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Color-detectors of hypergraphs

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Abstract. Let X be a set of cardinality k, $\mathcal{F}$ be a family of subsets of X. We say that a cardinal $\lambda, \lambda < k$, is a *color-detector* of the hypergraph $H = (X, \mathcal{F})$ if $card\, \chi(X) \leq \lambda$ for every coloring $\chi : X \to k$ such that $card\, \chi(F) \leq \lambda$ for every $F \in \mathcal{F}$. We show that the color-detectors of H are tightly connected with the covering number $cov\,(H) = \sup\{\alpha :$ *any* α*points of* X *are contained in some* $F \in \mathcal{F}\}$. In some cases we determine all of the color-detectors of H and their asymptotic counterparts. We put also some open questions.

Let X be a set, $\mathcal{F}$ be a family of subsets of X. The pair $H = (X, \mathcal{F})$ is called a *hypergraph* with the set of vertices X and the set of edges $\mathcal{F}$. We suppose that $\bigcup \mathcal{F} = X$.

Let λ be a cardinal such that $0 < \lambda < k = card\ X$. A coloring $\chi : X \to k$ is called λ-*admissible* if $card\ \chi(F) \leq \lambda$ for every $F \in \mathcal{F}$. We put

$$æ(H, \lambda) = sup\{card\ \chi(X) : \chi\ is\ a\ \lambda - admissible\ coloring\ of\ X\}.$$

Clearly, $æ(H, \lambda) \geq \lambda$. If $æ(H, \lambda) = \lambda$, we say that λ is a *detector* of H. If λ is a detector of H, then there exists $F \in \mathcal{F}$ such that $card\ F > \lambda$ (because, otherwise, a bijective coloring $\chi : X \to k$ is λ-admissible and $\chi(X) = k > \lambda$).

Proposition 1. *A cardinal λ is a detector of H if and only if, for every surjective coloring $\chi : X \to \lambda^+$, were λ^+ is the cardinal-successor of λ, there exists $F \in \mathcal{F}$ such that card $\chi(F) = \lambda^+$.*

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Proof. Let $æ(H,\lambda) = \lambda$ and let $\chi : X \to \lambda^+$ be a surjective coloring. Then χ is not λ-admissible, so there exists $F \in \mathcal{F}$ such that $card\, F = \lambda^+$.

Assume that $æ(H,\lambda) > \lambda$ and choose a λ-admissible coloring $\chi : X \to k$ such that card $\chi(X) > \lambda$. Identifying some colors, we get a surjective λ-admissible coloring $\chi' : X \to \lambda^+$, so $card\, \chi'(F) \leq \lambda$ for every $F \in \mathcal{F}$. □

We define the covering number of H as

$$cov(H) = sup\{\gamma : for\ every\ Y \subseteq X\ with\ card\ Y \leq \gamma \\ there\ exists\ F \in \mathcal{F}\ such\ that\ Y \subseteq F\}.$$

Proposition 2. *If $cov\ (H) \geq \lambda^+$, then λ is a detector of H.*

Proof. Let $\chi : X \to \lambda^+$ be a surjective coloring. Choose $Y \subseteq X$ such that $card\ Y = card\ \chi(Y) = \lambda^+$. Since $cov(H) \geq \lambda^+$, we can choose $F \in \mathcal{F}$ such that $Y \subseteq F$. Then $card\, \chi(F) = \lambda^+$. By Proposition 1, λ is a detector of H. □

Proposition 3. *If a natural number m is a detector of H, then cov $(H) \geq m$.*

Proof. We fix an arbitrary m-subset $Y = \{y_0, ..., y_{m-1}\}$ of X and put $\chi(y_i) = i$ and $\chi(x) = m$ for every $x \in X \setminus Y$. Since m is a detector, by Proposition 1, there exists $F \in \mathcal{F}$ such that $card\, \chi(F) = m + 1$. It follows that $Y \subseteq F$, so $cov\ (H) \geq m$. □

Proposition 4. *If a natural number m is a detector of H and $m' < m$, then m' is a detector of H.*

Proof. Assume, otherwise, and fix a surjective coloring $\chi : X \to m' + 1$ such that $card\, \chi(F) \leq m'$ for every $F \in \mathcal{F}$ (see Proposition 1). Since $m' + 1 < k$, there exist two elements $a, b \in X$ such that $\chi(a) = \chi(b)$. We define the new coloring $\chi' : X \to m' + 2$ such that $\chi'(x) = \chi(x)$ for every $x \in X \setminus \{a\}$, and $\chi'(a) = m' + 1$. Then $card\, \chi'(F) \leq m' + 1$ for every $F \in \mathcal{F}$, but $card\, \chi'(X) = m' + 2$, so $m' + 1$ is not a detector of H. Repeating the arguments, we conclude that m is not a detector of H, whence a contradiction. □

The following example shows that the finiteness assumption for m can not be omitted in Propositions 3 and 4.

Example 1. *Let Y, Z be disjoint infinite sets, card $Y = k$, card $Z = \lambda$ and $\lambda < k$. We put $X = Y \bigcup Z$ and $F_z = Y \bigcup \{z\}$ for every $z \in Z$. Then we consider the hypergraph $H = (X, \mathcal{F})$, where $\mathcal{F} = \{F_z : z \in Z\}$.*

Clearly, cov $(H) = 1$, λ *is a detector of* H, *but every cardinal* λ' *such that* $1 < \lambda' < \lambda$ *is not a detector of* H. □

For every hypergraph $H = (X, \mathcal{F})$, we consider the *graph* $\Gamma(H)$ *of intersections* of H with the set of vertices X and the set of edges defined by the rule: $(x_1, x_2) \in X \times X$ is an edge if and only if $x_1 \neq x_2$ and there exist $F_1, F_2 \in \mathcal{F}$ such that $x_1 \in F_1, x_2 \in F_2$ and $F_1 \bigcap F_2 \neq \emptyset$.

Proposition 5. *For every hypergraph* $H = (X, \mathcal{F})$, 1 *is a detector of* H *if and only if the graph* $\Gamma(H)$ *is connected.*

Proof. Assume that $\Gamma(H)$ is connected and take an arbitrary coloring $\chi : X \to k$ such that *card* $\chi(F) = 1$ for every $F \in \mathcal{F}$. Given any $x, y \in X$, we choose a path $x_1, x_2, ..., x_n$ in $\Gamma(H)$ such that $x = x_1, y = x_n$. Then $\chi(x_1) = \chi(x_2) = ... = \chi(x_n)$, so $\chi(x) = \chi(y)$.

If $\Gamma(H)$ is not connected, we take a connected component Y of $\Gamma(H)$ and, for every $x \in X$, we put

$$\chi(x) = \begin{cases} 0, & if \;\; x \in Y; \\ 1, & if \;\; x \in X \setminus Y; \end{cases}$$

Then the coloring χ is 1-admissible, but *card* $\chi(X) > 1$. □

Proposition 6. *If* $H = (X, \mathcal{F})$ *is a graph and card* $X > 1$, *then the only possible detector of* H *is 1, and 1 is a detector of* H *if and only if* H *is connected.*

Proof. We fix a bijection $\chi : X \to k$. Since *card* $\chi(F) = 2$ for every $F \in \mathcal{F}, \chi$ is λ-admissible for every $\lambda \geq 2$. It follows that if $1 < \lambda < k$, then λ is not a detector of H. On the other hand, by Proposition 5, 1 is a detector of H if and only if $\Gamma(H)$ is connected. It is easy to see that $\Gamma(H)$ is connected if and only if H is connected. □

Let $\Gamma = (V, E)$ be a connected graph with the set of vertices V and the set of edges E. For any $u, v \in V$, we denote by $d(u, v)$ the length of a shortest path between u and v. Given any $u \in V$, $r \in \mathbb{N}$, we put $B_d(u, r) = \{v \in V : d(u, v) \leq r\}$. Let $\mathcal{B}$ be the family of all unit balls in Γ. Call the hypergraph $H = (V, \mathcal{B})$ to be the *ball hypergraph* of Γ. By Proposition 5, 1 is a detector of H.

Problem 1. *Given a natural number* $n > 1$, *characterize the class* τ_n *of connected graphs such that* $\Gamma \in \tau_n$ *if and only if* n *is a detector of the ball hypergraph of* Γ.

For every natural number $n > 1$, we denote by $\mathcal{C}_n$ the class of all connected graphs such that $\Gamma \in \mathcal{C}_n$ if and only if $V(\Gamma) \geq n+1$ and any $\leq n$ vertices of Γ are contained in some unit ball in Γ. Note that $\mathcal{C}_2$ is the class of graphs of diameter ≤ 2, where $diam\,\Gamma = sup\{d(u,v) : u, v \in V\}$. By Propositions 2 and 3, we have

$$\mathcal{C}_2 \supseteq \tau_2 \supseteq \mathcal{C}_3 \supseteq \tau_3 \supseteq ...$$

The next two examples show that $\mathcal{C}_2 \supset \tau_2$ and $\mathcal{C}_{2n-1} \supset \tau_{2n-1}$ for every $n \geq 2$.

Example 2. *We consider a pentagone Γ with the set of vertices $\{a_1, a_2, a_3, a_4, a_5\}$ and the set of edges $\{(a_1,a_2),(a_2,a_3),(a_3,a_4),(a_4,a_5),(a_5,a_1)\}$. Since $diam\,\Gamma = 2$, we have $\Gamma \in \mathcal{C}_2$. On the other hand, a coloring χ, defined by the rule*

$$\chi(a_1) = 1, \chi(a_2) = 2, \chi(a_3) = 2, \chi(a_4) = 3, \chi(a_5) = 2,$$

is 2-admissible, so $\Gamma \notin \tau_2$. □

Example 3. *Let n be a natural number $> 1, A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}$ be disjoint sets. We consider the graph Γ with the set of vertices $V = A \bigcup B$ and the set of edges*

$$E = (A \times B) \setminus \{(a_i, b_i) : i \in \{1, ..., n\}\}.$$

Let V' be a subset of V such that card $V' \leq 2n-1$. Then there exists $i \in \{1,...,n\}$ such that either $a_i \notin V'$ or $b_i \notin V'$. If $a_i \notin V'$, then $V' \subseteq B(b_i, 1)$. If $b_i \notin V'$, then $V' \subseteq B(a_i, 1)$. Hence, $\Gamma \in \mathcal{C}_{2n-1}$. On the other hand, a coloring χ, defined by the rule

$$\chi(a_1) = 1, ..., \chi(a_n) = n, \chi(b_1) = n+1, ..., \chi(b_n) = 2n,$$

is $(2n-1)$-admissible, so $\Gamma \notin \tau_{2n-1}$. □

Question 1. *Is $\mathcal{C}_{2n} \supset \tau_{2n}$ for every $n \geq 2$?*

Question 2. *Is $\tau_n \supset \mathcal{C}_{n+1}$ for every $n \geq 2$?*

Proposition 7. *Let G be a group with the unit e, $Y \subseteq G$, $e \in Y$. Then 1 is a detector of the hypergraph $G_Y = (G, \{gY : g \in G\})$ if and only if $G = < Y >$, where $< Y >$ is the smallest subgroup of G containing Y.*

Proof. Let Γ be the intersection graph of G_Y. In view of Proposition 5, it suffices to show that Γ is connected if and only if $G = < Y >$.

Assume that Γ is connected and let g be an arbitrary element of G. Then there exist the elements $x_1, ..., x_n$ of G such that $x_1 = e, x_n = g$ and $x_iY \bigcap x_{i+1}Y \neq \emptyset$ for every $i \in \{1, ..., n-1\}$. It follows that $x_1 \in YY^{-1}, x_2 \in YY^{-1}YY^{-1}, ..., x_n \in (YY^{-1})^n$, so $g \in < Y >$.

Assume that $G = < Y >$ and let g be an arbitrary element of G. It suffices to show that the vertices e and g of Γ are connected. Let $g = y_1^{i_1} y_2^{i_2} ... y_n^{i_n}$, where $i_1, ..., i_n \in \{\pm 1\}$. We put $x_0 = e, x_1 = y_1^{i_1}, x_{k+1} = x_k y_{k+1}^{i_{k+1}}, k \in \{1, ..., n-1\}$. Since either $x_{k+1} \in x_kY$ or $x_k \in x_{k+1}Y$, then x_k, x_{k+1} are incident in Γ. □

Problem 2. *Let G be a group, $Y \subseteq G, e \in Y$ and let n be a natural number. Find necessary and sufficient conditions on Y under which n is a detector of G_Y?*

Proposition 8. *Let V be a vector space over some field F, γ be a cardinal such that $1 \leq \gamma < dimV, A(V, \gamma)$ be the family of all γ-dimensional affine subspaces of V. Let $H(V, \gamma)$ be the hypergraph $(V, A(V, \gamma))$ and let λ be a cardinal such that $\lambda \leq dimV$ if dim V is finite and $\lambda < dimV$ if V is infinite. If γ is finite, then λ is a detector of $H(V, \gamma)$ if and only if $\lambda \leq \gamma$. If γ is infinite, then λ is a detector of $H(V, \gamma)$ if and only if $\lambda < \gamma$.*

Proof. If γ is finite, then $cov\ (H(V, \gamma)) = \gamma + 1$. If $\lambda \leq \gamma$, by Proposition 2, λ is a detector of $H(V, \gamma)$. If γ is infinite, then $cov\ (H(V, \gamma)) = \gamma$. If $\lambda < \gamma$, by Proposition 2, λ is a detector of $H(V, \gamma)$.

Let $dim\ V = \delta$. We fix some basis $\{v_\alpha : \alpha < \delta\}$ of V and put $\chi(v_\alpha) = \alpha$ and $\chi(v) = \delta$ for every $v \in V \setminus \{v_\alpha : \alpha < \delta\}$. If γ is finite, then $|card\ \chi(S)| \leq \gamma + 1$ for every $S \in A(V, \gamma)$. Hence, if $\lambda > \gamma$, then λ is not a detector of $H(V, \gamma)$. If γ is infinite, then $|card\ \chi(S)| \leq \gamma$ for every $S \in A(V, \gamma)$. Hence, if $\lambda \geq \gamma$, then λ is not a detector of $H(V, \gamma)$. □

Problem 3. *Detect æ$(H(V, \gamma), \lambda)$ for every vector space V and any cardinals γ, λ.*

For example, if n, m are natural numbers, then

$$æ(H(\mathbb{R}^n, 1), m) = \begin{cases} 1, & if \quad m = 1; \\ n+1, & if \quad m = 2; \\ 2^{\aleph_0}, & if \quad m \geq 3. \end{cases}$$

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}, P$ is called the *set of radiuses.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\}, \quad B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha).$$

A ball structure $\mathcal{B}$ is called a *ballean* if
• for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'), \quad B^*(x,\beta) \subseteq B(x,\beta');$$

• for any α, $\beta \in P$ there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

• for any $x, y \in X$ there exists $\alpha \in P$ such that $y \in B(x,\alpha)$.

We note that the balleans arouse independently in asymptotic topology [2] and in combinatorics [3].

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset $A \subseteq X$ is called *bounded* if there exist $x \in X, \alpha \in P$ such that $A \subseteq B(x,\alpha)$.

Let Y be an arbitrary set, $f : X \to Y$. We define the asymptotic cardinality of $f(X)$ as

$$ascard\ f(X) = min\{card\ f(X \setminus V) : V\ is\ a\ bounded\ subset\ of\ X\}.$$

If $Y = X$ and f is the indentity mapping, we write *ascard* X instead of *ascard id* X.

Let $H = (X, \mathcal{F})$ be a hypergraph such that every subset $F \in \mathcal{F}$ is bounded in $\mathcal{B}$, λ be a cardinal, $\lambda < ascard X$. A coloring $\chi : X \to ascard X$ is called *asymptotically* λ-*admissible*, if *ascard* $\chi(F) \leq \lambda$ for every $F \in \mathcal{F}$. We put

$$æ_{as}(H,\lambda) = sup\{ascard\ \chi(X) : \chi\ is\ asymptotically\ \lambda - admissible\}$$

and say that λ is an *asymptotic detector* of H if $æ_{as}(H,\lambda) = \lambda$.

We define a *graph* $A\Gamma(H)$ of *asymptotic intersections* of hypergraph $H = (X, \mathcal{F})$ as a graph with the set of vertices $\mathcal{F}$ and the set of edges $\{(F,F') : F, F' \in \mathcal{F}, F \neq F'\ and\ F \bigcap F'\ is\ unbounded\}$.

Proposition 9. *Let $\mathcal{B} = (X, P, B)$ be a ballean such that X is a union of some family $\{B_n : n \in \omega\}$ of bounded subsets. Let $\mathcal{F} = \{F_n : n \in \omega\}$ be a family of unbounded subset of X. Then 1 is an asymptotic detector of $H = (X, \mathcal{F})$ if and only if there exists a finite subset $\mathcal{F}' \subset \mathcal{F}$ such that $G \setminus \bigcup \mathcal{F}'$ is bounded and $A\Gamma(H)$ is connected.*

Proof. Assume that 1 is an asymptotic detector of H, but $X \setminus \bigcup \mathcal{F}'$ is unbounded for every finite subset $\mathcal{F}' \subset \mathcal{F}$. Then we can choose an injective sequence $(x_n)_{n\in\omega}$ in X such that

$$x_n \in F_n \setminus (B_0 \bigcup ... \bigcup B_n \bigcup F_0 \bigcup ... \bigcup F_{n-1}).$$

We put $\chi(x_n) = 1$ for every $n \in \omega$, and $\chi(x) = 0$ if $x \neq \{x_n : n \in \omega\}$. Clearly, the coloring χ is asymptotically 1-admissible, but *ascard* $\chi(X) = 2$. Hence, $X = F_0 \bigcup ... \bigcup F_n \bigcup V$ for some $n \in \omega$ and some bounded subset V. Assume that $A\Gamma(H)$ is not connected and let C be a connected component of $A\Gamma(H)$. Put $X_0 = V \bigcup \{F_i : F_i \in C\}, X_1 = X \setminus X_0$, and let χ be the coloring of X, defined by the partition $X = X_0 \bigcup X_1$. If $F \in \mathcal{F}$, then either $F \bigcap X_0$ is bounded or $F \bigcap X_1$ is bounded, so χ is 1-asymptotically admissible, but *ascard* $\chi(X) = 2$.

Assume that $X \setminus \{F_0, ..., F_n\}$ is bounded for some $n \in \omega$ and $A\Gamma(H)$ is connected, but 1 is not an asymptotical detector of H. Then there exist an asymptotically 1-admissible coloring $\chi : X \to \{0, 1\}$ and $i, j \in \{0, ..., n\}, i \neq j$ such that $\chi|_{F_i \setminus V} \equiv 0, \chi|_{F_j \setminus V} \equiv 1$ for some bounded subset V of G. Then F_i, F_j are not distinct connected components of $A\Gamma(H)$, so $A\Gamma(H)$ is not connected. □

Let $\mathcal{B} = (X, P, B)$ be a ballean, $f : X \to \mathbb{R}, Y \subseteq X$. We say that $r \in \mathbb{R}$ is a limit of $f(Y)$ with respect to $\mathcal{B}$ if r is the limit of the filter with the base $\{f(Y \setminus V) : V \ \textit{is bounded subset of} \ X\}$. The next definition is inspired by [2]. A hypergraph $H = (X, \mathcal{F})$ is called *limit-detecting* if, given $f : X \to \mathbb{R}$, $f(X)$ has a limit provided that every $f(F), F \in \mathcal{F}$ has a limit.

Proposition 10. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $\mathcal{F}$ be a family of unbounded subsets of X. If $H = (X, \mathcal{F})$ is limit-detecting, then* 1 *is an asymptotic detector of H.*

Proof. Assume that H is limit detecting, but 1 is not an asymptotic detector of H. Then there exist an asymptotically 1-admissible coloring $\chi : X \to$ *ascard* X, the ordinals $\alpha, \beta, \alpha < \beta <$ *ascard* X and $F, F' \in \mathcal{F}$ such that

$$\chi|_{F\setminus V} = \alpha, \chi|_{F'\setminus V} = \beta$$

for some bounded subset V of X. We consider a mapping $f : X \to \{0, 1\}$, defined by the rule $f(x) = 0$ if $x \in \chi^{-1}(\alpha), f(x) = 1$ if $x \notin \chi^{-1}(\alpha)$. Then $f(Y)$ has a limit for every $Y \in \mathcal{F}$, but $f(X)$ has not a limit. □

Proposition 11. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $\mathcal{F}$ be a family of unbounded subsets of X such that $X \setminus \bigcup \mathcal{F}'$ is bounded for some finite subset $\mathcal{F}' \subseteq \mathcal{F}$. If* 1 *is an asymptotic detector of H, then H is limit-detecting.*

Proof. Let $\mathcal{F}' = \{F_0, ..., F_n\}$. Assume that 1 is a detector of H, but H is not limit-detecting. Then there exists a mapping $f : X \to \mathbb{R}$ such that every subset $f(F_i)$ has some limit r_i with respect to $\mathcal{B}$, but $f(X)$ has no limit. We may suppose that $r_0, ..., r_m$ are all distinct numbers from $\{r_0, ..., r_n\}$. Clearly, $m > 1$. Choose $\varepsilon > 0$ such that

$$(r_0 - \varepsilon, r_1 + \varepsilon) \bigcap (r_i - \varepsilon, r_i + \varepsilon) = \emptyset$$

for every $i \in \{1, .., n\}$. Put $\chi(x) = 0$ if $f(x) \in (r_0 - \varepsilon, r_0 + \varepsilon)$, and $\chi(x) = 1$ otherwise. Clearly, χ is an asymptotically 1-admissible, but ascard $\chi(X) = m > 1$, a contradiction. □

Question 3. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $\mathcal{F}$ be a family of unbounded subsets of X, $H = (X, \mathcal{F})$. Let 1 is an asymptotic detector of H. Is H limit detecting?*

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