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ON THE LIE ALGEBRA STRUCTURES CONNECTED WITH HAMILTONIAN DYNAMICAL SYSTEMS

ПРО СТРУКТУРИ АЛГЕБР ЛІ, ПОВ'ЯЗАНИХ З ГАМІЛЬТОНОВИМИ ДИНАМІЧНИМИ СИСТЕМАМИ

We construct the hierarchies of master symmetries constituting Virasoro-type algebras for the Hamiltonian vector fields preserving a recursion operator. Similarly repeatedly contracting a Hamiltonian vector field with the corresponding recursion operator, we define an Abelian Lie algebra of thus obtained hierarchy of vector fields. The approach is shown to be applicable for the Volterra and Toda lattices.

Для гамільтонових систем з рекурсивним оператором ієрархії будується мастер симетрій, які формують алгебри Лі типу Вірасоро. Аналогічно, повторно діючи рекурсивним оператором на гамільтонів потік, одержується ієрархія векторних полів, що складають абелеву алгберу Лі. Цей підхід застосовано до систем Вольтерра і Тода..

1. Introduction. We shall study the dynamical systems possessing Hamiltonian structure on an evendimensional Poisson manifold (M^{2n}, P):

$$X_H^i = P^{ik}\frac{\partial H}{\partial x^k} \tag{1}$$

(we use the Einstein summation convention), where P^{ij} is a Poisson bivector, i.e., a skew-symmetric 2-contravariant tensor field with the vanishing Schouten bracket given by (in a local coordinate chart)Û

$$[P, P]^{ijk} := P^{il}\frac{\partial P^{jk}}{\partial x^l} + P^{kl}\frac{\partial P^{ij}}{\partial x^l} + P^{jl}\frac{\partial P^{ki}}{\partial x^l} = 0, \tag{2}$$

and H is the corresponding Hamiltonian. Both P and H are preserved by the vector field X_H: $L_{X_H}P = L_{X_H}H = 0$ (here, L_{X_X} denotes the Lie derivative with respect to X_H). The Poisson bivector P naturally endows the manifold (M^{2n}, P) with the Poisson bracket $\{\,,\}_P$ defined for an arbitrary pair of functions $f, g \in \mathcal{F}(M^{2n})$

$$\{f, g\}_P := P^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}. \tag{3}$$

Condition (2) garantees that the Jacobi identity for bracket (3) is satisfied.

A vector field Y commuting with the initial Hamiltonian vector field X_H: $[Y, X_H] = 0$ is called a *symmetry* of the Hamiltonian system (1). The notion of a *master symmetry* was introduced in [1]. We define it as a vector field Z satisfying $[[Z, X_H], X_H] = 0$, provided that $[Y, X_H] \neq 0$. This is the case, for example, when Z is a *conformal invariance* for X_H, i.e., $L_Z X_H = kX_H$, $k \in \mathbb{R}$. Here, L_Z is the Lie derivative with respect to the vector field X_H. Assume that the Hamiltonian vector field (1) preserves along with the Poisson bivector P a (1, 1) tensor field $A(x)$, $x \in M^{2n}$, $L_{X_H}P = L_{X_H}A = 0$. Then we can construct an infinite hierarchy of vector fields $\{A^n, X_H\}$, $n \in \mathbb{Z}_+$. Note that $(AX_H)^i = A_l^i X_H^l$. Analogously, if, in addition, X_H has a conformal invariance Z_0, we come up with a similar hierarchy $\{A^n, Z_0\}$,

$n \in \mathbb{Z}_+$. Under certain assumptions for the operator A, both of these hierarchies have remarkable properties, namely, the former one becomes a commuting Lie algebra of vector fields, while the latter one becomes a Lie algebra isomorphic to the Virasoro algebra defined over $\mathbb{C}$ with the basis L_n, $n \in \mathbb{Z}$, c (the central element) and the following commutator:

$$[L_m, L_n] = (m-n)L_{m-n} + \delta_{m,-n}\frac{(m^3-m)c}{12}, \tag{4}$$

$$[c, L_n] = 0.$$

This is the subject of considerations that follow.

2. The Main Result.

Definition. *We call a* (1, 1) *tensor* $A(x)$, $x \in M^{2n}$, *a recursion operator if its Nijenhuis tensor vanishes identically, i.e.,*

$$N_A := A^2[X, Y] + [AX, AY] - A([X, AY] + [AX, Y]) = 0, \tag{5}$$

where $X, Y \in T(M^{2n})$.

If we consider the Lie derivatives instead of commutators, equation (5) is equivalent to

$$N_A(X, Y) = (L_{AX}A - AL_X A)Y = 0. \tag{6}$$

Theorem 1. *Let* $Z_0 \in T(M^{2n})$ *be a conformal invariance for a recursion operator* $A(x)$, $x \in M^{2n}$, *and a vector field* $X_0 \in T(M^{2n})$:

$$L_{Z_0} X = \alpha X_0, \quad L_{Z_0} A = \beta A, \quad \alpha, \beta \in \mathbb{R}.$$

In addition, $L_{X_0} A = 0$. *Define the following hierarchies of vector fields:* $\{X_n\}_{n\geq 0}$, $\{Z_n\}_{n\geq 0}$, *where* $X_n = A^n X_0$ *and* $Z_n = A^n Z_0$, $n \in \mathbb{Z}_+$.

Then the hierarchy $\{X_n\}_{n\geq 0}$ *constitutes a commutative Lie algebra, while* $\{Z_n\}_{n\geq 0}$ *is a Lie algebra with the Virasoro commutator relation* (4) *(with zero central element). Moreover,*

$$L_{X_n} A = 0 \quad \text{and} \quad L_{Z_n} A = \beta A^{n+1}.$$

Proof. Let us show first that $L_{X_n} A = 0$. Indeed, repeatedly applying relation (6), we derive the following equalities:

$$L_{A^n X} A = AL_{A(A^{n-1}X)} A = AL_{A^{n-1}X} A = \ldots = A^{n-1} L_{AX_0} A = A^n L_{X_0} A = 0.$$

Consider the hierarchy $\{X_n\}_{n\geq 0}$, where $X_n = A^n X_0$. Then, for an arbitrary $n \in \mathbb{Z}_+$,

$$[X_0, X_n] = A^n L_{X_0} X_0 + (L_{X_0} A^n) X_0 = 0.$$

Now assume that, for any $m \neq n$, we have $[X_n, X_m] = 0$. Then using the Leibniz rule for the Lie derivative, we obtain

$$[X_n, X_{m+1}] = [X_n, AX_m] = (L_{X_n} A) X_m + A[X_n, X_m] = (L_{X_n} A) X_m] = 0.$$

Hence, by induction, X_m commutes with all members of the hierarchy $\{X_n\}_{n\geq 0}$,

$n \in \mathbb{Z}_+$. And since m was picked out arbitrarily, $\{X_n\}_{n\geq 0}$ forms a commutative Lie algebra.

Analogously, for the hierarchy $\{Z_n\}_{n\geq 0}$, $Z_n := A^n Z_0$, $n \in \mathbb{Z}_+$, we first prove that $L_{Z_n} A = \beta A^{n+1} Z_0$. Indeed, employing the same technique again, we get

$$L_{Z_n} A = L_{A^n Z_0} A = L_{A(A^{n-1})Z_0} A = L_{A^{n-1} Z_0} A = \ldots = A^n L_{Z_0} A = \beta A^{n+1}.$$

Then,

$$[Z_0, Z_n] = L_{Z_0} A^n Z_0 = A^n L_{Z_0} Z_0 + (L_{Z_0} A^n) Z_0 = \beta n Z_n.$$

Now assume, as before, that, for any $m \neq n$,

$$[Z_n, Z_m] = \beta (m-n) Z_{n+m}. \tag{7}$$

Then,

$$[Z_n, Z_{m+1}] = [Z_n, A^{m+1} Z_0] = (L_{Z_n} A) A^m Z_0 + A[Z_n, Z_m] =$$

$$= \beta A^{n+1} A^m Z_0 + \beta (m-n) A Z_{n+m} = \beta Z_{n+1+m} + \beta (m-n) Z_{n+m+1} =$$

$$= \beta (m+1-n) Z_{n+m+1}.$$

Therefore, again by induction, for arbitrary integers n and m, we see that (7) takes place. This completes the proof.

Corollary. *If A is invertible, we can extend the hierarchy $\{Z_n\}_{n\geq 0}$ for negative n as well:*

$$\{\ldots, A^{-n} Z_0, \ldots, A^{-1} Z_0, Z_0, A Z_0, \ldots, A^n Z_0, \ldots\}.$$

Then, for $\beta = 1$, the Lie algebra of vector fields $\{Z_0, Z_1, \ldots, Z_n, \ldots\}$ is isomorphic to the Virasoro algebra with central element zero $c = 0$.

Proof. Indeed, the map $f\colon Z_n \to L_n$, $n \in \mathbb{R}$, preserved the algebraic structures (4), (7) and is bijective.

Remark. In the case of invertible A this Lie algebra possesses the following automorphism for any integer $n \in \mathbb{Z}$:

$$g\colon Z_{-n} \to -Z_n.$$

Example: The Volterra lattice.

Consider the finite nonperiodic Volterra lattice [2], i.e., the system of the following n equations:

$$\frac{dR_1}{dt} = -e^{-R_2(t)},$$

$$\frac{dR_k}{dt} = e^{-R_{k-1}(t)} - e^{-R_{k+1}(t)}, \quad k = 2, \ldots, n-1, \tag{8}$$

$$\frac{dR_n}{dt} = e^{-R_{n-1}(t)}.$$

It has the Hamiltonian representation (1) for the vector field

$$X_0^i := \frac{dR_i}{dt}, \quad i = 1, \ldots, n,$$

the Hamiltonian

$$H_0 = \sum_{i=1}^{n} e^{R_i(t)},$$

and the Poisson bivector P_0 defined by the $n \times n$ matrix $\| p_0^{ij} \|$ with the following nonzero entries:

$$p_0^{i,i+1} = 1, \quad p_0^{i,i-1} = -1, \quad i = 1, \dots, n-1.$$

We shall use Theorem 1 in order to construct a hierarchy of master symmetries for system (8) connected by the Virasoro relation (4).

Consider the vector field

$$Z_0 = \lambda \sum_{i=1}^{n} \frac{\partial}{\partial R_i}, \quad \lambda \in \mathbb{Z},$$

and the operator tensor field A defined by the $n \times n$ matrix $\| a_j^i \|$ with the following nonzero entries: $a_i^i = e^{-R_i(t)}$, $i = 1, \dots, n$. Note that the linear operator A is invertible. By the Nijenhuis theorem [3], A has the vanishing Nijenhuis tensor N_A (5) in the coordinates $R_1, \dots, R_n$, since it is defined by a diagonal matrix with the property that each eigenvalue depends only on the corresponding coordinate. By virtue of tensorial properties of N_A, we conclude that A is a recursion operator in any system of coordinates. Direct calculation show that the vector field Z_0 is a conformal invariance for both X_0 and A:

$$L_{Z_0} X_0 = -\lambda X_0, \quad L_{Z_0} A = -\lambda A.$$

Applying Theorem 1 for $X_n := A^n X_0$ and $Z_n := A^n Z$, $n \in \mathbb{Z}$, we get

$$L_{X_n} A = 0, \quad L_{Z_n} A = -\lambda A^{n+1},$$

and

$$\begin{aligned} [X_n, X_m] &= 0, \\ [Z_n, Z_m] &= -\lambda (m-n) Z_{n+m}. \end{aligned} \tag{9}$$

The hierarchy of symmetries Z_m, $m \in \mathbb{Z}$, forms a Lie algebra with the Virasoro commutator relation (9) and, in view of Corollary 1, for $\lambda = -1$, this algebra is isomorphic to the Virasoro algebra with central element zero, while the vector fields X_n, $n \in \mathbb{Z}$, form a commutative Lie algebra.

3. The Bi-Hamiltonian Case. Lie algebra properties connected with the chains of vector fields considered above were based on the existence of a (1, 1) tensor field A satisfying the invariance equation:

$$L_{X_0} A = 0. \tag{10}$$

For arbitrary Hamiltonian vector field (1), this condition is not always satisfied. The situation is different when we deal with the bi-Hamiltonian case, namely, when the dynamical system (1) has two Hamiltonian forms

$$X = P_0 dH_0 = P_1 dH_1. \tag{11}$$

Here, P_0, P_1 are compatible Poisson bivectors, i.e., their Schouten bracket vanishes identically,

$$[P_1, P_2]^{ijk} := \frac{\partial P_1^{ij}}{\partial x^\mu} P_2^{\mu k} + \frac{\partial P_2^{ij}}{\partial x^\mu} P_1^{\mu k} + (\text{cycle}) = 0 \tag{12}$$

(here, cycle means cyclic permutation of i, j, and k), and H_0, H_1 are the corresponding Hamiltonians. The compatibility condition (12), which guarantees the integrability of system (11) [4–6], can be reformulated in an alternative way. Since either of the Poisson bivectors P_0, P_1 is nondegenerate (e.g., P_0), we can construct a (1, 1) tensor $A := P_1 P_0^{-1}$. Then condition (12) is equivalent to the fact that A is a recursion operator, i.e., satisfies relation (5) [5]. In this case, the matrix of the operator A has doubly degenerate eigenvalues as a product of two skew-symmetric matrices P_1 and P_0^{-1}. Assuming that all these eigenvalues are functionally independent, we conclude that system (11) is completely integrable in the Arnol'd–Liouville sense [4–6]. The functions $H_n := 1/n \operatorname{Tr}(A^n)$ are the first invariants of the vector field X, in involution with respect to the Poisson brackets defined by the Poisson bivectors P_0, P_1. Obviously, A satisfies the invariance equation (10). In this case, the recursion operator A appears rather naturally and, if an appropriate conformal invariance is found, we can formulate a kindred of Theorem 1.

Theorem 2. *Let us have a bi-Hamiltonian dynamical system* (11) *defined by the vector field X_0 along with Poisson bivectors P_0 and P_1*

$$X_0 := P_0 dH_0 = P_1 dH_1,$$

integrable in the Arnol'd–Liouville sense. Assume that there be a vector field Z_0 generating a conformal invariance for X_0, P_0 and $\omega_1 := P_0^{-1}$ (provided that P_0 is nondegenerate);

$$L_{Z_0} X_0 = \alpha X_0, \quad L_{Z_0} P_0 = \beta P_0, \quad L_{Z_0} \omega_1 = \gamma \omega_1, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Then, defining $A := P_0 \omega_1$ and $Z_n := A^n Z_0$, one finds, for all n, m,

$$L_{X_n} X_m = 0, \tag{13}$$

$$L_{Z_n} Z_m = (m-n)(\beta+\gamma) Z_{n+m}. \tag{14}$$

Proof. The first part (13) coincides with the analogous statement of Theorem 1, since $A := P_0 \omega_1$ is a recursion operator and satisfies condition (10). The part about the vector fields $\{Z_n\}_{n \geq 0}$ can also be derived from the previous theorem. Indeed, applying the Leibniz rule, we get

$$L_{Z_0} A = P_0 L_{Z_0} \omega_1 + (L_{Z_0} P) \omega_1 = (\beta+\gamma) A,$$

and the result follows.

Note that equation (10) can be interpreted in terms of the Lax formalism. Indeed, in a local system of coordinates $(x_0, x_1, \ldots, x_n)$, (10) can be rewritten as

$$(L_X A)^i_j = X^k \frac{\partial A^i_j(x)}{\partial x^k} + A^i_k(x) \frac{\partial X^k}{\partial x_j} - A^k_j(x) \frac{\partial X^i}{\partial x^k} = 0,$$

or

$$X^k \frac{\partial A^i_j(x)}{\partial x^k} = A^k_j(x) \frac{\partial X^i}{\partial x^k} - A^i_k(x) \frac{\partial X^k}{\partial x_j}. \tag{15}$$

Define now the linear operator

$$B^i_j := \frac{\partial X^i}{\partial x^j}.$$

Taking into account that

$$\frac{dx^k}{dt} = \dot{x}^k(t) = X^k(x^1(t), \ldots, x^n(t)),$$

it easily follows that (15) is equivalent to

$$\frac{d}{dt} A^i_j(x) = A^k_j(x) B^i_k(x) - B^k_j(x) A^i_k(x),$$

which is the following matrix equation:

$$\dot{A} = [A, B]. \tag{16}$$

Equation (16) is reminiscent of the Lax equation and, thus, possesses certain algebraic structures (see, for example, [7]).

Example: the Toda lattice.

Consider the finite, nonperiodic Toda lattice, i.e., the system of equations that describes dynamics of a one-dimensional lattice of particles with exponential interaction of nearest neighbors. In terms of the canonical coordinates q^i and moments p_i, $i = 1, 2, \ldots, n$, it is given by

$$\begin{aligned} \frac{dq^i}{dt} &= p_i, \\ \frac{dp_i}{dt} &= e^{q^{i-1}-q^i} - e^{q^i-q^{i-1}}, \end{aligned} \tag{17}$$

where $q^i(t)$ can be interpreted as the coordinate of the i th particle in the lattice. This system takes the Hamiltonian form (1), and its Hamiltonian function H_0 is defined by the formula

$$H_0 := \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i-q^{i+1}},$$

while the corresponding Poisson bivector P_0 is defined by the canonical symplectic form $\omega_0 := P_0^{-1}$,

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

This particular case of the Toda lattice was thoroughly studied by A. Das and S. Okubo in [8] from the bi-Hamiltonian point of view. There, the second symplectic form ω_1 was found to be

$$\omega_1 = \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} dq^i \wedge dq^{i+1} + \sum_{i=1}^{n} p_i dq^i \wedge dp_i + \frac{1}{2} \sum_{i<j}^{n} dp_i \wedge dp_j$$

and

$$H_1(q, p) = \frac{1}{3} \sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1} (p_i + p_{i+1}) e^{q^i-q^{i-1}}$$

as the corresponding Hamiltonian. Furthermore, the corresponding operator $A := \omega_1 \omega_0^{-1}$ given by the formula

$$A = \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \otimes dq^i + \sum_{i=1}^{n-1} e^{q^i - q^{i+1}} \left(\frac{\partial}{\partial p_{i+1}} \otimes dq^i - \frac{\partial}{\partial p_i} \otimes dq^{i+1} \right) +$$

$$+ \frac{1}{2} \sum_{i<j}^{n} \left(\frac{\partial}{\partial q^i} \otimes dp_j - \frac{\partial}{\partial q^j} \otimes dp_i \right) + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \otimes dp_i$$

was proved to be a recursion operator [8]. Moreover, the linear operator A is invertible. This fact leads to the integrability of system (15) in the Arnol'd–Liouville sense as a bi-Hamiltonian system (see [4–6]). Consider now the vector field Z_0 given by

$$Z_0 = \sum_{i=1}^{n} \left[2(n+1-i) \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial p_i} \right], \tag{18}$$

for which one finds

$$L_{Z_0} X_0 = -X_0, \quad L_{Z_0} \omega_1 = 2\omega_1, \quad L_{Z_0} P_0 = -P_0,$$

where $P_0 := \omega_0^{-1}$ and X_0 is the vector field of system (15). Note that Z_0 was introduced in [9] as a nontrivial master symmetry for the vector field generating system (15).

Setting $Z_n = A^n Z_0$, $X_n = A^n X_0$, $\omega_{n+1} = \omega_1 A^n$, and $P_n := A^n P_0$ and applying Theorem 2, we arrive at the relations

$$[X_n, X_m] = 0, \quad [Z_n, Z_m] = (m-n) Z_{n+m}$$

for all integers n, m. Note that, in this case, the master symmetries Z_n, $z \in \mathbb{Z}$, form a Lie algebra isomorphic to the Virasoro algebra with central element zero (see Corollary 1 of Theorem 1).

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