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RESEARCH ARTICLE

Lattice rings: an interpretation of L-fuzzy rings as habitual algebraic structures

Leonid A. Kurdachenko, Igor Ya. Subbotin, Viktoriia S. Yashchuk

In memoriam Professor Vitaly Sushchansky

Abstract. In this paper, we introduce some algebraic structure associated with rings and lattices. It appeared as the result of our new approach to the fuzzy rings and L-fuzzy rings where L is a lattice. This approach allows us to employ more convenient language of algebraic structures instead of currently accepted language of functions.

Introduction

The purpose of this work is to look at some algebraic structures that relate to defined-on-a group functions with a somewhat different angle. If S is a set, then for each of its subset, M, there is a corresponding characteristic function, namely the mapping $\chi_M \colon S \to \{0, 1\}$ such that $\chi_M(y) = 1$ for all $y \in M$ and $\chi_M(y) = 0$ for all $y \notin M$. In many cases, the subset M is equated with its characteristic function. With the classical work of L.A. Zadeh [12] fuzzy mathematics, which is based on a generalization of characteristic functions, begins. A fuzzy set on a set S is a sort of generalized "characteristic function" on S, whose "degrees of membership" may be more general than "yes" or "no". In fact, we assume the existence of a set (from here it will be denoted by L) of degrees of membership. In an optimization problem, L may express the degree of optimality of the choice (in S), while in a classification problem,

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it may express the degree of membership in a pattern class. In other contexts, other terminologies appear. In fuzzy mathematics, a habitual step was to address the situation where $L = [0,1]$ is an ordinary closed interval of real numbers with its natural order. The motivation for this is the following interpretation: we can consider the value of a generalized characteristic function as the probability of the belonging of this element to this subset. Algebraic fuzzy structures appeared here as follows. To every algebraic structure A one ties an appropriate fuzzy structure, which described the specific functions of A in $[0,1]$ associated with the ordinary algebraic structure A (see, for example [9]). Thus fuzzy group theory studies the functions $\gamma\colon G \to [0,1]$ where G is a group, satisfying the following conditions:

$$\gamma(xy) \geqslant \gamma(x) \wedge \gamma(y) \text{ for all } x, y \in G; \text{ and } \gamma(x^{-1}) \geqslant \gamma(x) \text{ for every } x \in G$$

(see, for example, [10], §1.2).

Fuzzy ring theory considers the functions $\kappa\colon R \to [0,1]$ where R is a ring satisfying the following conditions:

$$\kappa(x-y) \geqslant \kappa(x) \wedge \kappa(y) \quad \text{and} \quad \kappa(xy) \geqslant \kappa(x) \wedge \kappa(y) \quad \text{for all } x, y \in R.$$

Fuzzy ring theory takes its roots in the works [5], [11]. Immediately, some generalizations appeared. More concretely, the functions $\gamma\colon A \to \mathfrak{L}$ where A is some algebraic structure and $\mathfrak{L}$ is a distributive lattice have been considered [1]. In particular, in the paper [6], the L-fuzzy rings were introduced (see, also the book [7]). Fuzzy algebra theory has been developed very rapidly, but it was the upswing in breadth rather than in depth. Outwardly, it looks like a collection of separate results. The most advanced here is the theory of fuzzy groups. There is the monograph [10] specifically devoted to fuzzy groups. But even in the book [10], it was no attempt to systematize the results: a large array of results on fuzzy groups have just been collected in this book. Concerning L-fuzzy groups, L-fuzzy rings and other L-fuzzy algebraic structures. However, except most common results, significant progress is not observed here. Note that interpretation of the algebraic structure as a certain function is quite inconvenient.Therefore, quite often the function $\gamma\colon A \to \mathfrak{L}$ is treated as a set of all point functions. More accurately it is viewed as a union of all point functions $\chi(g, \gamma(g)), g \in A$. (Here $\chi(g, \mathfrak{a})$ is a function such that $\chi(g, \mathfrak{a})(g) = \mathfrak{a}$, $\chi(g, \mathfrak{a})(y) = 0$ whenever $y \neq g$). In this sense, there is the following subtlety. There is a natural order on the set of such functions which is entered by the rule: $\gamma \leqslant \kappa$ if and only if $\gamma(x) \leqslant \kappa(x)$ for all $x \in A$. In other words, we can say about the subfunctions, and then a subfunction must be treated as a subset. Based on this logic one should consider the function γ as the union of all point functions $\chi(g, \mathfrak{a})$ for all $g \in G$ and $\mathfrak{a} \leqslant \gamma(g)$. This approach was taken, for example, in the papers [2], [3], which allowed to obtain natural fuzzy analogues of some important concepts of group theory. However, its point functions $\chi(g, \mathfrak{a})$, actually acting as elements, are formally subfunctions of γ, so that every time it is necessary to resort to special reservations. In the paper [4], the interpretation of an L-fuzzy group as a set with operations was offered. With this approach, the basic concepts and results acquire its natural algebraic form, and the process of their appearance becomes more meaningful. Moreover, a construction giving a very clear idea of these objects' structures has been demonstrated. In this paper, we consider a similar interpretation for L-fuzzy rings. The resulting structure is formally different, and therefore different terminology will be used for it. Furthermore, the term an L-fuzzy ring does not reflect the facts. We do not seek maximum generality, it seems more natural to consider the case, when the lattice $\mathfrak{L}$ is distributive and finite, even though all of the obtained results could be extended to the case of an arbitrary complete distributive lattice. In the paper [4] we did not touch the concept of homomorphism. Here this concept will be discussed in great detail.

1. Preliminary results

Let R be a ring and $\mathfrak{L}$ be a finite distributive lattice. Being finite, it has the greatest element $\mathfrak{m}^{\uparrow}$ and the least element $\mathfrak{m}_{\downarrow}$. Consider the Cartesian product $A = R \times \mathfrak{L}$. Define the operations on A by the following rule:

$$(u, \mathfrak{a}) + (v, \mathfrak{b}) = (u + v, \mathfrak{a} \wedge \mathfrak{b}) \quad \text{and} \quad (u, \mathfrak{a})(v, \mathfrak{b}) = (uv, \mathfrak{a} \wedge \mathfrak{b})$$

for all $u, v \in R$ and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$.

Clearly the operation of addition is commutative and associative, because the addition in R and the operation $\wedge$ in $\mathfrak{L}$ are commutative and associative. Pair $(0, \mathfrak{m}^{\uparrow})$ is a zero element. If the multiplication on R is associative, then the multiplication on A is also associative. If R has an multiplicative identity element e, then a pair $(e, \mathfrak{m}^{\uparrow})$ is a an identity element in A. If the multiplication on R is commutative, then the multiplication on A is also commutative. We can define the operation of subtraction on A in a usual way

$$(u, \mathfrak{a}) - (v, \mathfrak{b}) = (u - v, \mathfrak{a} \wedge \mathfrak{b}) \quad \text{for all } u, v \in R,\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{L}.$$

A nonempty subset K of $R \times \mathfrak{L}$ is called a *lattice ring over* $\mathfrak{L}$ if it satisfies the following conditions:

$$\begin{aligned} &\text{(LR 1)} \quad \text{if } (x, \mathfrak{a}) \in K \text{ and } \mathfrak{b} \leqslant \mathfrak{a}, \text{ then } (x, \mathfrak{b}) \in K; \\ &\text{(LR 2)} \quad \text{if } (x, \mathfrak{a}), (y, \mathfrak{b}) \in K, \text{ then } (x, \mathfrak{a}) - (y, \mathfrak{b}) \in K; \\ &\text{(LR 3)} \quad \text{if } (x, \mathfrak{a}), (y, \mathfrak{b}) \in K, \text{ then } (x, \mathfrak{a})(y, \mathfrak{b}) \in K; \end{aligned}$$

If K is a lattice ring and $(y, \mathfrak{b}) \in K$, then using (LR 2) we obtain that $(y, \mathfrak{b}) - (y, \mathfrak{b}) = (y - y, \mathfrak{b}) = (0, \mathfrak{b}) \in K$. It follows that $(0, \mathfrak{b}) - (y, \mathfrak{b}) = (-y, \mathfrak{b}) \in K$, and hence, if $(x, \mathfrak{a}), (y, \mathfrak{b}) \in K$, then

$$(x, \mathfrak{a}) + (y, \mathfrak{b}) = (x + y, \mathfrak{a} \wedge \mathfrak{b}) = (x - (-y), \mathfrak{a} \wedge \mathfrak{b}) = (x, \mathfrak{a}) - (-y, \mathfrak{b}) \in K.$$

Thus every lattice ring K is closed by multiplication and contains $(0, \mathfrak{a})$ for each element $a \in \mathrm{pr}_{\mathfrak{L}}(K)$.

Let K, Σ be the lattice rings over $\mathfrak{L}$. If K includes Σ, then we will say that Σ is a *lattice subring* of K and will denote this by $\Sigma \leqslant K$.

Clearly $R \times \mathfrak{L}$ is the greatest lattice ring over $\mathfrak{L}$, and $\{(0, \mathfrak{m}_{\downarrow})\}$ is the least lattice ring over $\mathfrak{L}$. The last lattice ring is called *trivial.* Furthermore, if $\mathfrak{a} \in \mathfrak{L}$, then $\{(0, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{a}\}$ is a lattice ring over $\mathfrak{L}$.

Every lattice ring K includes $\mathrm{pr}_R(K) \times \{\mathfrak{m}_{\downarrow}\}$. For every subring S of R the subset $S \times \{\mathfrak{m}_{\downarrow}\}$ is a lattice ring.

Proposition 1. *Let R be a ring, $\mathfrak{L}$ be a finite distributive lattice and $\mathfrak{S}$ be a family of lattice subrings over $\mathfrak{L}$. Then and the intersection $\cap\mathfrak{S}$ is a lattice subring.*

Proof. The proof is almost obvious.

Recall that a subset $\mathfrak{M}$ of $\mathfrak{L}$ is called a *lower* (respectively *upper*) *segment* of $\mathfrak{L}$ if from $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{b} \leqslant \mathfrak{a}$ (respectively $\mathfrak{a} \leqslant \mathfrak{b}$) it follows that $\mathfrak{b} \in \mathfrak{M}$.

If $\mathfrak{a} \in \mathfrak{L}$ then $\{\mathfrak{x} | \mathfrak{x} \in \mathfrak{L} \text{ and } \mathfrak{x} \leqslant \mathfrak{a}\}$ (respectively $\{\mathfrak{x} | \mathfrak{x} \in \mathfrak{L} \text{ and } \mathfrak{x} \geqslant \mathfrak{a}\}$) is a lower segment (respectively upper segment) of $\mathfrak{L}$. It called *principal lower* (respectively *upper*) *segment* of $\mathfrak{L}$, generated by $\mathfrak{a}$.

Let $\mathfrak{a} \in \mathfrak{L}$, put $K[\mathfrak{a}] = \{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in K\}$ and $\mathrm{H}(\mathfrak{a}) = \mathrm{pr}_R(K[\mathfrak{a}])$. We note that $K[\mathfrak{a}] = \mathrm{H}(\mathfrak{a}) \times \{\mathfrak{a}\}$.

For every element $x \in \mathrm{pr}_R(\Lambda)$ and a subset M of K put $\mathfrak{C}_M(x) = \{\mathfrak{a} \in \mathfrak{L} | (x, \mathfrak{a}) \in M\}$. □

Consider some preliminaries properties of lattice rings.

Proposition 2. *Let R be a ring, $\mathfrak{L}$ be a finite distributive lattice and K a lattice ring.*

(i) $\mathrm{pr}_{\mathfrak{L}}(K)$ *is a semigroup by the operation* $\wedge$ *with identity* $\mathfrak{e}(K) = \vee\mathfrak{C}_K(0)$ *and zero* $\mathfrak{m}_{\downarrow}$. *Moreover,* $\mathrm{pr}_{\mathfrak{L}}(K)$ *is a principal lower segment of* $\mathfrak{L}$ *generated by* $\mathfrak{e}(K)$.

(ii) $\mathrm{pr}_R(K)$ *is a subring of* R. *Conversely, for every subring* S *of* $\mathrm{pr}_R(K)$ *the subset* $\{(x,\mathfrak{a})|(x,\mathfrak{a}) \in K$ *and* $x \in S\}$ *is a lattice subring of* K.

(iii) *If* $\mathfrak{M} \subseteq \mathfrak{L}$ *and* $\mathfrak{M}$ *is a lower segment of* $\mathfrak{L}$, *then the subset* $\{(x,\mathfrak{a})|(x,\mathfrak{a}) \in K$ *and* $\mathfrak{a} \in \mathfrak{M}\}$ *is a lattice subring of* K. *In particular, the subset* $\{(x,\mathfrak{b})|(x,\mathfrak{b}) \in K$ *and* $\mathfrak{b} \leqslant \mathfrak{a}\}$ *is a lattice subring of* K *for every element* $\mathfrak{a} \in \mathfrak{L}$.

(iv) *Suppose that* S *is a subring of* $\mathrm{pr}_R(K)$ *and* $\mathfrak{M}$ *is a lower segment of* $\mathfrak{L}$. *Then the subset* $\{(x,\mathfrak{a})|(x,\mathfrak{a}) \in K, x \in S$ *and* $\mathfrak{a} \in \mathfrak{M}\}$ *is a lattice subring of* K.

(v) *For every element* $\mathfrak{a} \in \mathfrak{L}$ *the subset* $K[\mathfrak{a}]$ *is closed by operations of addition and multiplication of pairs and it is a ring by the restrictions of these operations. The subset* $\mathrm{H}(\mathfrak{a})$ *is a subring of* R *and is isomorphic to* $K[\mathfrak{a}]$.

(vi) *If* $\mathfrak{a},\mathfrak{b} \in \mathfrak{L}$ *and* $\mathfrak{a} \leqslant \mathfrak{b}$, *then* $\mathrm{H}(\mathfrak{b}) \leqslant \mathrm{H}(\mathfrak{a})$.

(vii) *A closed by addition and multiplication subset* M *of* K *is an ordinary ring by restrictions of addition and multiplication if and only if* $M \leqslant K[\mathfrak{a}]$ *for some element* $\mathfrak{a} \in \mathfrak{L}$ *(and hence* M *is an ordinary subring of* $K[\mathfrak{a}]$*). Furthermore,* K *is an ordinary ring by addition and multiplication if and only if* $K = K[\mathfrak{m}_{\downarrow}]$.

Proof. (i) Let $\mathfrak{a},\mathfrak{b} \in \mathrm{pr}_{\mathfrak{L}}(K)$, then there are elements $x, y \in R$ such that $(x,\mathfrak{a}),(y,\mathfrak{b}) \in K$. Since K is a lattice ring, $(x+y,\mathfrak{a}\wedge\mathfrak{b}) = (x,\mathfrak{a})+(y,\mathfrak{b}) \in K$, and it implies that $\mathfrak{a}\wedge\mathfrak{b} \in \mathrm{pr}_{\mathfrak{L}}(K)$.

Put $\mathfrak{e} = \mathfrak{e}(K)$. Since $\mathfrak{e} \in \mathrm{pr}_{\mathfrak{L}}(K)$, there exists an element $v \in R$ such that $(v,\mathfrak{e}) \in K$. Let $\mathfrak{a} \in \mathrm{pr}_{\mathfrak{L}}(K)$ and x be an element of R such that $(x,\mathfrak{a}) \in K$. As we have seen above, it follows that $(0,\mathfrak{a}) \in K$. In turn, it follows that $\mathfrak{a} \in \mathfrak{C}_K(0)$. Then $\mathfrak{a} \leqslant \mathfrak{e}$ and $\mathfrak{a}\wedge\mathfrak{e} = \mathfrak{a}$.

Finally, let $\mathfrak{c}$ be an arbitrary element of $\mathfrak{L}$ such that $\mathfrak{c} \leqslant \mathfrak{e}$. The fact that $(0,\mathfrak{e}) \in K$ and condition (LR 1) imply $(0,\mathfrak{c}) \in K$ and $\mathfrak{c} \in \mathrm{pr}_{\mathfrak{L}}(K)$.

(ii) Indeed, let $x, y \in \mathrm{pr}_R(K)$. Then there are elements $\mathfrak{a},\mathfrak{b} \in \mathfrak{L}$ such that $(x,\mathfrak{a}),(y,\mathfrak{b}) \in K$. The fact that K is a lattice ring implies

$$(x-y,\mathfrak{a}\wedge\mathfrak{b}) = (x,\mathfrak{a})-(y,\mathfrak{b}) \in K, (xy,\mathfrak{a}\wedge\mathfrak{b}) = (x,\mathfrak{a})(y,\mathfrak{b}) \in K$$

which implies that $x-y, xy \in \mathrm{pr}_R(K)$.

Conversely, let S be a subring of R and put

$$\Sigma = \{(x,\mathfrak{a})|(x,\mathfrak{a}) \in K \text{ and } x \in S\}.$$

Choose arbitrary pairs $(x,\mathfrak{a}),(y,\mathfrak{b}) \in \Sigma$. Since S is a subring of R, $x-y, xy \in S$, so that

$$(x,\mathfrak{a})-(y,\mathfrak{b})=(x-y,\mathfrak{a}\wedge\mathfrak{b})\in\Sigma \quad \text{and} \quad (x,\mathfrak{a})(y,\mathfrak{b})=(xy,\mathfrak{a}\wedge\mathfrak{b})\in\Sigma.$$

Let $(x,\mathfrak{a}) \in \Sigma$ and $\mathfrak{b} \leqslant \mathfrak{a}$. By (LR 1) $(x,\mathfrak{b}) \in K$, which implies that $(x,\mathfrak{b}) \in \Sigma$.

(iii) Indeed, let $\Sigma = \{(x,\mathfrak{a})|(x,\mathfrak{a})\in K \text{ and } \mathfrak{a}\in\mathfrak{M}\}$, and $(x,\mathfrak{a}),(y,\mathfrak{b})\in\Sigma$. By (ii) $x-y, xy \in \mathrm{pr}_R(K)$. Since K is a lattice ring,

$$(x,\mathfrak{a})-(y,\mathfrak{b})=(x-y,\mathfrak{a}\wedge\mathfrak{b})\in K \quad \text{and} \quad (x,\mathfrak{a})(y,\mathfrak{b})=(xy,\mathfrak{a}\wedge\mathfrak{b})\in K.$$

The fact $\mathfrak{a}\wedge\mathfrak{b}\leqslant\mathfrak{a}$ implies that $\mathfrak{a}\wedge\mathfrak{b}\in\mathfrak{M}$, so that $(x,\mathfrak{a})-(y,\mathfrak{b})\in\Sigma$, $(x,\mathfrak{a})(y,\mathfrak{b})\in\Sigma$. If $(x,\mathfrak{a})\in\Sigma$ and $\mathfrak{b}\leqslant\mathfrak{a}$, then $\mathfrak{b}\in\mathfrak{M}$. By (LR 1) $(x,\mathfrak{b})\in K$, which implies that $(x,\mathfrak{b})\in\Sigma$.

(iv) Is a immediate consequence of (ii) and (iii).

(v) Indeed,

$$(x,\mathfrak{a})-(y,\mathfrak{a})=(x-y,\mathfrak{a}\wedge\mathfrak{a})=(x-y,\mathfrak{a})\in K[\mathfrak{a}],$$
$$(x,\mathfrak{a})(y,\mathfrak{a})=(xy,\mathfrak{a})\in K[\mathfrak{a}].$$

The mapping $(x,\mathfrak{a})\to x$, $(x,\mathfrak{a})\in K[\mathfrak{a}]$, is a ring monomorphism and its image coincides with $\mathrm{H}(\mathfrak{a})$.

(vi) Suppose that $x\in\mathrm{H}(\mathfrak{b})$. We have $(x,\mathfrak{b})\in K$ and condition (LR 1) implies that $(x,\mathfrak{a})\in K$. It follows that $x\in\mathrm{H}(\mathfrak{a})$.

(vii) By (v), $K[\mathfrak{a}]$ is a ring by addition and multiplication for each $\mathfrak{a}\in\mathfrak{L}$. Suppose now that M is a subset of K and M is closed by addition and multiplication. Assume also that M is an ordinary ring by the restrictions of these operations. In particular, M contains zero element. This element is an idempotent by addition. As we have seen above, every idempotent of K has a form $(0,\mathfrak{b})$ for some element $\mathfrak{b}\in\mathfrak{L}$. Let $\mathfrak{a}=\vee\mathfrak{C}_M(0)$, then M contains a pair $(0,\mathfrak{a})$. Assume that M contains a pair $(x,\mathfrak{b})$ where $\mathfrak{b}\neq\mathfrak{a}$. Then M contains a pair $(x,\mathfrak{b})-(x,\mathfrak{b})=(0,\mathfrak{b})$. The pair $(0,\mathfrak{b})$ is an idempotent by addition. Since M is a ring, it contains only one idempotent by addition. It follows that $\mathfrak{b}=\mathfrak{a}$, so that $M\leqslant K[\mathfrak{a}]$. □

By (v), $K[\mathfrak{m}_\downarrow]$ is a ring by addition and multiplication of pair. In particular, it satisfies the conditions (LR 2) and (LR 3). Since $\mathfrak{m}_\downarrow$ is the least element of $\mathfrak{L}$, $K[\mathfrak{m}_\downarrow]$ satisfied the condition (LR 1). Thus $K[\mathfrak{m}_\downarrow]$ is a lattice subring of K.

Suppose now that K is an ordinary ring by addition and multiplication. Let $e=\vee\mathfrak{C}_K(0)$, and assume that $\mathfrak{e}\neq\mathfrak{m}_\downarrow$. Then both pairs $(0,\mathfrak{e})$ and

$(0, \mathfrak{m}_\downarrow)$ are the idempotents by addition. However an ordinary ring has only one idempotent. This contradiction shows that $\mathfrak{e} = \mathfrak{m}_\downarrow$.

If U is a subset of R and $\mathfrak{V}$ is a subset of $\mathfrak{L}$, then put

$$K[U] = \{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in K \text{ and } x \in U\},$$
$$K[\mathfrak{V}] = \{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in K \text{ and } \mathfrak{a} \in \mathfrak{V}\}.$$

By above proved, if S is a subring of R, then $K[S]$ is a lattice subring of K, and if $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, then $K[\mathfrak{M}]$ is a lattice subring of K. $K[\mathfrak{M}]$ is called a $\mathfrak{M}$-*level of* K.

The subset $K[\mathfrak{a}]$ is called the *layer* of K (more precisely, $\mathfrak{a}$-*layer*), and $\mathrm{H}(\mathfrak{a})$ is called the $\mathfrak{a}$-*hoop* of K.

Clearly $K[\mathfrak{a}] \cap K[\mathfrak{b}] = \varnothing$ whenever $\mathfrak{a} \neq \mathfrak{b}$ and $K = \cup_{\mathfrak{a} \in \mathfrak{L}} K[\mathfrak{a}]$. In other words, the family $\{K[\mathfrak{a}], \mathfrak{a} \in \mathfrak{L}\}$ is a partition of a lattice ring K. Moreover, it organizes some graduation of K by ordinary rings, because $K[\mathfrak{a}] + K[\mathfrak{b}] \subseteq K[\mathfrak{a} \wedge \mathfrak{b}]$ and $K[\mathfrak{a}]K[\mathfrak{b}] \subseteq K[\mathfrak{a} \wedge \mathfrak{b}]$.

Suppose that $\mathfrak{b} \leqslant \mathfrak{a}$ and $x \in \mathrm{H}(\mathfrak{a})$. Then $(x, \mathfrak{a}) \in K[\mathfrak{a}]$. By (LR 1) the fact that $(x, \mathfrak{a}) \in K$ implies that $(x, \mathfrak{b}) \in K$. It follows that $(x, \mathfrak{b}) \in K[\mathfrak{b}]$ and hence $x \in \mathrm{H}(\mathfrak{b})$. Thus $\mathfrak{b} \leqslant \mathfrak{a}$ implies $\mathrm{H}(\mathfrak{b}) \leqslant \mathrm{H}(\mathfrak{a})$.

Let K be a lattice ring. As we already mentioned, lattice ring can contains more than one idempotent (by addition). Moreover, K contains a pair $(0, \mathfrak{a})$ for each element $\mathfrak{a} \in \mathrm{pr}_{\mathfrak{L}}(K)$. Indeed, let u be an element of R such that $(u, \mathfrak{a}) \in K$. Since K is a lattice ring, $(0, \mathfrak{a}) = (u, \mathfrak{a}) - (u, \mathfrak{a}) \in K$.

If K is a lattice ring over $\mathfrak{L}$, then put $O(K) = \{(0, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{e}(K)\}$. Clearly $O(K)$ is a lattice subring of K. If $(x, \mathfrak{a})$ is an idempotent by addition, then x is an idempotent by addition on a ring R, so that $x = 0$. Thus $O(K)$ contains all idempotents by the addition of lattice ring K.

Let Λ be a lattice subring of K. The pair $(0, \mathfrak{e}(K))$ is a zero element of K and $(0, \mathfrak{e}(\Lambda))$ is a zero element of Λ. Since $\Lambda \leqslant K$, Proposition 2 shows that $\mathfrak{e}(\Lambda) \leqslant \mathfrak{e}(K)$. We say that Λ is a *complete lattice subring* of K if $(0, \mathfrak{e}(K)) \in \Lambda$. Every lattice subring of K can be extended to a complete lattice subring. Indeed, put $\Lambda^+ = \Lambda \cup O(K)$, then Λ^+ is a lattice ring. In fact, if $(u, \mathfrak{a}) \in \Lambda$, then $(u, \mathfrak{a}) - (0, \mathfrak{b}) = (u, \mathfrak{a} \wedge \mathfrak{b}), (u, \mathfrak{a})(0, \mathfrak{b}) = (0, \mathfrak{a} \wedge \mathfrak{b})$. Since $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{a}$, $(u, \mathfrak{a} \wedge \mathfrak{b}) \in \Lambda$. Clearly, Λ^+ satisfies the condition (LR 3).

Let Λ be a lattice subring of K. We say that Λ is an *lattice ideal* of K, if $(x, \mathfrak{a})(y, \mathfrak{b}), (y, \mathfrak{b})(x, \mathfrak{a}) \in \Lambda$ for all pairs $(x, \mathfrak{a}) \in K, (y, \mathfrak{b}) \in \Lambda$.

We remark that $(x, \mathfrak{a})(y, \mathfrak{b}) = (xy, \mathfrak{a} \wedge \mathfrak{b}), (y, \mathfrak{b})(x, \mathfrak{a}) = (yx, \mathfrak{a} \wedge \mathfrak{b})$. This shows at once that if Λ a lattice ideal of K, then $\mathrm{pr}_R(\Lambda)$ is an ideal of $\mathrm{pr}_R(K)$. Conversely, suppose that H is an ideal of R, then $K[H]$ is a lattice ideal of K. Indeed, by Proposition 2 (ii) $K[H]$ is a lattice subring of K. Furthermore, if $(x, \mathfrak{a}) \in K, (y, \mathfrak{b}) \in K[H]$, then

$$(x, \mathfrak{a})(y, \mathfrak{b}) = (xy, \mathfrak{a} \wedge \mathfrak{b}) \in K \quad \text{and} \quad (y, \mathfrak{b})(x, \mathfrak{a}) = (yx, \mathfrak{a} \wedge \mathfrak{b}) \in K.$$

Since H is an ideal of $\mathrm{pr}_R(K), xy, yx \in H$, so that

$$(x, \mathfrak{a})(y, \mathfrak{b}), (y, \mathfrak{b})(x, \mathfrak{a}) \in K[H].$$

Similarly, if $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, then $K[\mathfrak{M}]$ is a lattice ideal of K. Indeed, by Proposition 2 (iii) $K[\mathfrak{M}]$ is a lattice subring of K. Furthermore, let $(x, \mathfrak{a}) \in K, (y, \mathfrak{b}) \in K[\mathfrak{M}]$. Since $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, then from $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{b}$ implies that $\mathfrak{a} \wedge \mathfrak{b} \in \mathfrak{M}$. Then

$$(x, \mathfrak{a})(y, \mathfrak{b}) = (xy, \mathfrak{a} \wedge \mathfrak{b}) \in K[\mathfrak{M}] \quad \text{and} \quad (y, \mathfrak{b})(x, \mathfrak{a}) = (yx, \mathfrak{a} \wedge \mathfrak{b}) \in K[\mathfrak{M}].$$

2. Translation to fuzzy language

Let $\mathfrak{L}$ be a lattice and R be a ring. We will consider the set $\mathfrak{L}^R$ of all functions $\lambda \colon R \to \mathfrak{L}$. We define operations $\wedge$ and $\vee$ on this set by the following rules: if $\lambda, \mu \in \mathfrak{L}^R$, then put

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) \text{ and } (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) \text{ for each } x \in R.$$

Clearly the operations $\wedge$ and $\vee$ are commutative and associative,

$$(\lambda \wedge (\lambda \vee \mu))(x) = \lambda(x) \wedge (\lambda \vee \mu)(x) = \lambda(x) \wedge (\lambda(x) \vee \mu(x)) = \lambda(x)$$

and

$$(\lambda \vee (\lambda \wedge \mu))(x) = \lambda(x) \vee (\lambda \wedge \mu)(x) = \lambda(x) \vee (\lambda(x) \wedge \mu(x)) = \lambda(x)$$

so that $\lambda \wedge (\lambda \vee \mu) = \lambda$ and $\lambda \vee (\lambda \wedge \mu) = \lambda$. Clearly $\lambda \wedge \lambda = \lambda$ and $\lambda \vee \lambda = \lambda$. Hence the set $\mathfrak{L}^R$ is a lattice.

If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$, then $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b}$ is equivalent to $\mathfrak{a} \leqslant \mathfrak{b}$. Therefore we can define an order on $\mathfrak{L}^R$ if for $\lambda, \mu \in \mathfrak{L}^R$ we will put $\lambda \leqslant \mu$ if and only if $\lambda(x) \leqslant \mu(x)$ for each element $x \in R$.

Suppose now that a lattice $\mathfrak{L}$ is distributive and finite. Note that instead of the finiteness of the lattice $\mathfrak{L}$ sometime one uses another condition: *a lattice $\mathfrak{L}$ must be complete.* Since we do not aim maximum generality, the case of finiteness of lattice $\mathfrak{L}$ is more transparent for our consideration. Nevertheless, the considerations below can be extended to the case when $\mathfrak{L}$ is a complete lattice.

Being finite, it has the greatest element $\mathfrak{m}^{\uparrow}$ and the least element $\mathfrak{m}_{\downarrow}$. For every function $f\colon R \to \mathfrak{L}$ define $\mathrm{Supp}(f)$ as a subset of all elements $x \in R$ such that $f(x) \neq \mathfrak{m}_{\downarrow}$.

Let Y be a subset of R and $\mathfrak{a} \in \mathfrak{L}$. We define a function $\chi(Y, \mathfrak{a})$ as follows:

$$\chi(Y, \mathfrak{a}) = \begin{cases} \mathfrak{a} & \text{if } x \in Y, \\ \mathfrak{m}_{\downarrow} & \text{if } x \notin Y. \end{cases}$$

If $Y = \{y\}$, then instead of $\chi(\{y\}, \mathfrak{a})$ we will $\chi(y, \mathfrak{a})$. The function $\chi(y, \mathfrak{a})$ is called a *point function* or just *point*. By its definition $\chi(y, \mathfrak{a}) \in \mathfrak{L}^R$. Furthermore, let $f \in \mathfrak{L}^R$. If $\mathrm{Supp}(f) = \{g_1, \ldots, g_n\}$ and $f(g_j) = \mathfrak{a}_j$, $1 \leqslant j \leqslant n$, then clearly $f = \chi(g_1, \mathfrak{a}_1) \vee \ldots \vee \chi(g_n, \mathfrak{a}_n)$.

Let R be a ring, $\kappa \in \mathfrak{L}^R$. Then a function κ is said to be an $\mathfrak{L}$*-fuzzy ring on* R if it satisfies the following conditions:

$$\begin{aligned} &\text{(RF 1)} \quad \kappa(x-y) \geqslant \kappa(x) \wedge \kappa(y) \text{ for all } x, y \in R, \\ &\text{(RF 2)} \quad \kappa(xy) \geqslant \kappa(x) \wedge \kappa(y) \text{ for all } x, y \in R. \end{aligned}$$

Let γ, κ be L-fuzzy rings on R. If $\gamma \leqslant \kappa$, then we will say that γ *is a* $\mathfrak{L}$*-fuzzy subring of* κ. This fact we will denoted $\gamma \preccurlyeq \kappa$.

Define now the binary operation $\oplus$ on $\mathfrak{L}^R$ by the following rule. Let $\mu, \nu \in \mathfrak{L}^R$ and x be an arbitrary element of a ring R. Consider the subset

$$\{\mu(y) \wedge \nu(z) | u, v \text{ are the elements of } R \text{ such that } y + z = x\}$$

of the lattice $\mathfrak{L}$. Since $\mathfrak{L}$ is finite, this subset is finite. Therefore we can talk about its least upper bound. Put

$$(\mu \oplus \nu)(x) = \vee_{y,z \in R, y+z=x}(\mu(y) \wedge \nu(z)).$$

We remark that

$$(\mu \oplus \nu)(x) = \vee_{y \in R}(\mu(y) \wedge \nu(x-y)) = \vee_{z \in R}(\mu(x-z) \wedge \nu(z)).$$

Consider now the basic properties of this product.

Proposition 3. *The following assertions hold.*

(i) *Operation* $\oplus$ *is associative.*
(ii) *Operation* $\oplus$ *is commutative.*
(iii) *Function* $\chi(0, \mathfrak{m}_{\downarrow})$ *is a zero element of the operation* $\oplus$.
(iv) $\lambda \oplus (\mu \vee \nu) = (\lambda \oplus \mu) \vee (\lambda \oplus \nu)$ *for all functions* $\lambda, \mu, \nu \in \mathfrak{L}^R$.
(v) *if* $x, y \in R, \mathfrak{a} \in \mathfrak{L}$, *then* $(\chi(y, \mathfrak{a}) \oplus \lambda)(x) = \mathfrak{a} \wedge \lambda(x-y)$. *In particular, if* $\mathfrak{a} = \vee_{x \in R}\lambda(x)$, *then* $(\chi(y, \mathfrak{a}) \oplus \lambda)(x) = \lambda(x-y)$.

(vi) *If* $x, y, u \in R, \mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$, *then* $(\chi(y, \mathfrak{a}) \oplus \chi(u, \mathfrak{b}))(y + u) = \mathfrak{a} \wedge \mathfrak{b}$ *and* $(\chi(y, \mathfrak{a}) \oplus \chi(u, \mathfrak{b}))(x) = \mathfrak{m}_{\downarrow}$ *if* $x \neq y + u$. *In other words,* $\chi(y, \mathfrak{a}) \oplus \chi(u, \mathfrak{b}) = \chi(y + u, \mathfrak{a} \wedge \mathfrak{b})$,*in particular,* $\chi(y, \mathfrak{a}) \oplus \chi(u, \mathfrak{a}) = \chi(y + u, \mathfrak{a})$.

(vii) *If* $\lambda, \mu, \nu \in \mathfrak{L}^R$ *and* $\lambda \leqslant \mu$, *then* $\lambda \oplus \nu \leqslant \mu \oplus \nu$.

Proof. We will prove only assertions (ii) and (vii). The proofs of other assertions can be done in exactly same way as proofs of corresponding assertions of Proposition 1 of the paper [4].

(ii) Let $\lambda, \mu \in \mathfrak{L}^R$. Put $\kappa = \lambda \oplus \mu$ and $\eta = \mu \oplus \nu$. We have

$$(\lambda \oplus \mu)(x) = \vee_{y,z \in R,\ y+z=x}(\lambda(y) \wedge \mu(z)),$$
$$(\mu \oplus \lambda)(x) = \vee_{y,z \in R,\ y+z=x}(\mu(y) \wedge \lambda(z)).$$

Put

$$R_x = \{(y, z) | y + z = x\},$$
$$\mathfrak{D}_1 = \{\lambda(y) \wedge \mu(z) | (y, z) \in R_x\} \quad \text{and} \quad \mathfrak{D}_2 = \{\mu(y) \wedge \lambda(z) | (y, z) \in R_x\}.$$

Since the addition on R is commutative, $(y, z) \in R_x$ implies that $(z, y) \in R_x$. Hence if $\lambda(y) \wedge \mu(z) \in \mathfrak{D}_1$ (respectively $\mu(y) \wedge \lambda(z) \in \mathfrak{D}_2$) then $\mu(y) \wedge \lambda(z) = \lambda(z) \wedge \mu(y) \in \mathfrak{D}_1$ (respectively $\lambda(y) \wedge \mu(z) = \mu(z) \wedge \lambda(y) \in \mathfrak{D}_2$), which proves that $\mathfrak{D}_1 = \mathfrak{D}_2$. In turn it follows that $(\lambda \oplus \mu)(x) = (\mu \oplus \lambda)(x)$ for each $x \in R$ and hence $\lambda \oplus \mu = \mu \oplus \lambda$.

(vii) We have

$$(\lambda \oplus \nu)(x) = \vee_{y,z \in R,\ y+z=x}(\lambda(y) \wedge \nu(z)) \leqslant \vee_{y,z \in R,\ y+z=x}(\mu(y) \wedge \nu(z)) = (\mu \oplus \nu)(x).$$

It follows that $\lambda \oplus \nu \leqslant \mu \oplus \nu$.

Define now the binary operation $\odot$ on $\mathfrak{L}^R$ by the following rule. Let $\mu, \nu \in \mathfrak{L}^R$ and x be an arbitrary element of a ring R. Consider the subset

$$\{\mu(y) \wedge \nu(z) | u, v \text{ are the elements of } R \text{ such that } yz = x\}$$

of the lattice $\mathfrak{L}$. Since $\mathfrak{L}$ is finite, this subset is finite. Therefore we can say about its least upper bound. Put

$$(\mu \odot \nu)(x) = \vee_{y,z \in R, yz=x}(\mu(y) \wedge \nu(z)).$$

Consider now the basic properties of this product. The proofs here are exactly same as proofs of corresponding assertions of Proposition 1 of the paper [4] and in Proposition 3. Therefore we omit them. □

Proposition 4. *The following assertions hold.*

(i) *If a multiplication on R is associative, then the operation $\odot$ is associative.*

(ii) *If a multiplication on R is commutative, then the operation $\odot$ is commutative.*

(iii) *If a ring R has an identity element e, then a function $\chi(e, \mathfrak{m}_{\downarrow})$ is an identity element of an operation $\odot$.*

(iv) $(\lambda \odot (\mu \vee \nu) = (\lambda \odot \mu) \vee (\lambda \odot \nu)$ *and* $(\mu \vee \nu) \odot \lambda = (\mu \odot \lambda) \vee (\nu \odot \lambda)$ *for all functions $\lambda, \mu, \nu \in \mathfrak{L}^R$.*

(v) *If $x, y, u \in R$, and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$, then $(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(yu) = \mathfrak{a} \wedge \mathfrak{b}$ and $(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(x) = \mathfrak{m}_{\downarrow}$ if $x \neq yu$. In other words, $\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}) = \chi(yu, \mathfrak{a} \wedge \mathfrak{b})$, in particular, $\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{a}) = \chi(yu, \mathfrak{a})$.*

(vi) *If $\lambda, \mu, \nu \in \mathfrak{L}^R$ and $\lambda \leqslant \mu$, then $\lambda \odot \nu \leqslant \mu \odot \nu$ and $\nu \odot \lambda \leqslant \nu \odot \mu$.*

Corollary 1. *Let R be a ring, $\mathfrak{L}$ be a finite distributive lattice, $\kappa \in \mathfrak{L}^G$ and suppose that κ is an L-fuzzy ring on R. If $\lambda, \nu \leqslant \kappa$, then $\lambda \oplus \nu \leqslant \kappa$ and $\lambda \odot \nu \leqslant \kappa$, in particular, $\kappa \oplus \kappa \leqslant \kappa$ and $\kappa \odot \kappa \leqslant \kappa$.*

Proof. Let x be an arbitrary element of R. The inclusions $\lambda, \nu \leqslant \kappa$ imply $\lambda(y) \wedge \nu(z) \leqslant \kappa(y) \wedge \kappa(z)$. Since κ is an L-fuzzy ring group function, then $\kappa(y) \wedge \kappa(z) \leqslant \kappa(y+z), \kappa(y) \wedge \kappa(z) \leqslant \kappa(yz)$, thus

$$(\lambda \oplus \nu)(x) = \vee_{y,z \in R, y+z=x}(\lambda(y) \wedge \nu(z)) \leqslant \vee_{y,z \in R, y+z=x}\kappa(y+z) = \kappa(x),$$
$$(\lambda \odot \nu)(x) = \vee_{y,z \in R, yz=x}(\lambda(y) \wedge \nu(z)) \leqslant \vee_{y,z \in R, yz=x}\kappa(yz) = \kappa(x). \qquad \square$$

Proposition 5 (The points criterion)**.** *Let R be a ring, $\mathfrak{L}$ be a finite distributive lattice and $\kappa \in \mathfrak{L}^G$. Then κ is an L-fuzzy ring on R if and only if the following assertions hold.*

(RF3) $\chi(x, \kappa(x)) \oplus \chi(y, \kappa(y)) \leqslant \kappa$ *for all $x, y \in R$.*

(RF4) $\chi(-x, \kappa(x)) \leqslant \kappa$ *for every $x \in R$.*

(RF5) $\chi(x, \kappa(x)) \odot \chi(y, \kappa(y)) \leqslant \kappa$ *for all $x, y \in R$.*

Proof. Suppose first that κ is an L-fuzzy ring on R. Let x, y be the arbitrary elements of R. Clearly $\chi(x, \kappa(x)) \leqslant \kappa$ and $\chi(y, \kappa(y)) \leqslant \kappa$ for every elements $x, y \in R$. Using Corollary 1 we obtain that

$$\chi(x, \kappa(x)) \oplus \chi(y, \kappa(y)) \leqslant \kappa \quad \text{and} \quad \chi(x, \kappa(x)) \odot \chi(y, \kappa(y)) \leqslant \kappa.$$

Let x be an arbitrary element of R. We have $(\chi(-x, \kappa(x)))(-x) = \kappa(-x)$. Since κ is an L-fuzzy ring on R, $\kappa(-x) = \kappa(x)$. We note that if $y \neq -x$, then $(\chi(-x, \kappa(x)))(y) = \mathfrak{m}_{\downarrow}$, so that $(\chi(-x, \kappa(x)))(y) \leqslant \kappa(y)$ for every $y \in R$. This means that $\chi(-x, \kappa(x)) \leqslant \kappa$.

Conversely, suppose that κ satisfies the conditions (RF 3)–(RF 5). Let x, y be the arbitrary elements of R. Then (RF 4) shows that $\chi(-y, \kappa(y)) \leqslant \kappa$. It follows that $(\chi(-y, \kappa(y)))(-y) = \kappa(y) \leqslant \kappa(-y)$. Because of symmetry, $\kappa(-y) \leqslant \kappa(y)$, so that $\kappa(y) = \kappa(-y)$. Using condition (RF 3), we obtain that $\chi(x, \kappa(x)) \oplus \chi(-y, \kappa(-y)) \leqslant \kappa$. By Proposition 3 (vi)

$$(\chi(x, \kappa(x)) \oplus \chi(-y, \kappa(-y)))(x - y) = \kappa(x) \wedge \kappa(-y) = \kappa(x) \wedge \kappa(y).$$

The inclusion $\chi(x, \kappa(x)) \oplus \chi(-y, \kappa(-y)) \leqslant \kappa$ implies that

$$(\chi(x, \kappa(x)) \oplus \chi(-y, \kappa(-y)))(x - y) = \kappa(x) \wedge \kappa(-y) \leqslant \kappa(x - y),$$

and κ satisfies (RF 1).

Using condition (RF 5), we obtain that $\chi(x, \kappa(x)) \oplus \chi(y, \kappa(y)) \leqslant \kappa$. By Proposition 4 (v)

$$(\chi(x, \kappa(x)) \odot \chi(y, \kappa(y)))(xy) = \kappa(x) \wedge \kappa(y).$$

The inclusion $\chi(x, \kappa(x)) \odot \chi(y, \kappa(y)) \leqslant \kappa$ implies that

$$(\chi(x, \kappa(x)) \odot \chi(y, \kappa(y)))(xy) = \kappa(x) \wedge \kappa(y) \leqslant \kappa(xy),$$

and κ satisfies (RF 2). □

Proposition 5 shows the following. We can look on the L-fuzzy ring κ as a L-fuzzy ring consisting of point functions $\chi(x, \mathfrak{a})$ where $\mathfrak{a} \leqslant \kappa(x)$. As Propositions 3 and 4 show, these point functions satisfy the following rules of addition and multiplication

$$\chi(y, \mathfrak{a}) \oplus \chi(u, \mathfrak{b}) = \chi(y + u, \mathfrak{a} \wedge \mathfrak{b}) \quad \text{and} \quad \chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}) = \chi(yu, \mathfrak{a} \wedge \mathfrak{b}).$$

This interpretation directly points to a connection of L-fuzzy rings and lattice rings over $\mathfrak{L}$, and therefore makes possible the transition from the language of fuzzy functions to the language of algebraic structures. This connection is in fact a one–one. A lattice ring K defines an L-fuzzy ring on R. Indeed, for every element $x \in \mathrm{pr}_R(K)$ the set $\mathfrak{C}_K(x)$ is not empty. Put $\kappa(x) = \vee\mathfrak{C}_\Lambda(x)$. If $x \notin \mathrm{pr}_R(K)$, then put $\kappa(x) = \mathfrak{m}_\downarrow$. Then κ is a function from R in $\mathfrak{L}$. If $u, v \in R$ and $\kappa(u) = \mathfrak{a}, \kappa(v) = \mathfrak{b}$, then $(uv, \mathfrak{a} \wedge \mathfrak{b}) \in K$ by condition (LR 3). It follows that

$$\kappa(uv) \geqslant \mathfrak{a} \wedge \mathfrak{b} = \kappa(u) \wedge \kappa(v),$$

so that κ satisfies (RF 2). And similarly, using condition (LR 2) we obtain that $(u - v, \mathfrak{a} \wedge \mathfrak{b}) \in K$. It follows that $\kappa(u - v) \geqslant \mathfrak{a} \wedge \mathfrak{b} = \kappa(u) \wedge \kappa(v)$, so that κ satisfies (RF 1).

3. Homomorphisms

The notion of a homomorphism of L-fuzzy rings is formulated rather cumbersome, we will not bring it here, we start to work directly with lattice rings over $\mathfrak{L}$ where this concept looks quite natural. Note at once the following remark. Since we will focus on mappings preserving the structure of lattice rings, it is not enough just to demand that they retain the operations of addition and multiplication. The following simple example demonstrates this. Let R be an arbitrary ring, a lattice $\mathfrak{L}$ is the set $\{1, 2\}$ with a natural order. Consider a lattice ring $K = R \times \{1, 2\}$. Then $\Sigma = R \times \{1\}$ is a lattice subring. The mapping $f\colon (x, 1) \to (x, 2), x \in R$, preserves addition and multiplication. However, $\mathrm{Im}(f) = R \times \{2\}$ is a not lattice ring. These considerations lead us to the next concept. Let R be a ring, $\mathfrak{L}$ be a finite distributive lattice and K be a lattice ring over $\mathfrak{L}$. Since $K \subseteq \mathrm{pr}_R(K) \times \mathfrak{L}$, we will assume farther that $R = \mathrm{pr}_R(K)$.

Let R, T be rings and $\mathfrak{L}$ be a finite distributive lattice, and let $K \subseteq R \times \mathfrak{L}$ (respectively $\Theta \subseteq T \times \mathfrak{L}$) be a lattice rings over $\mathfrak{L}$. Then the mapping $f\colon K \to \Theta$ is called a *homomorphism*, if it satisfies the following conditions:

- $f(u, \mathfrak{a}) + f(v, \mathfrak{b}) = f((u, \mathfrak{a}) + (v, \mathfrak{b}))$ and $f(u, \mathfrak{a}) f(v, \mathfrak{b}) = f((u, \mathfrak{a})(v, \mathfrak{b}))$ for all $(u, \mathfrak{a}), (v, \mathfrak{b}) \in K$;
- if $(z, \mathfrak{c}) \in \mathrm{Im}(f)$ and $\mathfrak{d} \leqslant \mathfrak{c}$, then $(z, \mathfrak{d}) \in \mathrm{Im}(f)$.

As usual, an injective homomorphism is called a *monomorphism*, a surjective homomorphism is called an *epimorphism*, and a bijective homomorphism is called an *isomorphism*.

Now we will obtain some properties of homomorphism. We will denote by 0_R the zero element of a ring R.

Lemma 1. *Let R, T be the rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$, respectively $\Theta \subseteq T \times \mathfrak{L}$, be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Then $f(O(K)) \leqslant O(\Theta)$. Moreover, if $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{b})$, $f(0_R, \mathfrak{c}) = (0_T, \mathfrak{d})$ and $\mathfrak{a} \leqslant \mathfrak{c}$, then $\mathfrak{b} \leqslant \mathfrak{d}$. In particular, $f(0_R, \mathfrak{m}_\downarrow) = (0_T, \mathfrak{m}_\downarrow)$.*

Proof. Let $\mathfrak{a}$ be an arbitrary element of $\mathfrak{L}$. The equality $(0_R, \mathfrak{a}) + (0_R, \mathfrak{a}) = (0_R, \mathfrak{a})$ implies

$$f(0_R, \mathfrak{a}) = f((0_R, \mathfrak{a}) + (0_R, \mathfrak{a})) = f(0_R, \mathfrak{a}) + f(0_R, \mathfrak{a}).$$

It shows that $f(0_R, \mathfrak{a})$ is an idempotent by addition on lattice ring Θ. As we have seen above, every idempotent by addition of Θ has a form $(0_T, \mathfrak{b})$ for some element $\mathfrak{b} \in \mathfrak{L}$, so that $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{b})$. Let $\mathfrak{c} \in \mathfrak{L}$ and suppose that $f(0_R, \mathfrak{c}) = (0_T, \mathfrak{d})$. We have

$$(0_T, \mathfrak{b} \wedge \mathfrak{d}) = (0_T, \mathfrak{b}) + (0_T, \mathfrak{d}) = f(0_R, \mathfrak{a}) + f(0_R, \mathfrak{c})$$
$$= f((0_R, \mathfrak{a}) + (0_R, \mathfrak{c})) = f(0_R, \mathfrak{a} \wedge \mathfrak{c}).$$

In particular, if $\mathfrak{a} \leqslant \mathfrak{c}$, then $\mathfrak{a} \wedge \mathfrak{c} = \mathfrak{a}$, so that $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{b}) = (0_T, \mathfrak{b} \wedge \mathfrak{d})$. This means that $\mathfrak{b} \leqslant \mathfrak{d}$. By the definition of homomorphism $(0_T, \mathfrak{m}_\downarrow) \in \operatorname{Im}(f)$, that is $(0_T, \mathfrak{m}_\downarrow) = f(0_R, \mathfrak{u})$ for some element $\mathfrak{u} \in \mathfrak{L}$. Let $f(0_R, \mathfrak{m}_\downarrow) = (0_T, \mathfrak{q})$, then $\mathfrak{m}_\downarrow \leqslant \mathfrak{u}$ implies $\mathfrak{q} \leqslant \mathfrak{m}_\downarrow$. Since $\mathfrak{m}_\downarrow$ is the least element of $\mathfrak{L}$, $\mathfrak{q} = \mathfrak{m}_\downarrow$, so that $f(0_R, \mathfrak{m}_\downarrow) = (0_T, \mathfrak{m}_\downarrow)$. □

Corollary 2. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$, respectively $\Theta \subseteq T \times \mathfrak{L}$, be lattice rings over $\mathfrak{L}$ and $f \colon K \to \Theta$ be a homomorphism. If $\mathfrak{a} \in \mathfrak{L}$, and $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{b})$, then the mapping $f^L \colon \mathfrak{L} \to \mathfrak{L}$, defined by the rule $f^L(\mathfrak{a}) = \mathfrak{b}$ satisfies the following conditions:*
- (i) *$f^L(\mathfrak{a}\wedge\mathfrak{b}) = f^L(\mathfrak{a})\wedge f^L(\mathfrak{b})$, in particular, if $\mathfrak{a} \leqslant \mathfrak{b}$, then $f^L(\mathfrak{a}) \leqslant f^L(\mathfrak{b})$;*
- (ii) *if $\mathfrak{b} \in \operatorname{Im}(f^L)$ and $\mathfrak{a} \leqslant \mathfrak{b}$, then $\mathfrak{a} \in \operatorname{Im}(f^L)$.*

Corollary 3. *Let R, T be the rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f \colon K \to \Theta$ be a homomorphism. If $\mathfrak{a}, \mathfrak{b}$ elements of $\mathfrak{L}$ such that $\mathfrak{a} \leqslant \mathfrak{b}, x \in R$ and $f(x, \mathfrak{a}) = (u, \mathfrak{c}), f(x, \mathfrak{b}) = (v, \mathfrak{d})$, then $\mathfrak{c} \leqslant \mathfrak{d}$. In particular, $f(x, \mathfrak{m}_\downarrow) \in T[\mathfrak{m}_\downarrow]$.*

Proof. We have $(x, \mathfrak{a}) - (x, \mathfrak{a}) = (0_R, \mathfrak{a})$, which implies

$$(0_T, \mathfrak{c}) = (u, \mathfrak{c}) - (u, \mathfrak{c}) = f(x, \mathfrak{a}) - f(x, \mathfrak{a}) = f((x, \mathfrak{a}) - (x, \mathfrak{a})) = f(0_R, \mathfrak{a}).$$

And similarly, $(0_T, \mathfrak{d}) = f(0_R, \mathfrak{b})$. By Lemma 1 we obtain that $\mathfrak{c} \leqslant \mathfrak{d}$. □

Corollary 4. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f \colon K \to \Theta$ be a homomorphism. If $x \in R$ and $f(x, \mathfrak{m}_\downarrow) = (u, \mathfrak{m}_\downarrow)$ where $u \in T$, then the mapping $f^R \colon R \to T$ defined by the rule $f^R(x) = u$ is an ordinary ring homomorphism.*

Proof. Indeed, let $x, y \in R$ and $f(x, \mathfrak{m}_\downarrow) = (u, \mathfrak{m}_\downarrow), f(y, \mathfrak{m}_\downarrow) = (v, \mathfrak{m}_\downarrow)$. Then $f(x+y, \mathfrak{m}_\downarrow) = f((x, \mathfrak{m}_\downarrow)+(y, \mathfrak{m}_\downarrow)) = f(x, \mathfrak{m}_\downarrow)+f(y, \mathfrak{m}_\downarrow) = (u, \mathfrak{m}_\downarrow)+(v, \mathfrak{m}_\downarrow) = (u+v, \mathfrak{m}_\downarrow)$, which follows that $f^R(x+y) = u+v = f^R(x)+f^R(y)$.

And similarly,

$$f(xy, \mathfrak{m}_\downarrow) = f((x, \mathfrak{m}_\downarrow)(y, \mathfrak{m}_\downarrow)) = f(x, \mathfrak{m}_\downarrow)f(y, \mathfrak{m}_\downarrow)$$
$$= (u, \mathfrak{m}_\downarrow)(v, \mathfrak{m}_\downarrow) = (uv, \mathfrak{m}_\downarrow),$$

which follows that $f^R(xy) = uv = f^R(x)f^R(y)$. □

Lemma 2. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Let $x \in R, \mathfrak{a} \in \mathfrak{L}$ and suppose that $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{c})$, where $\mathfrak{c} \in \mathfrak{L}$. Then $f(x, \mathfrak{a}) = (v, \mathfrak{c})$ for some element $v \in T$.*

Proof. Suppose that $f(x, \mathfrak{a}) = (v, \mathfrak{d})$ for some element $\mathfrak{d} \in \mathfrak{L}$. We have

$$\begin{aligned}(0_T, \mathfrak{d}) &= (v, \mathfrak{d}) - (v, \mathfrak{d}) = f(x, \mathfrak{a}) - f(x, \mathfrak{a}) = f((x, \mathfrak{a}) - (x, \mathfrak{a})) \\ &= f(0_R, \mathfrak{a}) = (0_T, \mathfrak{c}).\end{aligned}$$

It follows that $\mathfrak{d} = \mathfrak{c}$. □

Corollary 5. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Then $f(x, \mathfrak{a}) - f(y, \mathfrak{b}) = f((x, \mathfrak{a}) - (y, \mathfrak{b}))$ for all $(x, \mathfrak{a}), (y, \mathfrak{b}) \in K$.*

Proof. Suppose that $f(x, \mathfrak{a}) = (u, \mathfrak{c}), f(-x, \mathfrak{a}) = (v, \mathfrak{d})$ where $\mathfrak{c}, \mathfrak{d} \in \mathfrak{L}$. By Lemma 2 $\mathfrak{c} = \mathfrak{d}$. We have $(0_R, \mathfrak{a}) = (x + (-x), \mathfrak{a}) = (x, \mathfrak{a}) + (-x, \mathfrak{a})$, which implies $(0_T, \mathfrak{c}) = f(x, \mathfrak{a}) + f(-x, \mathfrak{a}) = (u, \mathfrak{c}) + (v, \mathfrak{c}) = (u + v, \mathfrak{c})$. It follows that $v = -u$. Put $f(y, \mathfrak{b}) = (w, \mathfrak{m})$ where $\mathfrak{m} \in \mathfrak{L}$. Now we have

$$f(x, \mathfrak{a}) - f(y, \mathfrak{b}) = (u, \mathfrak{c}) - (w, \mathfrak{m}) = (u - w, \mathfrak{c} \wedge \mathfrak{m}),$$

and

$$\begin{aligned}f((x, \mathfrak{a}) - (y, \mathfrak{b})) &= f((x, \mathfrak{a}) + (-y, \mathfrak{b})) = f(x, \mathfrak{a}) + f(-y, \mathfrak{b}) \\ &= (u, \mathfrak{c}) + (-w, \mathfrak{m}) = (u - w, \mathfrak{c} \wedge \mathfrak{m}),\end{aligned}$$

so that $f((x, \mathfrak{a}) - (y, \mathfrak{b})) = f(x, \mathfrak{a}) - f(y, \mathfrak{b})$. □

Corollary 6. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Then $\mathrm{Im}(f)$ is a lattice subring of Θ.*

Proof. Indeed, Corollary 5 shows that $\mathrm{Im}(f)$ satisfies condition (LR 2). Let $(x, \mathfrak{a}), (y, \mathfrak{b})$ be arbitrary elements of K, then

$$f(x, \mathfrak{a})f(y, \mathfrak{b}) = f((x, \mathfrak{a})(y, \mathfrak{b})) \in \mathrm{Im}(f),$$

so that $\mathrm{Im}(f)$ satisfies condition (LR 3). Finally, $\mathrm{Im}(f)$ satisfies condition (LR 1) by the definition of homomorphism.

Let $K \subseteq R \times \mathfrak{L}$, $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings and $f\colon K \to \Theta$ be a homomorphism. Put $\mathrm{Ker}(f) = \{(x, \mathfrak{a}) | f(x, \mathfrak{a}) \in O(\Theta)\}$. □

Lemma 3. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Then* $\mathrm{Ker}(f)$ *is a lattice ideal of K.*

Proof. Indeed, since $O(\Theta)$ is a lattice ideal of Θ, it is not difficult to check that $(x, \mathfrak{a}), (y, \mathfrak{b}) \in \mathrm{Ker}(f), (z, c) \in K$, then $(x, \mathfrak{a}) - (y, \mathfrak{b}) \in \mathrm{Ker}(f)$, $(z, \mathfrak{c})(x, \mathfrak{a}), (x, \mathfrak{a})(z, \mathfrak{c}) \in \mathrm{Ker}(f)$. Let $(x, \mathfrak{a}) \in \mathrm{Ker}(f), f(x, \mathfrak{a}) = (0_T, \mathfrak{u})$, where $\mathfrak{u} \in \mathfrak{L}$ and $\mathfrak{d}$ be an element of $\mathfrak{L}$ such that $\mathfrak{d} \leqslant \mathfrak{a}$. We have

$$f(0_R, \mathfrak{d}) = f((x, \mathfrak{d}) - (x, \mathfrak{a})) = f(x, \mathfrak{d}) - f(x, \mathfrak{a}) = f(x, \mathfrak{d}) - (0_T, \mathfrak{u}).$$

By above proved $f(O(K)) \leqslant O(\Theta)$, which implies that $f(x, \mathfrak{d}) \in O(\Theta)$, so that $(x, \mathfrak{d}) \in \mathrm{Ker}(f)$.

The lattice ideal $\mathrm{Ker}(f)$ arises in the following way. The intersection $K[\mathfrak{m}_\downarrow] \cap \mathrm{Ker}(f)$ is an ordinary ideal in a ring $K[\mathfrak{m}_\downarrow]$. By our agreement $R = \mathrm{pr}_R(K) = \mathrm{pr}_R(K[\mathfrak{m}_\downarrow])$, and moreover R and $K[\mathfrak{m}_\downarrow]$ are isomorphic as ordinary rings. It follows that $R_f = \mathrm{pr}_R(K[\mathfrak{m}_\downarrow] \cap \mathrm{Ker}(f)) = \mathrm{pr}_R(\mathrm{Ker}(f))$ is an ideal of R. □

Proposition 6. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Then* $\mathrm{Ker}(f) = K[R_f]$.

Proof. Indeed, if $(x, \mathfrak{a}) \in \mathrm{Ker}(f)$, then the fact that $\mathrm{Ker}(f)$ is a lattice ideal implies that $(x, \mathfrak{m}_\downarrow) \in \mathrm{Ker}(f)$, which implies that $x \in R_f$.

Conversely, assume that $(x, \mathfrak{a}) \in K[R_f]$, then $x \in R_f$, so that $f(x, \mathfrak{m}_\downarrow) \in O(\Theta)$. Lemma 3.1 shows that $f(x, \mathfrak{m}_\downarrow) = (0_T, \mathfrak{m}_\downarrow)$, since $\mathfrak{m}_\downarrow \leqslant \mathfrak{a}$,

$$f(0_R, \mathfrak{m}_\downarrow) = f((x, \mathfrak{a}) - (x, \mathfrak{m}_\downarrow)) = f(x, \mathfrak{a}) - f(x, \mathfrak{m}_\downarrow) = f(x, \mathfrak{a}) - (0_T, \mathfrak{m}_\downarrow).$$

By above proved, $f(O(K)) \leqslant O(\Theta)$, which implies that $f(x, \mathfrak{a}) \in O(\Theta)$, so that $(x, \mathfrak{a}) \in \mathrm{Ker}(f)$. □

Lemma 4. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. Let $x \in R$, and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}, \mathfrak{a} \leqslant \mathfrak{b}$, and suppose that $f(x, \mathfrak{b}) = (y, \mathfrak{c})$ for some $y \in T, \mathfrak{c} \in \mathfrak{L}$. Then $f(x, \mathfrak{a}) = (y, \mathfrak{d})$ where $f(0_R, \mathfrak{a}) = (0_T, \mathfrak{d})$.*

Proof. By Lemma 1 $\mathfrak{d} \leqslant \mathfrak{c}$. Since f is a homomorphism, $(y, \mathfrak{d}) \in \mathrm{Im}(f)$, so that there is an element $u \in T$ such that $f(u, \mathfrak{a}) = (y, \mathfrak{d})$. We have

$$2(y, \mathfrak{d}) = (y, \mathfrak{d}) + (y, \mathfrak{c}) = f(x, \mathfrak{b}) + f(u, \mathfrak{a}) = f((x, \mathfrak{b}) + (u, \mathfrak{a})) = f(x + u, \mathfrak{a}).$$

On the other hand, $2(y,\mathfrak{d}) = 2f(u,\mathfrak{a}) = f(2u,\mathfrak{a})$. Thus $f(x+u,\mathfrak{a}) = f(2u,\mathfrak{a})$. Using Corollary 5 we obtain

$$\begin{aligned} f(x,\mathfrak{a}) - f(u,\mathfrak{a}) &= f((x,\mathfrak{a}) - (u,\mathfrak{a})) = f(x-u,\mathfrak{a}) \\ &= f((x+u,\mathfrak{a}) - (2u,\mathfrak{a})) = f(x+u,\mathfrak{a}) - f(2u,\mathfrak{a}) \\ &= f(2u,\mathfrak{a}) - f(2u,\mathfrak{a}) = (0_T,\mathfrak{d}). \end{aligned}$$

It follows that $f(x,\mathfrak{a}) = f(u,\mathfrak{a})$, so that $f(x,\mathfrak{a}) = (y,\mathfrak{d})$. □

Corollary 7. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f\colon K \to \Theta$ be a homomorphism. If $(x,\mathfrak{a}) \in K$, then $f(x,\mathfrak{a}) = (f^R(x), f^L(\mathfrak{a}))$.*

Proof. By Lemma 2, $f(x,\mathfrak{a}) = (u, f^L(\mathfrak{a}))$. By Lemma 4, $f(x,\mathfrak{m}_\downarrow) = (u,\mathfrak{m}_\downarrow)$, which follows that $u = f^R(x)$.

Whereas obtained above, we arrive at the following two natural mappings. We define the mapping $\mathbf{s}(f)\colon K \to T \times \mathfrak{L}$ by the following. Let $f\colon R \to T$ be an ordinary ring homomorphism of R in T. If $(x,\mathfrak{a}) \in K$, then put $\mathbf{s}(f)(x,\mathfrak{a}) = (f(x),\mathfrak{a})$. Then this mapping is a homomorphism. In fact, for $(x,\mathfrak{a}), (y,\mathfrak{b}) \in K$ we have

$$\begin{aligned} \mathbf{s}(f)((x,\mathfrak{a}) + (y,\mathfrak{b})) = \mathbf{s}(f)(x+y,\mathfrak{a}\wedge\mathfrak{b}) = (f(x+y),\mathfrak{a}\wedge\mathfrak{b}), \\ \mathbf{s}(f)(x,\mathfrak{a}) + \mathbf{s}(f)(y,\mathfrak{b}) = (f(x),\mathfrak{a}) + (f(y),\mathfrak{b}) = (f(x)+f(y),\mathfrak{a}\wedge\mathfrak{b}). \end{aligned}$$

Since $f(x+y) = f(x) + f(y)$,

$$\mathbf{s}(f)((x,\mathfrak{a}) + (y,\mathfrak{b})) = \mathbf{s}(f)(x,\mathfrak{a}) + \mathbf{s}(f)(y,\mathfrak{b}).$$

Similarly we can prove that $\mathbf{s}(f)((x,\mathfrak{a})(y,\mathfrak{b})) = \mathbf{s}(f)(x,\mathfrak{a})\mathbf{s}(f)(y,\mathfrak{b})$.

Let $(z,\mathfrak{c}) \in \mathrm{Im}(\mathbf{s}(f))$ and $\mathfrak{d} \leqslant \mathfrak{c}$. The fact that $(z,\mathfrak{c}) \in \mathrm{Im}(\mathbf{s}(f))$ means that K contains a pair $(x,\mathfrak{a})$ such that $\mathbf{s}(f)(x,\mathfrak{a}) = (z,\mathfrak{c})$. On the other hand, $\mathbf{s}(f)(x,\mathfrak{a}) = (f(x),\mathfrak{a})$. It follows that $\mathfrak{a} = \mathfrak{c}$. Since K is a lattice ring, $\mathfrak{d} \leqslant \mathfrak{c} = \mathfrak{a}$ implies that $(x,\mathfrak{d}) \in K$. Then $(z,\mathfrak{d}) = (f(x),\mathfrak{d}) = \mathbf{s}(f)(x,\mathfrak{d}) \in \mathrm{Im}(\mathbf{s}(f))$, so that all conditions of definition of homomorphism hold.

Consider now the mapping $g\colon \mathfrak{L} \to \mathfrak{L}$, satisfying the following conditions:

(i) $g(\mathfrak{a}\wedge\mathfrak{b}) = g(\mathfrak{a}) \wedge g(\mathfrak{b})$,

(ii) If $\mathfrak{b} \in \mathrm{Im}(g)$ and $\mathfrak{a} \leqslant \mathfrak{b}$, then $\mathfrak{a} \in \mathrm{Im}(g)$.

In other words, $\mathrm{Im}(g)$ is a lower segment of lattice $\mathfrak{L}$.

Define the mapping $\mathbf{p}(g)\colon K \to R \times \mathfrak{L}$ by the following rule. If $(x,\mathfrak{a}) \in K$, then put $\mathbf{p}(g)(x,\mathfrak{a}) = (x, g(\mathfrak{a}))$. Then this mapping is a homomorphism. In fact, for $(x, \mathfrak{a}), (y, \mathfrak{b}) \in K$ we have

$$\mathbf{p}(g)((x, \mathfrak{a}) + (y, \mathfrak{b})) = \mathbf{p}(g)(x + y, \mathfrak{a} \wedge \mathfrak{b}) = (x + y, g(\mathfrak{a} \wedge \mathfrak{b})),$$
$$\mathbf{p}(g)(x, \mathfrak{a}) + \mathbf{p}(g)(y, \mathfrak{b}) = (x, g(\mathfrak{a})) + (y, g(\mathfrak{b})) = (x + y, g(\mathfrak{a}) \wedge g(\mathfrak{b})).$$

By our conditions, $g(\mathfrak{a} \wedge \mathfrak{b}) = g(\mathfrak{a}) \wedge g(\mathfrak{b})$, so that

$$\mathbf{p}(g)((x, \mathfrak{a}) + (y, \mathfrak{b})) = \mathbf{p}(g)(x, \mathfrak{a}) + \mathbf{p}(g)(y, \mathfrak{b}).$$

Similarly we can prove that

$$\mathbf{p}(g)((x, a)(y, b)) = \mathbf{p}(g)(x, a)\mathbf{p}(g)(y, b).$$

Let $(z, \mathfrak{c}) \in \mathrm{Im}(\mathbf{p}(g))$ and $\mathfrak{d} \leqslant \mathfrak{c}$. The fact that $(z, \mathfrak{c}) \in \mathrm{Im}(\mathbf{p}(g))$ means that K contains a pair $(x, \mathfrak{a})$ such that $\mathbf{p}(g)(x, \mathfrak{a}) = (z, \mathfrak{c})$. On the other hand, $\mathbf{p}(g)(x, \mathfrak{a}) = (x, g(\mathfrak{a}))$. It follows that $x = z$ and $g(\mathfrak{a}) = \mathfrak{c}$, i.e. $\mathfrak{c} \in \mathrm{Im}(g)$. By **(ii)** it follows that $\mathfrak{d} \in \mathrm{Im}(g)$. In other words, there is an element $\mathfrak{b} \in \mathfrak{L}$ such that $g(\mathfrak{b}) = \mathfrak{d}$. Since $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{a}$, $(x, \mathfrak{a} \wedge \mathfrak{b}) \in K$. Then

$$\mathbf{p}(g)(x, \mathfrak{a} \wedge \mathfrak{b}) = (x, g(\mathfrak{a} \wedge \mathfrak{b})) = (x, g(\mathfrak{a}) \wedge g(\mathfrak{b})) = (x, \mathfrak{c} \wedge \mathfrak{d}) = (x, \mathfrak{d}) = (z, \mathfrak{d}).$$

Hence $(z, \mathfrak{d}) \in \mathrm{Im}(\mathbf{p}(g))$, so that all of the conditions of the definition of a homomorphism are fulfilled. □

Proposition 7. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f \colon K \to \Theta$ be a homomorphism. Then $f = \boldsymbol{p}(f^L) \circ \boldsymbol{s}(f^R)$.*

Proof. Let $(x, \mathfrak{a})$ be an arbitrary element of K and $f(x, \mathfrak{a}) = (u, \mathfrak{c})$, where $u \in T$ and $\mathfrak{c} \in \mathfrak{L}$. By Corollary 7 $(u, \mathfrak{c}) = (f^R(x), f^L(\mathfrak{a}))$. By Corollary 2 and Lemma 2 $(f^R(x), f^L(\mathfrak{a})) = \mathbf{p}(f^L)(f^R(x), \mathfrak{a})$. Using Corollary 4 we obtain that $(f^R(x), \mathfrak{a}) = \mathbf{s}(f^R)(x, \mathfrak{a})$, so that

$$f(x, \mathfrak{a}) = \mathbf{p}(f^L)(f^R(x), \mathfrak{a}) = \mathbf{p}(f^L)(\mathbf{s}(f^R)(x, \mathfrak{a})) = (\mathbf{p}(f^L) \circ \mathbf{s}(f^R))(x, \mathfrak{a}).$$

A homomorphism $f \colon K \to \Theta$ is called a layer-preserved, if $f(K[\mathfrak{a}]) \leqslant \Theta[\mathfrak{a}]$ for each element $\mathfrak{a} \in \mathfrak{L}$. □

Note that the mapping $\mathbf{p}(f^L)$ is completely defined by a transformation of a lattice L satisfying the conditions (i) and (ii). Thus Proposition 7 shows that layer-preserved homomorphisms plays here a major role.

For layer-preserved homomorphisms we have the following direct analog of the theorem of ring homomorphisms.

Theorem 1. *Let R, T be rings, $\mathfrak{L}$ be a finite distributive lattice. Let $K \subseteq R \times \mathfrak{L}$ and $\Theta \subseteq T \times \mathfrak{L}$ be lattice rings over $\mathfrak{L}$ and $f \colon K \to \Theta$ be a* *layer-preserved homomorphism. Define the lattice ring $K_f \subseteq R/R_f \times \mathfrak{L}$ by the rule: the pair $(x + R_f, \mathfrak{a}) \in K_f$ if and only if $(x, \mathfrak{a}) \in K$. Then* $\mathrm{Im}(f)$ *is a lattice subring of* Θ *and* $\mathrm{Im}(f)$ *is isomorphic to* K_f.

Proof. The fact that $\mathrm{Im}(f)$ is a lattice subring of Θ was proved in Corollary 6. We show that K_f is a lattice subring of $R/R_f \times \mathfrak{L}$. Let $(x+R_f, \mathfrak{a})$, $(y + R_f, \mathfrak{b}) \in K_f$. Then $(x, \mathfrak{a}), (y, \mathfrak{b}) \in K$. Since K is a lattice ring, $(x, \mathfrak{a}) - (y, \mathfrak{b}) \in K$, $(x, \mathfrak{a})(y, \mathfrak{b}) \in K$. We have $(x, \mathfrak{a}) - (y, \mathfrak{b}) = (x - y, \mathfrak{a} \wedge \mathfrak{b})$, $(x, \mathfrak{a})(y, \mathfrak{b}) = (xy, \mathfrak{a} \wedge \mathfrak{b})$. It follows that $(x - y + R_f, \mathfrak{a} \wedge \mathfrak{b}) \in K_f$, and $(xy + R_f, \mathfrak{a} \wedge \mathfrak{b}) \in K_f$. Moreover,

$$(x - y + R_f, \mathfrak{a} \wedge \mathfrak{b}) = (x + R_f, \mathfrak{a}) - (y + R_f, \mathfrak{b})$$

and

$$(xy + R_f, \mathfrak{a} \wedge \mathfrak{b}) = (x + R_f, \mathfrak{a})(y + R_f, \mathfrak{b}).$$

It follows that K_f satisfies the conditions (LR 2), (LR 3). Finally suppose that $(z + R_f, \mathfrak{c}) \in K_f$ and $\mathfrak{d} \leqslant \mathfrak{c}$. The fact that $(z + R_f, \mathfrak{c})$ implies that $(z, \mathfrak{c}) \in K$. Since K is a lattice ring, $(z, \mathfrak{d}) \in K$. It follows that $(z + R_f, \mathfrak{d}) \in K_f$.

Define now the mapping $f^{\uparrow} \colon K_f \to \mathrm{Im}(f)$ by the rule $f^{\uparrow}(x + R_f, \mathfrak{a}) = f(x, \mathfrak{a})$ for each $(x + R_f, \mathfrak{a}) \in K_f$. This definition is correct. In fact, let $(x + R_f, \mathfrak{a}) = (y + R_f, \mathfrak{b})$ and $f(x, \mathfrak{a}) = (u, \mathfrak{a})$ for some $\mathfrak{m} \in \mathfrak{L}$. Then $\mathfrak{a} = \mathfrak{b}$ and $x + R_f = y + R_f$. The last equality implies that $y = x + z$ for some element $z \in R_f$. We obtain $f(y, \mathfrak{a}) = f(x + z, \mathfrak{a}) = f(x, \mathfrak{a}) + f(z, \mathfrak{a})$. By $z \in R_f$ and Proposition 6 $(z, \mathfrak{a}) \in \mathrm{Ker}(f)$. Lemma 2 shows that $f(z, \mathfrak{a}) = (0_T, \mathfrak{a})$. Now we obtain that

$$f(y, \mathfrak{a}) = f(x, \mathfrak{a}) + f(z, \mathfrak{a}) = (u, \mathfrak{a}) + (0_T, \mathfrak{a}) = (u, \mathfrak{a}) = f(x, \mathfrak{a}).$$

The mapping $f^{\uparrow}$ is a homorphism. Indeed

$$\begin{aligned} f^{\uparrow}((x + R_f, \mathfrak{a}) + (y + R_f, \mathfrak{b})) &= f^{\uparrow}(x + y + R_f, \mathfrak{a} \wedge \mathfrak{b}) = f(x + y, \mathfrak{a} \wedge \mathfrak{b}), \\ f^{\uparrow}(x + R_f, \mathfrak{a}) + f^{\uparrow}(y + R_f, \mathfrak{b}) &= f(x, \mathfrak{a}) + f(y, \mathfrak{b}) = f((x, \mathfrak{a}) + (y, \mathfrak{b})) \\ &= f(x + y, \mathfrak{a} \wedge \mathfrak{b}), \end{aligned}$$

so that

$$f^{\uparrow}((x + R_f, \mathfrak{a}) + (y + R_f, \mathfrak{b})) = f^{\uparrow}(x + R_f, \mathfrak{a}) + f^{\uparrow}(y + R_f, \mathfrak{b}).$$

Similarly

$$f^{\uparrow}((x + R_f, \mathfrak{a})(y + R_f, \mathfrak{b})) = f^{\uparrow}(x + R_f, \mathfrak{a}) f^{\uparrow}(y + R_f, \mathfrak{b}).$$

By the choice of $f^{\uparrow}$ we have $\mathrm{Im}(f^{\uparrow}) = \mathrm{Im}(f)$. By Corollary 6, $\mathrm{Im}(f^{\uparrow})$ is a lattice subring of Θ. Let $(z, \mathfrak{c}) \in \mathrm{Im}(f^{\uparrow})$ and $\mathfrak{d} \leqslant \mathfrak{c}$. The equality $\mathrm{Im}(f^{\uparrow}) = \mathrm{Im}(f)$ and the fact that f is a homomorphism imply that $(z, \mathfrak{d}) \in \mathrm{Im}(f^{\uparrow})$.

Finally, suppose that $f^{\uparrow}(x + R_f, \mathfrak{a}) = f^{\uparrow}(y + R_f, \mathfrak{b})$. Then $f(y, \mathfrak{b}) = f(x, \mathfrak{a}) \in f(K[\mathfrak{a}])$, which follows that $\mathfrak{b} = \mathfrak{a}$. Further

$$(0_T, \mathfrak{a}) = f(x, \mathfrak{a}) - f(y, \mathfrak{a}) = f((x, \mathfrak{a}) - (y, \mathfrak{a})) = f(x - y, \mathfrak{a}),$$

so that $(x - y, \mathfrak{a}) \in \mathrm{Ker}(f)$. By Proposition 6, $\mathrm{Ker}(f) = K[R_f]$. In turn out, it follows that $x - y \in R_f$. It follows that $x + R_f = y + R_f$ and therefore $(x + R_f, \mathfrak{a}) = (y + R_f, \mathfrak{b})$. Hence $f^{\uparrow}$ is injective epimorphism, the mapping $f^{\uparrow} \colon K_f \to \mathrm{Im}(f)$ is an isomorphism. □

Theorem 1 shows that a layer-preserved homomorphism f is defined by the ordinary ideal R_f of a ring R, and R_f is defined by the kernel of f, and the latter is a lattice ideal of K. Thus, to every homomorphism a lattice ideal of K is assigned. The question about the feedback naturally arises here. We now consider options for this connection.

Let $K \subseteq R \times \mathfrak{L}$ be a lattice ring and Λ be a lattice ideal of K. Then $\Lambda[\mathfrak{m}_{\downarrow}] = \{(x, \mathfrak{m}_{\downarrow}) | (x, \mathfrak{m}_{\downarrow}) \in \Lambda\} = K[\mathfrak{m}_{\downarrow}] \cap \Lambda$ is an ordinary ideal of the ring $K[\mathfrak{m}_{\downarrow}]$. Then $\mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}) = \mathrm{pr}_R(\Lambda[\mathfrak{m}_{\downarrow}])$ is an ordinary ideal of a ring R and $\Lambda[\mathfrak{m}_{\downarrow}] = \mathrm{H}_{\Lambda}(\mathfrak{a}) \times \{\mathfrak{m}_{\downarrow}\}$. Hence we can consider n ordinary factor-ring $R/\mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow})$. Define the lattice ring $K_{\Lambda} \subseteq R/\mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}) \times \mathfrak{L}$ by the rule: the pair $(x + \mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}), \mathfrak{a}) \in K_{\Lambda}$ if and only if $(x, \mathfrak{a}) \in K$. Repeating the arguments given above, it is easy to show that K_{Λ} is a lattice subring of $R/\mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}) \times \mathfrak{L}$, and the mapping $h \colon K \to K_{\Lambda}$ defined by the rule: $h(x, \mathfrak{a}) = (x + \mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}), \mathfrak{a}), x \in K$ is a layer-preserved homomorphism. Let $(x, \mathfrak{a}) \in \mathrm{Ker}(h)$, then $(x + \mathrm{H}_{\Lambda}(\mathfrak{m}_{\downarrow}), \mathfrak{a}) = h(x, \mathfrak{a}) \in O(K_{\Lambda})$ for each $\mathfrak{a} \leqslant \mathfrak{e}(K) = \mathfrak{e}(K_{\Lambda})$. For the element $\mathfrak{a} \in \mathfrak{L}$ we put $\Lambda[\mathfrak{a}] = \{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in \Lambda\} = K[\mathfrak{a}] \cap \Lambda$. Then $(x, \mathfrak{a}) \in \Lambda[\mathfrak{a}]$ for each $\mathfrak{a} \leqslant \mathfrak{e}(K)$. Taking into account the inequality $K = \cup_{\mathfrak{a} \leqslant \mathfrak{e}(K)} K[\mathfrak{a}]$, we obtain that $(x, \mathfrak{a}) \in \cup_{\mathfrak{a} \leqslant \mathfrak{e}(K)} \Lambda[\mathfrak{a}] = \Lambda$, which shows that $\mathrm{Ker}(h) = \Lambda$.

We can consider a lattice ring K_{Λ} over $\mathfrak{L}$ as a factor of a lattice ring K. Here we have a difference from ordinary rings. In the case of the ordinary quotient ring, the ring is a set whose elements are some subsets of K, on which the operations of addition and multiplication are entered in a special way. In our case, we have an opportunity to get a partition of a lattice ring, which can be viewed as "internal" analogue of a factor-ring.

Put $\mathrm{H}_{\Lambda}(\mathfrak{a}) = \mathrm{pr}_R(\Lambda[\mathfrak{a}])$. We note that $\Lambda[\mathfrak{a}] = \mathrm{H}_{\Lambda}(\mathfrak{a}) \times \{\mathfrak{a}\}$. As we have seen above, $K[\mathfrak{a}]$ is closed by addition and multiplication and is an ordinary ring by the restrictions of these operations, and $\Lambda[\mathfrak{a}]$ is an ideal of $K[\mathfrak{a}]$.

Thus we can consider the (ordinary) factor-ring $K[\mathfrak{a}]/\Lambda[\mathfrak{a}]$. Do it for each $\mathfrak{a} \in \mathfrak{L}$ and consider a set K/Λ, whose elements are all of the resulting cosets $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}]$. We remark that either $((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) \cap ((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]) = \varnothing$ or $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] = (y, \mathfrak{b}) + \Lambda[\mathfrak{b}]$. Indeed, if $\mathfrak{a} \neq \mathfrak{b}$, then by $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] \subseteq K[\mathfrak{a}]$, $(y, \mathfrak{b}) + \Lambda[\mathfrak{b}] \subseteq K[\mathfrak{b}]$ and by $K[\mathfrak{a}] \cap K[\mathfrak{b}] = \varnothing$ we conclude that $((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) \cap ((y, \mathfrak{b}) + \Lambda[b]) = \varnothing$. Suppose that $\mathfrak{a} = \mathfrak{b}$. The subset $K[\mathfrak{a}]$ is an ordinary ring by addition and multiplication and $\Lambda[\mathfrak{a}]$ is an ordinary ideal of $K[\mathfrak{a}]$. Then either

$$((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) \cap ((y, \mathfrak{a}) + \Lambda[\mathfrak{a}]) = \varnothing \quad \text{or} \quad (x, \mathfrak{a}) + \Lambda[\mathfrak{a}] = (y, \mathfrak{a}) + \Lambda[\mathfrak{a}].$$

Taking into account the equality $K = \cup_{\mathfrak{a} \in \mathfrak{L}} K[\mathfrak{a}]$, we obtain that the family $\{(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] | x \in R, \mathfrak{a} \in \mathfrak{L}\}$ is a partition of K.

On the set K/Λ we define the addition and multiplication by the following rules. Let $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}]$, $(y, \mathfrak{b}) + \Lambda[\mathfrak{b}]$ be arbitrary cosets. Then put

$$\begin{aligned}(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] + (y, \mathfrak{b}) + \Lambda[\mathfrak{b}] &= (x + y, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}],\\ ((x, \mathfrak{a}) + \Lambda[\mathfrak{a}])((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]) &= (xy, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}],\end{aligned}$$

Show now that these operations are defined correctly. Suppose that

$$(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] = (u, \mathfrak{a}) + \Lambda[\mathfrak{a}] \quad \text{and} \quad (y, \mathfrak{b}) + \Lambda[\mathfrak{b}] = (v, \mathfrak{b}) + \Lambda[\mathfrak{b}].$$

Then

$$(u, \mathfrak{a}) = (x, \mathfrak{a}) + (w, \mathfrak{a}) \quad \text{and} \quad (v, \mathfrak{b}) = (y, \mathfrak{b}) + (z, \mathfrak{b}),$$

where $(w, \mathfrak{a}) \in \Lambda[\mathfrak{a}]$, $(z, \mathfrak{b}) \in \Lambda[\mathfrak{b}]$. We have

$$\begin{aligned}(u, \mathfrak{a}) + (v, \mathfrak{b}) &= (x, \mathfrak{a}) + (w, \mathfrak{a}) + (y, \mathfrak{b}) + (z, \mathfrak{b})\\ &= (x + y, \mathfrak{a} \wedge \mathfrak{b}) + (w + z, \mathfrak{a} \wedge \mathfrak{b}),\\ (u, \mathfrak{a})(v, \mathfrak{b}) &= ((x, \mathfrak{a}) + (w, \mathfrak{a}))((y, \mathfrak{b}) + (z, \mathfrak{b}))\\ &= (xy, \mathfrak{a} \wedge \mathfrak{b}) + (xz + wy + wz, \mathfrak{a} \wedge \mathfrak{b}).\end{aligned}$$

Since Λ is a lattice ideal $(w+z, \mathfrak{a}\wedge\mathfrak{b}) = (w, \mathfrak{a})+(z, \mathfrak{b}) \in \Lambda$, more concretely, $(w + z, \mathfrak{a} \wedge \mathfrak{b}) \in \Lambda[\mathfrak{a} \wedge \mathfrak{b}]$. It follows that

$$\begin{aligned}&(u, \mathfrak{a}) + \Lambda[\mathfrak{a}] + (v, \mathfrak{b}) + \Lambda[\mathfrak{b}] = (u + v, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}]\\ &\quad = (x + y, \mathfrak{a} \wedge \mathfrak{b}) + (w + z, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}] = (x + y, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}]\\ &\quad = (x, \mathfrak{a}) + \Lambda[\mathfrak{a}] + (y, \mathfrak{b}) + \Lambda[\mathfrak{b}].\end{aligned}$$

And similarly,

$$(xz + wy + wz, \mathfrak{a} \wedge \mathfrak{b}) = (x, \mathfrak{a})(z, \mathfrak{b}) + (w, \mathfrak{a})(y, \mathfrak{b}) + (w, \mathfrak{a})(z, \mathfrak{b}) \in \Lambda,$$

which implies that, $(xz + wy + wz, \mathfrak{a} \wedge \mathfrak{b}) \in \Lambda[\mathfrak{a} \wedge \mathfrak{b}]$. Then we obtain

$$\begin{aligned}((u, \mathfrak{a}) + \Lambda[\mathfrak{a}])((v, \mathfrak{b}) + \Lambda[\mathfrak{b}]) &= (uv, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}] \\ &= (xy, \mathfrak{a} \wedge \mathfrak{b}) + (xz + wy + wz, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}] \\ &= (xy, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}] = ((x, \mathfrak{a}) + \Lambda[\mathfrak{a}])((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]).\end{aligned}$$

Consider now the mapping $\eta\colon K/\Lambda \to K_\Lambda$ defined by the rule: $\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) = (x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a})$ for each pair $(x, \mathfrak{a}) \in K$. This mapping is defined correctly. In fact, let again $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}] = (u, \mathfrak{a}) + \Lambda[\mathfrak{a}]$. Then $(u, \mathfrak{a}) = (x, \mathfrak{a}) + (w, \mathfrak{a}) = (x + w, \mathfrak{a})$ where $(w, \mathfrak{a}) \in \Lambda[\mathfrak{a}]$. From $\Lambda[\mathfrak{a}] = \mathrm{H}_\Lambda(\mathfrak{a}) \times \{\mathfrak{a}\}$ we obtain that $w \in \mathrm{H}_\Lambda(\mathfrak{a})$. The inclusion $H_\Lambda(\mathfrak{a}) \leqslant \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow)$ shows that $w \in \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow)$. Then

$$(u + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}) = (x + w + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}) = (x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}).$$

Let $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}], (y, \mathfrak{b}) + \Lambda[\mathfrak{b}]$ be arbitrary cosets. Then put

$$\begin{aligned}\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) + \eta((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]) &= (x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}) + (y + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{b}) \\ &= (x + y + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a} \wedge \mathfrak{b})\end{aligned}$$

and

$$\begin{aligned}\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}] + (y, \mathfrak{b}) + \Lambda[\mathfrak{b}]) &= \eta((x + y, \mathfrak{a} \wedge \mathfrak{b}) + \Lambda[\mathfrak{a} \wedge \mathfrak{b}]) \\ &= (x + y + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a} \wedge \mathfrak{b}),\end{aligned}$$

so that

$$\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}] + (y, \mathfrak{b}) + \Lambda[b]) = \eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) + \eta((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]).$$

Similarly

$$\eta(((x, \mathfrak{a}) + \Lambda[\mathfrak{a}])((y, \mathfrak{b}) + \Lambda[\mathfrak{b}])) = \eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}])\eta((y, \mathfrak{b}) + \Lambda[\mathfrak{b}]).$$

The mapping η is surjective. Indeed if $(x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}) \in K_\Lambda$, then $(x, \mathfrak{a}) \in K$ and therefore we can consider the coset $(x, \mathfrak{a}) + \Lambda[\mathfrak{a}]$. By definition of η we have $\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) = (x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a})$.

Finally: let $\eta((x, \mathfrak{a}) + \Lambda[\mathfrak{a}]) = \eta((y, \mathfrak{b}) + \Lambda[\mathfrak{b}])$. Then $(x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{a}) = (y + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow), \mathfrak{b})$. It immediately implies that $\mathfrak{a} = \mathfrak{b}$ and $x + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow) = y + \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow)$. Then $y = x + u$ for some element $u \in \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow)$. On the other hand, by $(x, \mathfrak{a}), (y, \mathfrak{a}) \in K$ we obtain that $(x, \mathfrak{a}), (y, \mathfrak{a}) \in K[\mathfrak{a}]$, and hence $x, y \in \mathrm{H}(\mathfrak{a})$. It follows that $u = y - x \in \mathrm{H}(\mathfrak{a})$, and therefore $u \in \mathrm{H}(\mathfrak{a}) \cap \mathrm{H}_\Lambda(\mathfrak{m}_\downarrow) = \mathrm{H}_\Lambda(\mathfrak{a})$. Then $(u, \mathfrak{a}) \in \Lambda[\mathfrak{a}]$ and we have

$$(y, \mathfrak{a}) + \Lambda[\mathfrak{b}] = (x, \mathfrak{a}) + (u, \mathfrak{a}) + \Lambda[\mathfrak{a}] = (x, \mathfrak{a}) + \Lambda[\mathfrak{a}],$$

which shows that η is injective. That is why η is an isomorphism.

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Contact information

Leonid A. Kurdachenko, Viktoriia S. Yashchuk
Department of Algebra, Oles Honchar Dnipro National University, 72 Gagarin Av., Dnipro, Ukraine 49010
E-Mail(s): lkurdachenko@i.ua, ViktoriiaYashchuk@mail.ua

Igor Ya. Subbotin
Department of Mathematics and Natural Sciences, National University, 5245 Pacific Concourse Drive, LA, CA 90045, USA
E-Mail(s): isubboti@nu.edu

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