(|x|)\,dx \; +$$

$$+ \; C_{12}(M_1, \varepsilon, \delta)\,\|v(x)\|_{L^s(\Omega_1)} \; + \; C_{13}(\varepsilon, \delta)\,\|v(x)\|_{L^s(\Omega_1)} \; + \; \frac{1}{\varepsilon}\|g\|^2_{W_\nu^*(\Omega_N)}.$$

By the definition of $\theta(s)$ and Hölder inequality it follows

$$\int_{\Omega_{1/2}} \nu(x)|\nabla u|^2\,dx + \int_{\Omega_{1/2}} \nu(x)|u|^{p+1}\,dx \le$$

$$\le C_{14}(M_1) + C_{15}\|v(x)\|^s_{L^s(\Omega_1)} + C_{16}\|g\|^2_{W^*_\nu(\Omega_N)}.$$

For the general case $R \ge 1$, we consider $\tilde g \in (H^1_\nu(\Omega_N))^*$ such that $(\tilde g, w) = (g, w)$ for any $w \in W_\nu(\Omega_N)$ and $\|\tilde g\|_{H^1_\nu(\Omega_N)} = \|g\|_{W^*_\nu(\Omega_N)}$. Then, there exist $f_i(x) \in L^2\left(\frac{1}{\nu}, \Omega_N\right)$ $(i = 0, 1, 2, \ldots, m)$, such that

$$(\tilde g, \omega) = \int_{\Omega_N}\left\{f_0\,\omega + \sum_{i=1}^m f_i\frac{\partial\omega}{\partial x_i}\right\}dx$$

for any $\omega \in H^1_\nu(\Omega_N)$.

If we put $x = Rx'$, then we have

$$\int_{\Omega_1'}\sum_{i,j=1}^m a_{i,j}(Rx', u(Rx'))\frac{\partial v}{\partial x_j'}(x')\frac{\partial}{\partial x_i'}(v(x')\,\theta^2(|x'|))\,dx' +$$

$$+ a_0\int_{\Omega_1'}\nu(Rx')|v(x')|^{p+1}\theta^2(|x'|)\,dx' \le$$

$$\le R^{(p+1)/(p-1)}\int_{\Omega_1'}\sum_{i=1}^m f_i(Rx')\frac{\partial}{\partial x_i'}(v(x')\,\theta^2(|x'|))\,dx' +$$

$$+ R^{2p/(p-1)}\int_{\Omega_1'} f_0(Rx')|v(x')|\,\theta^2(|x'|)\,dx'; \qquad (12)$$

here $v(x') = R^{2/(p-1)}u(Rx')$, $\Omega_1' = \Omega' \cap B_1$ where Ω' is the image of Ω by the above change of variable.

Define for $(x', s) \in \Omega_1' \times \mathbb{R}$

$$\tilde a_{i,j}(x', s) = a_{i,j}\left(Rx', \frac{s}{R^{2/(p-1)}}\right), \qquad \tilde\nu(x') = \nu(Rx'), \qquad \tilde f_i(x') = f_i(Rx').$$

From (12) it then follows, bearing in mind (1), that

$$\int_{\Omega_1'}\tilde\nu(x')|\nabla v|^2\theta^2(|x'|)\,dx' + a_0\int_{\Omega_1'}\tilde\nu(x')|v|^{p+1}\theta^2(|x'|)\,dx' \le$$

$$\le 2M_R\int_{\Omega_1'}\tilde\nu(x')\sum_{i,j=1}^m\left|\frac{\partial v}{\partial x_i'}\right||v||\theta(|x'|)|\left|\frac{\partial\theta}{\partial x_i'}(|x'|)\right|dx' +$$

$$+ \left\{\left(\int_{\Omega_1'} R^{4p/(p-1)}\frac{|\tilde f_0(x')|^2}{\tilde\nu(x')}\,dx'\right)^{1/2} + \sum_{i=1}^m\left(R^{2(p+1)/(p-1)}\frac{|\tilde f_i(x')|^2}{\tilde\nu(x')}\,dx'\right)^{1/2}\right\}\times$$

$$\times\left\{\int_{\Omega_1'}\tilde\nu(x')\sum_{i=1}^m\left|\frac{\partial(v\theta^2(|x'|))}{\partial x_i'}\right|^2 + \tilde\nu(x')|v\theta^2(|x'|)|^2\,dx'\right\}^{1/2}.$$

Hence, analogously to case $R = 1$, we have

$$\int_{\Omega_1'} \tilde{\nu}(x') |\nabla v|^2 \theta^2(|x'|) dx' + \int_{\Omega_1'} \tilde{\nu}(x') |v|^{p+1} \theta^2(|x'|) dx' \leq$$

$$\leq C_{17}(M_R) \left(\int_{\Omega_1'} \tilde{\nu}(x')^s dx' \right)^{1/s} + C_{18} \left[\left(\int_{\Omega_1'} R^{4p/(p-1)} \frac{|\tilde{f}_0(x')|^2}{\tilde{\nu}(x')} dx' \right)^{1/2} + \right.$$

$$\left. + \sum_{i=1}^{m} \left(R^{2(p+1)/(p-1)} \frac{|\tilde{f}_i(x')|^2}{\tilde{\nu}(x')} dx' \right)^{1/2} \right]^2 .$$

Now, taking into account that $x' = \dfrac{x}{R}$, $v\left(\dfrac{x}{R}\right) = R^{2/(p-1)} u(x)$, and definition of $\theta(s)$, we have

$$R^{2(p+1)/(p-1)} \left\{ \int_{\Omega_{R/2}} \nu(x) |\nabla u|^2 dx + \int_{\Omega_{R/2}} \nu(x) |u(x)|^{p+1} dx \right\} \leq$$

$$\leq C_{19}(M_R) R^m + C_{20} \|\nu(x)\|^s_{L^s(\Omega_R)} + C_{21} R^{4p/(p-1)} \sum_{i=0}^{m} \int_{\Omega_R} \frac{|f_i|^2}{\nu} dx ,$$

and therefore the required result follows.

Let $R > 0$ and $N > 2R$, by Lemma 5.1 it follows

$$\int_{\Omega_R} \nu(x) |\nabla u_N|^2 dx + \int_{\Omega_R} \nu(x) |u_N|^{p+1} dx \leq$$

$$\leq A(2R)^{m-2(p+1)/(p-1)} + B(2R)^{-2(p+1)/(p-1)} \|\nu(x)\|^s_{L^s(\Omega_{2R})} +$$

$$+ C(2R)^2 \|f\|^2_{\tilde{W}^*_\nu(\Omega)} = \hat{C}(2R).$$

By standard results we have that $\{u_N\}$ is bounded in $H^1_\nu(\Omega_R) \cap L^{p+1}(\nu, \Omega_R)$ and by diagonal extraction it is possible to find a subsequence $\{u_N\}$ and a function u such that $u_N \to u$ in $H^1_{\mathrm{loc},\nu}(\Omega)$ weakly, in $L^{p+1}_{\mathrm{loc}}(\nu, \Omega)$ weakly and a.e. in Ω. Obviously $u|_{\partial\Omega \cap B_R} = 0$ for any $R > 0$.

Let $\varphi \in \tilde{W}_\nu(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be, φ with compact support. Then there exists $\overline{N} \in \mathbb{N}$ such that $\varphi \in H^1_0(\nu, \Omega_N) \cap L^\infty(\Omega_N)$ for any $N \geq \overline{N}$; moreover, the compact support, C_φ, is a subset of Ω_N. Since $u_N(x)$ is a solution of (P_N), we have

$$\int_{C_\varphi} \sum_{i,j=1}^{m} a_{i,j}(x, u_N) \frac{\partial u_N}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx +$$

$$+ \int_{C_\varphi} a(x) \nu(x) |u_N|^{p-1} u_N \varphi \, dx = (f, \varphi) \quad \text{for any} \quad N \geq \overline{N}. \tag{13}$$

Choose, now, a bounded open set A satisfying cone-property and such that

$$C_\varphi \subset A \subset \Omega_{\overline{N}}$$

then, we can write (13) replacing C_φ by A. As $u_N \to u$ weakly in $H^1(\nu, A)$, from Remark 3.4 and Lemma 4.3, we get

$$\lim_{N\to\infty} \int_A \sum_{i,j=1}^{m} a_{i,j}(x, u_N) \frac{\partial u_N}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = \int_A \sum_{i,j=1}^{m} a_{i,j}(x, u) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx. \qquad (14)$$

As $u_N \to u$ a.e. in A, from $\int_{\Omega_N} a(x)\,\nu(x)\,|u_N|^{p+1} dx \le C(\overline{N})$ for any $N \ge \overline{N}$, we have

$$\lim_{N\to\infty} \int_A a(x)\,\nu(x)\,|u_N|^{p-1} u_N \varphi\, dx = \int_A a(x)\,\nu(x)\,|u_N|^{p-1} u\varphi\, dx. \qquad (15)$$

We conclude from (14), (15) that function $u(x)$ is a weak solution of problem (*).

***Remark* 5.2.** If $a_{i,j}$ does not depend of r and $\frac{a_{i,j}(x)}{\nu(x)} \in L^{\infty}(\Omega)$ the same proof of Theorem 1 of [7, p. 790], gives us the uniqueness of weak solution of problem (*).

6. Example. In this section we give the example of weighted function $\nu(x)$ of such that the preceeding assumptions are valid.

In order to construct example we will take

$$\Omega \equiv \{x \in \mathbb{R}^m : |x| > 1\}$$

and

$$\nu(x)_\rho = (|x| - 1)^\rho, \qquad -\left(\frac{2g - m}{mg}\right) < \rho < \frac{2}{m};$$

here $g > \frac{m}{2}$. In this case it is easy to see that Hypothesys 2.1 is satisfied.

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