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On the difference between the spectral radius and the maximum degree of graphs*

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Abstract. Let G be a graph with the eigenvalues $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$. The largest eigenvalue of G, $\lambda_1(G)$, is called the spectral radius of G. Let $\beta(G) = \Delta(G) - \lambda_1(G)$, where $\Delta(G)$ is the maximum degree of vertices of G. It is known that if G is a connected graph, then $\beta(G) \geqslant 0$ and the equality holds if and only if G is regular. In this paper we study the maximum value and the minimum value of $\beta(G)$ among all non-regular connected graphs. In particular we show that for every tree T with $n \geqslant 3$ vertices, $n - 1 - \sqrt{n-1} \geqslant \beta(T) \geqslant 4\sin^2(\frac{\pi}{2n+2})$. Moreover, we prove that in the right side the equality holds if and only if $T \cong P_n$ and in the other side the equality holds if and only if $T \cong S_n$, where P_n and S_n are the path and the star on n vertices, respectively.

1. Introduction

Throughout this paper all graphs are simple, that is finite and undirected without loops and multiple edges. Let $G = (V, E)$ be a simple graph. The *order* of G denotes the number of vertices of G. For two vertices u and v by $e = uv$ we mean the edge e between u and v. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set

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$E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. A *regular graph* is a graph that all of its vertices have the same degree. By $\Delta(G)$ we mean the maximum degree of vertices of G. For two vertices u and v of connected graph G, the distance between u and v in G that is denoted by $d(u,v)$, is the length of a shortest path between u and v. The greatest distance between any two vertices of G is the *diameter of* G, denoted by $diam(G)$. The *complete graph*, the *cycle*, and the *path* of order n, are denoted by K_n, C_n and P_n, respectively. We denote the *complete bipartite graph* with part sizes m and n, by $K_{m,n}$. The *star* of order n that is denoted by S_n is the complete bipartite graph $K_{1,n-1}$.

Let G be a graph with vertex set $\{v_1,\dots,v_n\}$. The *adjacency matrix* of G, $A(G)=[a_{ij}]$, is an $n\times n$ matrix such that $a_{ij}=1$ if v_i and v_j are adjacent, and otherwise $a_{ij}=0$. Thus $A(G)$ is a symmetric matrix with zeros on the diagonal and all the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G)\geqslant\cdots\geqslant\lambda_n(G)$. By the *spectral radius* of G we mean $\lambda_1(G)$. We note that $\lambda_1(G)$ is also called the *index of* G. It is well known that $|\lambda_i(G)|\leqslant\lambda_1(G)$, for $i=1,\dots,n$. Many papers are devoted to study the characteristic polynomials and spectra of the adjacency matrix of graphs, in particular characterization of graphs by their eigenvalues and finding the location of eigenvalues of graphs, see [1]–[17] and the references therein. Studying the spectral radius of graphs has always been of great interest to researchers in graph theory, for instance see [1], [3], [5], [6], [11] and [13]–[17].

Let G be a graph. It is a well known fact that $\lambda_1(G)\leqslant\Delta(G)$. Moreover if G is connected, then the equality holds if and only if G is regular. Therefore it is natural to ask about the value of $\Delta(G)-\lambda_1(G)$. For a graph G by $\beta(G)$ we mean $\beta(G)=\Delta(G)-\lambda_1(G)$. Hence for every graph G, $\beta(G)\geqslant 0$. Also if G is connected, then G is regular if and only if $\beta(G)=0$. One can regard $\beta(G)$ as a parameter that indicates the measure of irregularity of G. There are some papers related this parameter. Cioabă [3] has proved that if G is a non-regular connected graph of order n, then $\Delta(G)-\lambda_1(G)>\frac{1}{nd}$, where $d=diam(G)$. In this paper we study the maximum value and the minimum value of $\beta(G)$ among all non-regular graphs. We show among all non-regular graphs the stars have the maximum value of β. We prove that for every tree T with $n\geqslant 3$ vertices, $n-1-\sqrt{n-1}\geqslant\beta(T)\geqslant 4\sin^2(\frac{\pi}{2n+2})$. Moreover we obtain that in the right side the equality holds if and only if $T\cong P_n$

and in the other side the equality holds if and only if $T \cong S_n$. Finally we conjecture that among all non-regular connected graph the paths have the minimum value of β.

2. Results

In this section we obtain the maximum and the minimum value of the difference between the spectral radius and the maximum degree of non-regular connected graphs. We need the following result.

Theorem 1. [4] *Let G be a connected graph. If H is a proper subgraph of G, then $\lambda_1(G) > \lambda_1(H)$.*

Now we show that among all graphs the stars attain the maximum value of β.

Theorem 2. *Let G be a graph of order n. Then*

$$\beta(G) \leqslant \beta(S_n) = n - 1 - \sqrt{n-1}.$$

Moreover the equality holds if and only if $G \cong S_n$.

Proof. First we prove the theorem for connected graphs. Let H be a connected graph of order t. Suppose that $H \ncong S_t$. We show that $\beta(H) < \beta(S_t)$. For $t = 1$, there is noting to prove. So let $t \geqslant 2$. Let $h = \Delta(H)$. Hence $h \geqslant 1$. Since S_{h+1} is a proper subgraph of H, by Theorem 1 we obtain that

$$\lambda_1(H) > \lambda_1(S_{h+1}) = \sqrt{h}. \tag{1}$$

For every $x > 0$, let $f(x) = x - \sqrt{x}$. This function is increasing on the interval $[\frac{1}{4}, \infty)$. Since $t - 1 \geqslant h \geqslant 1$ by (1),

$$\beta(S_t) = t-1-\sqrt{t-1} = f(t-1) \geqslant f(h) = h - \sqrt{h} > h - \lambda_1(H) = \beta(H).$$

So for connected graphs the result follows. Now assume that $G \ncong S_n$ be a non-connected graph of order n. So $\Delta(G) \leqslant n-2$. If $\Delta(G) = 0$, there is nothing to prove. Hence $1 \leqslant \Delta(G) \leqslant n-2$. On the other hand similar to above, one can see that $\lambda_1(G) \geqslant \sqrt{\Delta(G)}$. Since $n - 1 > \Delta(G) \geqslant 1$,

$$f(n-1) > f(\Delta(G)) = \Delta(G) - \sqrt{\Delta(G)} \geqslant \Delta(G) - \lambda_1(G) = \beta(G).$$

Therefore $\beta(G) < f(n-1) = n - 1 - \sqrt{n-1}$. The proof is complete. $\square$

Remark 1. For every $n \geqslant 2$, let $H_n = K_n + K_1$. Then $\Delta(H_n) = \lambda_1(H_n) = n-1$. Therefore $\beta(H_n) = 0$ while H_n is not regular. This example shows that the minimum value of $\beta(G)$ among all non-regular graphs G or order $n \geqslant 2$ is zero.

In sequel we study the minim value of $\beta(G)$ among the family of non-regular connected graphs G. We need the following nice upper bound on the spectral radius of trees.

Theorem 3. [14] *Let T be a tree with maximum degree Δ. Then*

$$\lambda_1(T) < 2\sqrt{\Delta - 1}.$$

Theorem 4. *Let T be a tree of order $n \geqslant 3$. Then*

$$\beta(T) \geqslant \beta(P_n) = 4\sin^2\left(\frac{\pi}{2n+2}\right).$$

Moreover the equality holds if and only if $T \cong P_n$.

Proof. Let $n \geqslant 3$. Since $\lambda_1(P_n) = 2\cos\frac{\pi}{n+1}$ and $\Delta(P_n) = 2$, $\beta(P_n) = 2 - 2\cos\frac{\pi}{n+1} = 4\sin^2(\frac{\pi}{2n+2})$. For every $x > 1$, let $f(x) = x - 2\sqrt{x-1}$. It is easy to see that f is an increasing function on the interval $[2, \infty)$. Therefore for every $x \geqslant 3$, $f(x) \geqslant 3 - 2\sqrt{2}$. On the other hand it is not hard to see that for every $n \geqslant 7$, $3 - 2\sqrt{2} > 4\sin^2(\frac{\pi}{2n+2})$. Hence for every $x \geqslant 3$ and $n \geqslant 7$ we obtain that

$$f(x) > 4\sin^2\left(\frac{\pi}{2n+2}\right). \tag{2}$$

One can check the result for $n \leqslant 6$. Now let $n \geqslant 7$. Let $T \not\cong P_n$ be a tree of order n. We show that $\beta(T) > \beta(P_n)$. Since $T \not\cong P_n$, $\Delta(T) \geqslant 3$. On the other hand by Theorem 3, $\lambda_1(T) < 2\sqrt{\Delta(T) - 1}$. Since $\Delta(T) \geqslant 3$ and $n \geqslant 7$, by (2),

$$\begin{aligned}\beta(T) &= \Delta(T) - \lambda_1(T) > \Delta(T) - 2\sqrt{\Delta(T) - 1} \\ &= f(\Delta(T)) > 4\sin^2\left(\frac{\pi}{2n+2}\right) = \beta(P_n).\end{aligned}$$

This completes the proof. $\square$

Using Theorems 2 and 4 we obtain the following result.

Theorem 5. *Let T be a tree of order n. Then*

$$\beta(P_n) \leqslant \beta(T) \leqslant \beta(S_n).$$

Moreover in the left side the equality holds if and only if $T \cong P_n$ and in the other side the equality holds if and only if $T \cong S_n$.

We think that among all non-regular connected graphs G of order n the path P_n has the minimum value of β. Hence we pose the following conjecture.

Conjecture 1. Let G be a non-regular connected graph of order n. If $G \ncong P_n$, then $\beta(G) > \beta(P_n)$.

Now we show that the Conjecture 1 is valid for graphs with small diameter.

Theorem 6. [3] *Let G be a non-regular connected graph of order n. Then*

$$\Delta(G) - \lambda_1(G) > \frac{1}{nd},$$

where d is the diameter of G.

Theorem 7. *Let G be a non-regular connected graph of order $n \geqslant 3$. If $d = diam(G) \leqslant \frac{n}{10}$, then $\beta(G) > \beta(P_n)$.*

Proof. Since $10 > \pi^2$, for every $n \geqslant 1$, $\frac{10}{\pi^2} > (\frac{n}{n+1})^2$. On the other hand for $n \geqslant 1$, $\frac{\pi}{2n+2} \geqslant \sin(\frac{\pi}{2n+2})$. Hence for every $n \geqslant 1$, $\frac{10}{n^2} > 4(\frac{\pi}{2n+2})^2 \geqslant 4\sin^2(\frac{\pi}{2n+2})$. This shows that $\frac{1}{nd} \geqslant \frac{10}{n^2} > 4\sin^2(\frac{\pi}{2n+2})$. Therefore for every $n \geqslant 3$, by Theorem 6 we obtain that

$$\Delta(G) - \lambda_1(G) > \frac{1}{nd} > 4\sin^2\left(\frac{\pi}{2n+2}\right) = \beta(P_n).$$

The proof is complete. $\square$

Remark 2. One can see that for every $n \geqslant 2$, $\lambda_1(K_n \setminus e) = \frac{n-3+\sqrt{n^2+2n-7}}{2}$ where e is an edge of K_n, see [8]. Therefore $\lim_{n\to\infty}(\beta(K_n \setminus e)) = 0$.

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