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RESEARCH ARTICLE

# On representation type of a pair of posets with involution

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*Dedicated to Yu.A. Drozd on the occasion of his 60th birthday*

Abstract. In this paper we consider the problem on classifying the representations of a pair of posets with involution. We prove that if one of these is a chain of length at least 4 with trivial involution and the other is with nontrivial one, then the pair is tame $\Leftrightarrow$ it is of finite type $\Leftrightarrow$ the poset with nontrivial involution is a $*$-semichain ($*$ being the involution). The case that each of the posets with involution is not a chain with trivial one was considered by the author earlier. In proving our result we do not use the known technically difficult results on representation type of posets with involution.

Our paper is devoted to study representations of pairs of posets with involution.

Representations of a pair of posets (without involution) were introduced, in the language of matrices, in [1]. In the same paper, all pairs of finite type were described. For tame type it did in [2, 3]. Representations of a pair of posets with involution (in fact, a more general situation) were studied in [4, Section 5]; as a consequence of Theorem 6, one has a criterion for such a pair to be tame in the case when each of the posets with involution is not a chain with trivial one. The case, when one of these is a chain with trivial involution, easily reduces to representations of posets with involution (see Remark at the end of this paper), and we can use the results from [5, 6] on classifying the tame and wild posets with involution. All statements of this paper are proved without using these, technically difficult, results of [5, 6].

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## 1. Formulation of the main result

Throughout the paper, $K$ denotes a fixed field. All posets are finite and all $K$-vector spaces are finite-dimensional. Singletons are always identified with the elements themselves. The category of (finite-dimensional) $K$-vector spaces is denoted as usual by $\mathrm{mod}\, K$. For simplicity, an involution on a poset is denoted by the same symbol $*$.

We stress that we keep the right-side notation (vector spaces are right; linear maps are wrote to the right of elements of spaces and are composed from left to right, etc.).

Before formulation of our main result we introduce some definitions and notation (see [4, 7]).

Let $X = (A, *)$ be a poset with involution. By an $X$-*graded vector space over* $K$ we mean the direct sum $U = \oplus_{a\in A} U_a$ of $K$-vector spaces $U_a$ such that $U_{a^*} = U_a$ for each $a \in A$. For $B \subseteq A$, put $U_B = \oplus_{y\in B} U_y$; for $B, C \subseteq A$ and a linear map $\varphi : U \to U'$ with $U$ and $U'$ being $X$-graded spaces, denote by $\varphi_{BC}$ the map of $U_B$ into $U'_C$ induced by $\varphi$. A linear map $\varphi : U \to U'$ between the $X$-graded spaces $U$ and $U'$ is said to be an $X$-*map* if $\varphi_{a^*a^*} = \varphi_{aa}$ for any $a$ and $\varphi_{bc} = 0$ for any $b \not\leq c$. The set of $X$-maps of $U$ into $U'$ is denoted by $\mathrm{Hom}_{X,K}(U, U')$. The category of $X$-graded vector spaces over $K$ (which has as objects the $X$-graded spaces and as morphisms the $X$-maps) is denoted by $\mathrm{mod}_X K$.

Since it is natural to identify a poset $A$ with trivial involution $*$ (when $a^* = a$ for any $a \in A$) with the poset itself, the above definitions involve the case of usual posets.

Let $S = (A, *)$ and $T = (B, *)$ be posets with involution and set $M = \mathrm{mod}_S K$, $N = \mathrm{mod}_T K$. *A representation of the pair* $(S, T)$ is a triple $(V, U, \lambda)$ formed by objects $V \in M$, $U \in N$ and a linear map $\lambda$ of $V$ into $U$. A morphism from $(V, U, \lambda)$ to $(V', U', \lambda')$ is determined by morphisms $\mu : V \to V'$ and $\nu : U \to U'$ such that $\lambda\nu = \mu\lambda'$. The category defined in this way is denoted by $\mathrm{rep}_K(S, T)$.

We say that a pair $(S, T)$ is *of finite type* if $\mathrm{rep}_K(S, T)$ has finitely many isomorphism classes of indecomposables, and *tame* or *wild* if so is the problem of classifying, up to isomorphism, the objects of $\mathrm{rep}_K(S, T)$. For the precise definitions of tame and wild pairs see [4, Section 5]; these are particular cases of Drozd's definitions [8, 9] (the definition of wild pairs will be recalled in Subsection 2.1).

A *semichain of length* $m$ is by definition a poset of the form $X = \bigcup_{i=1}^{m} X_i$, where each $X_i$ (called a *link* of $X$) consist of either one point or two incomparable points, and $x_1 < x_2 < \cdots < x_m$ for any $x_1 \in X_1, x_2 \in X_2, \ldots, x_m \in X_m$ (if every $X_i$ is a singleton, the poset $X$ is called a *chain*). A poset with involution $T = (Y, *)$ is called a semichain (respectively, chain) if $Y$ is a semichain (respectively, chain) and $*$ is trivial, and a $*$-*semichain* if $Y$ is a semichain and $x^* = x$ for each $x$ from the union of all two-point links (to say that $T$ is a $*$-chain means that $Y$ is a chain).

Under the assumption that $K$ is algebraic closed, we prove the following theorem.

**Theorem.** *Let $S$ and $T$ be posets with involution such that the first (respectively, second) one is a chain of length at least 4. Then the following conditions are equivalent:*

1) *$(S, T)$ is tame;*

2) *$(S, T)$ is of finite type;*

3) *$S$ (respectively, $T$) is a $*$-semichain.*

This theorem will be proved in Subsections 2.2 and 2.3.

## 2. Proof of Theorem

**2.1. The definition of wild pairs.** In this part of the paper we recall the definition of a wild pair $(S, T)$ ($S, T$ being posets with involution).

Let $X = (A, *)$ be a poset with involution and $\widehat{K} = K\langle x, y\rangle$ be the free (associative) $K$-algebras in two noncommuting variables $x$ and $y$. We can define the category $\mathrm{mod}_X \widehat{K}$ of (right) free $X$-graded $\widehat{K}$-modules similar to the category $\mathrm{mod}_X K$, considering finitely generated free $\widehat{K}$-modules instead of finite-dimensional $K$-vector spaces (such a module $U = \oplus_{a\in A} U_a$ is called free if so are all the modules $U_a, a \in A$).

Let $S = (A, *)$ and $T = (B, *)$ be posets with involution, and let $\mathcal{R}_{\widehat{K}}(S, T)$ denotes the set of all triples $(V, U, \lambda)$ formed by objects $V \in \mathrm{mod}_S \widehat{K}$, $U \in \mathrm{mod}_T \widehat{K}$ and a homomorphism $\lambda$ from $V$ into $U$. By $\mathcal{L}(\widehat{K})$ we denote the category of left finite-dimensional (over $K$) $\widehat{K}$-modules. The pair $(S, T)$ is called *wild* if there exists an element $M = (V, U, \lambda)$ of $\mathcal{R}_{\widehat{K}}(S, T)$ such that the functor

$$H(M) = M \otimes - : \mathcal{L}(\widehat{K}) \to \mathrm{rep}_K(S, T)$$

preserves indecomposability and isomorphism classes ($M \otimes Y = (V \otimes Y, \lambda \otimes \mathbf{1}_Y, V \otimes Y)$ with $(V \otimes Y)_a = V_a \otimes Y$ and $(U \otimes Y)_a = U_a \otimes Y$ for every $a$); here all tensor products are considered over $\widehat{K}$. An element $M \in \mathcal{R}_{\widehat{K}}(S, T)$ with such properties will be called *strict*.

For an element $M = (V, U, \lambda)$ of $\mathcal{R}_{\widehat{K}}(S, T)$ with $V = \widehat{K}^n$, $V_a = \widehat{K}^{n_a}$, $U = \widehat{K}^m$ and $U_b = \widehat{K}^{m_b}$ $(a \in A, b \in B)$, we identify the map $\lambda$ with the matrix $(\lambda_{ab})$, $a \in A, b \in B$, where $\lambda_{ab}$ is the map of $V_a$ into $U_b$ induced by $\lambda$; in turn, we identify each $\lambda_{ab}$ with the corresponding $n_a \times m_b$ matrix with entries in $\widehat{K}$. So $\lambda$ is identified with the block matrix $(\lambda_{ab})$, $a \in A, b \in B$, of order $n \times m$. We will indicate strict elements in this form.

**2.2. Subsidiary lemma.** When we determine a poset with involution $S = (A, *)$, the order relation is given up to transitivity and $a^*$ is indicated only if $a^* \neq a$. If the elements of $A$ are natural numbers, the order relation is denoted by $\prec$ (to distinguish between the given relation and the natural ordering $<$ of the integer numbers); the only exception is the case when $A$ is linear ordered. By $\langle m \rangle$ we denote the chain $\{1 < 2 < \ldots < m\}$.

**Lemma.** *Let $S = \langle m \rangle$ and $T = (B, *)$ be a poset with nontrivial involution. Then the pair $(S, T)$ is wild in the following cases:*

(*a*) $m = 2,\ B = \{1, 2\},\ 1^* = 2;$

(*b*) $m = 3,\ B = \{1, 2, 3, 4 \mid 1 \prec 3, 2 \prec 3 \prec 4\},\ 1^* = 3, 2^* = 4;$

(*c*) $m = 3,\ B = \{1, 2, 3, 4 \mid 1 \prec 2 \prec 3, 2 \prec 4\},\ 1^* = 3, 2^* = 4;$

(*d*) $m = 3,\ B = \{1, 2, 3, 4 \mid 1 \prec 2 \prec 4, 1 \prec 3 \prec 4\},\ 1^* = 3, 2^* = 4;$

(*e*) $m = 4,\ B = \{1, 2, 3 \mid 1 \prec 3, 2 \prec 3\},\ 1^* = 3;$

(*f*) $m = 4,\ B = \{1, 2, 3 \mid 1 \prec 2, 1 \prec 3\},\ 1^* = 3.$

*Proof.* We will use the definitions, notation and conventions of 2.1. Writing a strict element $M = (V, U, \lambda)$ of $\mathcal{R}_{\widehat{K}}(S, T)$, we assume that $r$th horizontal (respectively, vertical) band of $\lambda$ is situated above (respectively, to the left of) $s$th one if $r < s$.

As a strict element $L = (V, U, \lambda)$ of $\mathcal{R}_{\widehat{K}}(S, T)$ one can take the following one:

in case (a), $V = \widehat{K}^2$, $V_1 = V_2 = \widehat{K}$, $U = \widehat{K}^4$, $U_1 = U_2 = \widehat{K}^2$, and

$$\lambda = \left( \begin{array}{cc|cc} 1 & 0 & 0 & x \\ \hline 0 & 1 & 1 & y \end{array} \right);$$

in case (b), $V = \widehat{K}^{10}$, $V_1 = \widehat{K}^2$, $V_2 = \widehat{K}^3$, $V_3 = \widehat{K}^5$, $U = \widehat{K}^{16}$, $U_1 =$

$U_2 = U_3 = U_4 = \widehat{K}^4$, and

$$\lambda = \left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ \hline
0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\ \hline
1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&x\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&y
\end{array}\right);$$

in cases (c) and (d), $V, U$ and $\lambda$ are the same as these in case (b), but one must arrange the vertical bands $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $\lambda$ as follows: $\lambda_4, \lambda_3, \lambda_2, \lambda_1$ and $\lambda_3, \lambda_2, \lambda_1, \lambda_4$, respectively;

in case (e), $V = \widehat{K}^{12}$, $V_1 = \widehat{K}^2$, $V_2 = V_3 = \widehat{K}^3$, $V_4 = \widehat{K}^4$, $U = \widehat{K}^{16}$, $U_1 = \widehat{K}^6$, $U_2 = \widehat{K}^4$, $U_3 = \widehat{K}^6$, and

$$\lambda = \left(\begin{array}{cccccc|cccc|cccccc}
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ \hline
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&x\\ \hline
0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&y\\ \hline
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1
\end{array}\right);$$

in case (f), $V, U$ and $\lambda$ are the same as these in case (e), but one must arrange the vertical bands $\lambda_1, \lambda_2, \lambda_3$ of $\lambda$ in reverse order.

The fact that each of these elements of $\mathcal{R}_{\widehat{K}}(S, T)$ is strict can be proved in the same way as in the case when the poset with involution $S$ is not a chain (see [4, Section 5]). $\square$

**2.3. Completion of the proof.** We proceed now immediately to the proof of the theorem. By Proposition 7 of [4] $(S, T)$ is tame if and only if so is $(T^{\rm op}, S^{\rm op})$, and hence one may assume that $S$ is a chain of length at least 4 ($S^{\rm op}$, $T^{\rm op}$ denote the dual posets with involution).

The implication 2) $\Rightarrow$ 1) is trivial. The implication 3) $\Rightarrow$ 2) follows from the fact that, for any $*$-semichain $T = (B, *)$, the category $\mathrm{rep}_K(S, T)$ is isomorphic to the category of representations of a bundle of the chain $S$ and the semichain $B$ (with the involution on $S \cup B$ induced by $*$); this bundle is of finite type by the main classification theorem of [10, §1]. Finally, in view of the fact that any $(S, T)$ is either tame or wild (see the main result of [9]), and Lemma 8 of [4], the implication 1) $\Rightarrow$ 3) is equivalent to the assertion that $(S, T)$ with $S = \langle 4 \rangle$ is wild if $T = (B, *)$ is not a $*$-semichain. This assertion immediately follows from Lemma 12 of [4] and the above lemma.

**2.4. Remark.** One can easily show (in an analogous way as in [1, §2] for representations of pairs of posets without involution) that in the case $S = \langle m \rangle$, $T = (B, *)$ with $m > 1$ the problem of classifying, up to isomorphism, the objects of $\mathcal{R} = \mathrm{rep}(S, T)$ reduces to that for the category $\mathcal{R}'$ of representations of the poset with involution $T(m-1) = \langle m-1 \rangle \coprod T$ (here $\coprod$ denotes the direct sum of posets with involution), and we can apply the main results of the theory of representations of posets with involution [5, 6] (see also [11]).

All statements of this paper have proved without using the (technically difficult) results of [5, 6, 11]. For readers interested in all points of view to our theory, we give here a sketch of a proof of Lemma (respectively, of the implication 3) $\Rightarrow$ 1) of Theorem) with using ones.

In cases (b)–(f) of Lemma, using the algorithm from [6, §4], one can reduce the problem of classifying the representations of $T(m-1)$ to that of some poset with trivial involution $T'(m-1)$. Since $T'(m-1)$ contains (in each of cases (b)–(f)) a subposet isomorphic to $\langle 1 \rangle \coprod \langle 3 \rangle \coprod \langle 4 \rangle$, it is wild by [12]; then $T(m-1)$ is also wild and consequently so is the pair $(S, T)$. In case (a) the pair $(S, T)$ is wild, for example, by Main theorem of [6].

As to Theorem, the implication 3) $\Rightarrow$ 1) follows from the fact that, for any $*$-semichain $T = (B, *)$, the poset $T'(m-1)$ is of finite type.

## References

[1] M. M. Kleiner, *Partially ordered sets of finite type*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28** (1972), 32-41 (in Russian).

[2] M. M. Kleiner, *Pairs of partially ordered sets of tame representation type*, Linear Algebra Appl., **104** (1988), 103-115.

[3] A. S. Shkabara, *Quivers with relations and DGC*, Akad. Nauk Ukrain. SSR, Inst. Mat., Preprint No. 78.43 (1978), 3-41 (in Russian).

[4] V. M. Bondarenko, *Linear operators on S-graded vector spaces*, Linear Algebra Appl., **365** (2003), 45-90.

[5] V. M. Bondarenko, A. G. Zavadskij, *Posets with an equivalence relation of tame type and of finite growth*, Canad. Math. Soc. Conf. Proc., **11** (1991), 67-88.

[6] V. M. Bondarenko, A. G. Zavadskij, *Tame posets with equivalence relation*, Contem. Math., **131**, part 2 (1992), 237-251.

[7] V. M. Bondarenko, *On dispersing representations of quivers and their connection with representations of bundles of semichains*, Algebra and discrete math., **1** (2002), 19-31.

[8] Y. A. Drozd, *On tame and wild matrix problems*, Matrix problems, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1977, 104–114 (in Russian).

[9] Y. A. Drozd, *Tame and wild matrix problems*, Representations and Quadratic Forms, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979, 39-74 (English transl. in AMS Translations, **128** (1986), 31-55).

[10] V. M. Bondarenko, *Representations of bundles of semichained sets, and their applications*, Algebra i Analiz, **3** (1991), 38-61 (English transl. in St. Petersburg Math. J., **3** (1992) 973-996).

[11] L. A. Nazarova, A. V. Roiter, *Representations and forms of weakly completed partially ordered sets*, Linear algebra and the theory of representations, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1983, 19–54 (in Russian).

[12] L. A. Nazarova, *Partially ordered sets of infinite type*, Izv. Akad. Nauk SSSR, **39** (1975), 963-991 (in Russian).

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