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LYAPUNOV TRANSFORMATION AND STABILITY OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

ПЕРЕТВОРЕННЯ ЛЯПУНОВА І СТІЙКІСТЬ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ В БАНАХОВИХ ПРОСТОРАХ

A sufficient condition of exponential stability of regular linear systems with bifurcation on Banach space is proved.

Встановлено достатні умови експоненціальної стійкості регулярних лінійних систем з біфуркацією в банаховому просторі.

vd-Transformation and its properties. In this section we shall give the definition, examples and some properties of a vd-transformation on Banach spaces. It is an expansion of a vd-transformation on finite-dimensional spaces given by Yu. S. Bogdanov [1 – 5]. From that, we shall give the definition of regular linear equations which are applied to study the stability of regular linear equations with bifurcation on Banach spaces.

Let E be a Banach space and let G be an open simple connected domain containing the origin O of E.

We define H as follows: $H = G \times \mathbb{R} = \{\eta = (x, t) \colon x \in G,\ t \in \mathbb{R}\}$.

Consider a function

$$v_0 \colon \mathbb{R}^+ \to \mathbb{R}^+$$

that is continuous, monotone, and strictly increasing and satisfies the following conditions:

$$v_0(0) = 0; \quad v_0(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Let d be a given real function of two variables:

$$d \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R},$$

$$(\gamma_1, \gamma_2) \mapsto d(\gamma_1, \gamma_2).$$

We assume that d satisfies the following conditions for all $\gamma > 0$, $\gamma_3 > \gamma_2 > \gamma_1 > 0$:

d$_1$) $d(\gamma_2, \gamma_1) = -d(\gamma_1, \gamma_2)$;

d$_2$) $d(\gamma_2, \gamma) > d(\gamma_1, \gamma)$;

d$_3$) $d(\gamma_3, \gamma_2) + d(\gamma_2, \gamma_1) \geq d(\gamma_3, \gamma_1)$;

d$_4$) $\bigcup\limits_{\gamma \in \mathbb{R}^+} \{d(\gamma, \gamma_1)\} = \mathbb{R}$.

Assume that J is a diffeomorphism from H to H, namely

$$J \colon H \to H,$$

$$\eta = (x, t) \mapsto \eta' = (x', t'),$$

and let it satisfy the following equalities for all $t \in \mathbb{R}$:

$$J(0,t) = (0,t),$$
$$J(x,t) = (x',t).$$

It is easy to prove that the set L of all such transformations $L = \{J\}$ is a group with composition of maps.

Consider a real function

$$v : H^* \to \mathbb{R}^+,$$

$$\eta = (x,t) \mapsto v(\eta) = v_0(\|x\|),$$

where $H^* = G^* \times \mathbb{R} = (G \backslash \{0\}) \times \mathbb{R}$.

Since the function $v : H^* \to \mathbb{R}^+$ is independent of t, i. e. $v(x,t) = v(x,t')$ for all $t, t' \in \mathbb{R}$, we can denote by $v(x)$ the value of $v(x,t)$ for any $x \in G^*$ and $t \in \in \mathbb{R}$.

Definition. *The transformation $J \in L$ is called a vd-transformation iff*

$$\sup_{\eta \in H^*} |d\{v(\eta), v[J(\eta)]\}| < +\infty. \tag{1}$$

From the definition of the function d, we also have

$$\sup_{\eta' \in H^*} |d\{v(\eta'), v(J^{-1}(\eta'))\}| < +\infty.$$

Consequently, if we denote by L_{vd} the set of vd-transformations, then it is a subgroup of L.

Examples. **1.** Let $v_0(x,t) = \|x\|$, $d_0(\gamma_1, \gamma_2) = \ln \frac{\gamma_1}{\gamma_2}$, and let $J(x,t)$ (with fixed t) be a linear transformation having a bounded partial derivative with respect to t. Then J is a $v_0 d_0$-transformation if and only if it is a Lyapunov transformation [6].

2. If $v(x,t) = |x|^2$, $E = \mathbb{R}$,

$$d(\gamma_1, \gamma_2) = \begin{cases} \sqrt{\gamma_1} - \sqrt{\gamma_2} & \text{if } \gamma_1 \cdot \gamma_2 \geq 1, \\ \dfrac{1}{\sqrt{\gamma_2}} - \dfrac{1}{\sqrt{\gamma_1}} & \text{if } \gamma_1 \cdot \gamma_2 < 1, \end{cases}$$

then all conditions $d_1) - d_4)$ are satisfied.

Thus,

$$J(x,t) = \left(x + \frac{1}{2} \sin t \sin^2 x,\ t\right)$$

is vd-transformation.

3. We consider the following linear differential system in the J_∞-space:

$$\frac{dx_k}{dt} = -\cos t \cdot x_k + x_{k+1}, \quad k \in N. \tag{2}$$

The transformation $J(x', t) = e^{\sin t}(x,t)$ with $v(x,t) = \|x\|$; $d(\gamma_1, \gamma_2) = \ln \frac{\gamma_1}{\gamma_2}$ is a vd-transformation and it reduces (2) to the system

$$\frac{dy_k}{dt} = y_{k+1}, \quad k \in N.$$

From example 1, we can see that a $v\,d$-transformation is an expansion of the Lyapunov transformation, but it still keeps an important property, namely, the stability of the trivial solution of the following differential equation on the Banach space E:

$$\frac{dx}{dt} = f(x,t), \qquad f(0,t) \equiv 0. \tag{3}$$

We denote by $x(t;\xi)$ the solution of equation (3) that satisfies the initial condition $x(t_0;\xi) = \xi$ and assume that

$$\lambda = \lim_{\varepsilon\to 0^+} \sup_{\substack{\|\xi\|\le\varepsilon \\ t\ge t_0}} \|x(t;\xi)\|, \quad \lambda_1 = \lim_{\varepsilon\to 0^+} \sup_{\substack{v(\xi)\le\varepsilon \\ t\ge t_0}} v(x(t;\xi)).$$

Definition [7]. *The solution $x = 0$ of the differential equation* (3) *is said to be Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each solution $x(t)$ of* (3) *whose initial value $x(t_0) = \xi$ satisfies the condition $\|\xi\| < \delta(\varepsilon)$, the inequality $\|x(t;\xi)\| < \varepsilon$ hold for all $t \ge t_0$.*

From the definition, we can see that the solution $x = 0$ of the differential equation (3) is stable iff $\lambda = 0$.

Proposition 1. $\lambda = 0$ *if and only if* $\lambda_1 = 0$.

Proof. By the continuity of the function v, we immediately have $\lim_{\xi\to 0} v(\xi) = 0$. Since $v(\|x\|)$ is a monotone strictly increasing function, we can deduce $\lim_{v(\xi)\to 0} \xi = 0$. Therefore,

$$\lim_{k\to\infty} \xi_k = 0 \iff \lim_{k\to\infty} v(\xi_k) = 0. \tag{4}$$

We assume that $\lambda = 0$. Then

$$\lim_{k\to\infty} \|x(t_k;\xi_k)\| = 0$$

for all sequences $\{\varepsilon_k\} \subset \mathbb{R}^+ : \varepsilon_k \to 0$, $\{\xi_k\} \subset E : \|\xi_k\| < \varepsilon_k$, and $\{t_k\} \subset \mathbb{R}$, $t_k \ge t_0$. By virtue of (4), we have $\lim_{k\to\infty} \|x(t_k;\xi_k)\| = 0 \iff \lim_{k\to\infty} v(x(t_k;\xi_k)) = 0$.

It follows that $\lambda = 0 \iff \lambda_1 = 0$.

Proposition 2. *A $v\,d$-transformation preserves the stability of the trivial solution $x = 0$ of the differential equation* (3).

Proof. By vd-transformation

$$(x,t) \mapsto J(x,t) = (y,t),$$

equation (3) is transformed to the following one:

$$\frac{dy}{dt} = g(y,t). \tag{5}$$

By assumption, the solution $x = 0$ of equation (3) is stable, which means that

$$\lim_{\varepsilon\to 0^+} \sup_{\substack{\|x_0\|\le\varepsilon \\ t\ge t_0}} \|x(t;x_0)\| = 0 \iff \lim_{\varepsilon\to 0^+} \sup_{\substack{v(x_0)\le\varepsilon \\ t\ge t_0}} v[x(t;x_0)] = 0.$$

If the solution $y = 0$ of (5) is unstable, then

$$\lim_{\varepsilon \to 0^+} \sup_{\substack{v(y_0) \le \varepsilon \\ t \ge t_0}} v[y(t, y_0)] > 0.$$

This means that there exists a positive number δ such that

$$\exists \{\eta_n\} \subset E: \eta_n \to 0, \ \exists \{t_n\} \subset \mathbb{R}^+: t_n \ge t_0, \ \forall n \in N:$$

$$v[y(t_n, \eta_n)] \ge \delta. \tag{6}$$

Since $v[x(t_n; \xi_n)] \to 0$ as $n \to \infty$, where $(\xi_n, t_n) = J^{-1}(\eta_n, t_n)$, one can say

$$v[x(t_n; \xi_n)] < \delta \quad \forall n \in N. \tag{7}$$

From (6), (7) and d_4) we conclude that

$$|d\{v[x(t_n; \xi_n)], v[y(t_n; \eta_n)]\}| = d\{v[y(t_n; \eta_n)], v[x(t_n; \xi_n)]\} >$$

$$> d\{\delta, v[x(t_n; \xi_n)]\} \to +\infty \quad \text{as} \quad n \to \infty.$$

Consequently,

$$\sup_{n \in N} |d\{v[x(t_n; \xi_n)], v[J(x(t_n; \xi_n))]\}| = +\infty,$$

which contradicts the definition of J.

Regular system.

Definition. *A transformation $J \in L$ satisfying the condition*

$$d\{v(\eta), v[J(\eta)]\} = o(t) \quad \text{as} \quad t \to \pm\infty$$

for all $\eta \in H^*$ is called vd-transformation.

Definition. *A transformation $y = L(t)x$ is a generalized Lyapunov one if*

$$\chi[L(t)] = \chi[L^{-1}(t)] = 0, \tag{8}$$

where $\chi[L(t)] := \overline{\lim_{t \to \infty}} \frac{1}{t} \ln \| L(t) \|$ is called the characteristic exponent of $L(t)$.

By definition, we immediately have following remarks:

***Remark* 1.** Generalized Lyapunov transformations preserve Lyapunov exponents [6].

***Remark* 2.** A generalized Lyapunov transformation is a generalized vd-transformation if

$$v(x) = \|x\|, \quad d(\gamma_1, \gamma_2) = \ln \frac{\gamma_1}{\gamma_2},$$

and J is homogeneously linear for x (here, $J(x, t) = (L(t)x, t)$).

We consider the following linear differential system:

$$\frac{dx}{dt} = A(t)x, \tag{9}$$

where $x \in \mathbb{R}^n$, $A(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and is real and continuous for all $t \in \mathbb{R}$, and $\sup_t \|A(t)\| < \infty$.

Let $X(t)$ be a normal fundamental matrix of (9) and let $\sigma_X = \sum_{k=1}^m n_k \alpha_k$ be the sum of all its exponent numbers [6].

Definition [6]. *The linear system* (9) *is said to be regular iff*

$$\sigma_X = \varliminf_{t\to\infty} \frac{1}{t}\int_{t_n}^{t} \operatorname{Sp}A(\tau)\,d\tau.$$

We know the following proposition [6]:

Lemma. *A necessary and sufficient condition for system* (9) *to be regular one is that there exists a generalized Lyapunov transformation that reduces system* (9) *to the system with constant matrix* $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$:

$$\frac{dy}{dt} = By. \tag{10}$$

Definition. *A linear differential system*

$$\frac{dx}{dt} = A(t)x, \tag{11}$$

where $A(t) \in \mathcal{L}(E, E)$ *and is continuous for all* $t \in \mathbb{R}$ *and* $\sup_t \|A(t)\| < \infty$, *is said to be a regular one iff there is a generalized Lyapunov transformation* $y = L(t)x$ *that reduces it to a linear differential equation with constant operator:*

$$\frac{dy}{dt} = By. \tag{12}$$

We now present the main theorem for regular differential equations on Banach spaces.

Consider the differential equation

$$\frac{dx}{dt} = A(t)x + f(x, t), \tag{13}$$

where $A(t) \in \mathcal{L}(E, E)$, $\sup_{t\in\mathbb{R}} \|A(t)\| < \infty$, $f \in C^{(1,0)}(E\times\mathbb{R})$, $f(0, t) \equiv 0$, and

$$\|f(x, t)\| \le \psi(t)\|x\|^m, \quad m > 1, \quad \chi[\psi(t)] = 0.$$

Under these conditions, the following theorem is true:

Theorem. *If equation* (11) *is regular and all its characteristic exponents are not larger than* $-\lambda < 0$, *the trivial solution* $x = 0$ *of equation* (13) *is exponentially stable* [7], *i.e., there exist* $N > 0$ *and* $A > 0$ *such that*

$$\|x(t)\| \le A e^{-N(t-t_0)} \|x(t_0)\|$$

for all solutions $x(t)$ *of* (13).

Proof. We denote by $X(t)$ $(X(t_0) = Jd_E)$ the Cauchy operator of equation (11) [7, p. 147].

1. First, we estimate the resolvent operator $K(t, \tau) = X(t)X^{-1}(\tau)$, $\tau_0 \le \tau \le t$.

By virtue of the regularity of equation (11), there is a generalized Lyapunov transformation $y = L(x)x$ that reduces equation (11) to equation (12).

The operator $Y(t) = L(t)X(t)$ is the resolvent operator of equation (12).

If we put $H(t, \tau) = Y(t)Y^{-1}(\tau)$, then $K(t, \tau) = L(t)H(t, \tau)L^{-1}(\tau)$.

Assume that all characteristic exponents of equation (11) are not larger than α.

Hence, all characteristic exponents of equation (12) are not larger than α, i.e., for every solution $y(t) = Y(t)y_0$ and $\varepsilon > 0$, there exists $c > 0$ such that

$$\|y(t)\| \le c\,e^{(\alpha+\varepsilon/2)t} \quad \forall\, t \ge t_0.$$

Then, the operator family $\{e^{-(\alpha+\varepsilon/2)t}\,Y(t),\ t \ge t_0\}$ is point-bounded. By virtue of the Banach – Stenhauss theorem, there exists $c_1 \ge 0$ such that

$$\|e^{-(\alpha+\varepsilon/2)t}\,Y(t)\| \le c_1 \iff \|Y(t)\| \le c_1 e^{(\alpha+\varepsilon/2)t}$$

Therefore, $\|H(t,\tau)\| = \|Y(t-\tau)\| \le c_1 e^{(\alpha+\varepsilon/2)(t-\tau)}$ for the equation with constant operator (12).

On the other hand,

$$\chi[L(t)] = \chi[L^{-1}(t)] = 0 \iff \begin{cases} \|L(t)\| \le c_2 e^{\varepsilon t/2}, \\ \|L^{-1}(t)\| \le c_3 e^{\varepsilon\tau/2}. \end{cases}$$

It follows that

$$\|K(t,\tau)\| \le \|L(t)\|\,\|H(t,\tau)\|\,\|L^{-1}(\tau)\| \le$$
$$\le c_1 c_2 c_3 e^{(\alpha+\varepsilon)(t-\tau)} e^{\varepsilon\tau} = c(\varepsilon, t_0)\, e^{(\alpha+\varepsilon)(t-\tau)},$$

where $c = c_1 c_2 c_3$.

Since $K(t, t_0) = X(t)$, we have $\|X(t)\| \le c\,e^{(\alpha+\varepsilon)t}$.

In the case where $\alpha < 0$, there exists a positive number ε such that $\alpha + \varepsilon \le 0$, whence

$$\|K(t,\tau)\| \le c\,e^{\varepsilon t},$$
$$\|X(t)\| \le c.$$

2. We now prove the theorem. Denoting $y = x\,e^{\gamma(t-t_0)}$, where γ is a positive number such that $0 < \gamma < \lambda$, we transform equation (13) to the form

$$\frac{dy}{dt} = B(t)y + g(t,y) \tag{14}$$

with $B(t) = A(t) + \gamma J d_E$

$$g(t,y) = \exp(\gamma(t-t_0))\,f(t, y\,e^{-\gamma(t-t_0)}). \tag{15}$$

Let us show that the equation

$$\frac{d\eta}{dt} = B(t)\eta \tag{16}$$

is regular. Indeed, by virtue of the regularity of (11), there is a generalized Lyapunov transformation $z = L(t)\xi$ that reduces (11) to an equation witn constant operator

$$\frac{dz}{dt} = Cz,$$

where $C = L'(t)L^{-1}(t) + L(t)A(t)L^{-1}(t)$.

The transformation $\xi = L(t)\eta$ implies the following:

$$\frac{d\xi}{dt} = [L'(t)L^{-1}(t) + L(t)B(t)L^{-1}(t)]\xi = (C + \gamma J d_E)\xi.$$

The regularity of (16) is proved.

— We denote by $\eta(t)$ a solution of (16) and then $e^{-\gamma(t-t_0)}\eta(t)$ is a solution of (11). This yields

$$\chi[\eta(t)e^{-\gamma(t-t_0)}] \le -\lambda \Rightarrow$$

$$\Rightarrow \chi[\eta(t)] \le \chi[e^{\gamma(t-t_0)}] + \chi[\eta(t)e^{-\gamma(t-t_0)}] \le -\lambda + \gamma < 0.$$

By virtue of the estimation of the resolvent operator, the following inequality is true:

$$\| K(t,\tau)\| \le Ne^{\varepsilon\tau}, \quad t_0 \le \tau < \infty,$$

where $K(t,\tau)$ is the resolvent operator of (11).

Now considering the solution of (14)

$$y(t) = K(t,t_0)\, y(t_0) + \int_{t_0}^{t} K(t,\tau)\, g(\tau, y(\tau))\, d\tau,$$

we have

$$\| y(t)\| \le \|K(t,t_0)\| \|y(t_0)\| + \int_{t_0}^{t} \|K(t,\tau)\| \|g(\tau, y(\tau))\|\, d\tau \le$$

$$\le Ne^{\varepsilon t_0}\|y(t_0)\| + \int_{t_0}^{t} Ne^{\varepsilon\tau} e^{\gamma(\tau-t_0)}\psi(\tau)\|y(\tau))\|^m e^{-m\gamma(\tau-t_0)}\, d\tau \le$$

$$\le Ne^{\varepsilon t_0}\|y(t_0)\| + \int_{t_0}^{t} Ne^{\varepsilon\tau} e^{(1-m)\gamma(\tau-t_0)} c e^{\varepsilon\tau}\|y(\tau))\|^m\, d\tau =$$

$$= c_1\|y(t_0)\| + \int_{t_0}^{t} c_2 e^{[2\varepsilon-(m-1)\gamma](\tau-t_0)}\|y(\tau))\|^m\, d\tau,$$

where $c_1 = Ne^{\varepsilon t_0}$, $c_2 = cNe^{-2\varepsilon t_0}$.

Hence,

$$\| y(t)\| \le c_1\|y(t_0)\| + \int_{t_0}^{t} c_2 e^{-\delta(\tau-t_0)}\|y(\tau))\|^m\, d\tau,$$

where $\delta = (m-1)\gamma - 2\varepsilon$.

Let us find a positive number ε such that $\delta > 0$.

Since

$$\int_{t_0}^{t} e^{-\delta(\tau-t_0)}\, d\tau = \frac{1}{\delta} - \frac{1}{\delta}e^{-\delta(t-t_0)} < \frac{1}{\delta},$$

there is $\Delta > 0$ such that

$$N = (m-1)c_1^{m-1}\|y(t_0)\|^{m-1}\int_{t_0}^{t} c_2 e^{-\delta(\tau-t_0)}\, d\tau < 1,$$

provided that

$$\| y(t_0)\| < \Delta.$$

We apply here the Bihari lemma [8] and find

$$\|y(t)\| \le \frac{c_1 \|y(t_0)\|}{[1-N]^{1/(m-1)}} = A\|y(t_0)\|, \quad A = \frac{c_1}{[1-N]^{1/(m-1)}} \quad \Rightarrow$$

$$\Rightarrow \quad \|x(t)\| \le Ae^{-\gamma(t-t_0)}\|x(t_0)\|)^h, \quad x(t_0) = y(t_0),$$

which means the exponential stability of the solution $x = 0$ of (13), and the proof of the theorem is completed.

1. *Bogdanov Yu. S.* Application of generalized exponent numbers to the investigation of the stability of equilibrium point // Dokl. Acad. Nauk SSSR. – 1964. – **158**, № 1. – P. 9–12 (in Russian).
2. *Bogdanov Yu. S.* Generalized exponent numbers of nonautonomous systems // Different. Equat. – 1965. – **1**, № 9. – P. 1140–1148 (in Russian).
3. *Bogdanov Yu. S.* On the reveal of asymptotically stability by means of little $v\,d$-numbers // Ibid. – 1966. – **2**, № 3. – P. 309–313 (in Russian).
4. *Bogdanov Yu. S.* Approximate generalized exponent numbers of differential systems // Ibid. – № 7. – P. 927–933 (in Russian).
5. *Bogdanov Yu. S., Bogdanova M. P.* A nonlinear analog of Lyapunov transformation // Ibid. – 1967. – **3**, № 5. – P. 742–748 (in Russian).
6. *Demidovitch B. P.* Lectures on the mathematical theory of stability. – Moscow: Nauka, 1967. – 472 p. (in Russian).
7. *Daletski Ju. L., Krein M. G.* Stability of solutions of differential equations in a Banach space. – Moscow: Nauka, 1967. – 534 p. (in Russian).
8. *Bihari J.* A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations // Acta math. Acad. sci. hung. – 1956. – **7**, № 1. – P. 81–94.

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