**CONDENSED
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Helmholtz decomposition theorem and Blumenthal's extension by regularization

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Helmholtz decomposition theorem for vector fields is usually presented with too strong restrictions on the fields and only for time independent fields. Blumenthal showed in 1905 that decomposition is possible for any asymptotically weakly decreasing vector field. He used a regularization method in his proof which can be extended to prove the theorem even for vector fields asymptotically increasing sublinearly. Blumenthal's result is then applied to the time-dependent fields of the dipole radiation and an artificial sublinearly increasing field.

Key words: *Helmholtz theorem, vector field, electromagnetic radiation*

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1. Introduction

Regularization is nowadays a common method to modify the observable physical quantities in order to avoid infinities and make them finite. Especially in the modern treatment of phase transitions by renormalization theory [1] it is a tool in calculating, e.g., critical exponents. However, such regularization methods turned out to be also useful in university lectures on such classical fields as electrodynamics or hydrodynamics. Unfortunately, this method is rarely mentioned in this context. It is the aim of this paper to present a classical example known as Helmholtz decomposition theorem and to show the power of regularization in this case.

According to the above mentioned theorem, one can divide a given vector field $\vec{v}(\vec{x})$ into a sum of two vector fields $\vec{v}_l(\vec{x})$ and $\vec{v}_t(\vec{x})$ where $\vec{v}_l$ is irrotational (curl-free) and $\vec{v}_t$ solenoidal (divergence-free), if the vector field fulfills certain conditions on continuity and asymptotic decrease ($r \to \infty$). Here, $\vec{x}$ is the position vector in three-dimensional space and $r = |\vec{x}|$ is its absolute value. Helmholtz calls these two components *integrals of first class* for which a velocity potential exists and *integrals of second class* for which this is not the case ([2] first reference, p. 22).

Usually, these two integrals are constructed directly from the vector field by starting from the identity $\Delta \vec{a} = \vec{\nabla} \vec{\nabla} \cdot \vec{a} - \vec{\nabla} \times (\vec{\nabla} \times \vec{a})$ with $\vec{a} = -\vec{v}(\vec{x}')/4\pi|\vec{x}' - \vec{x}|$. Integrating $\Delta \vec{a} = \vec{v}(\vec{x}')\delta(\vec{x} - \vec{x}')$ over all space, one obtains for continuously differentiable vector field $\vec{v}$:

$$\vec{v}(\vec{x}) = \underbrace{-\frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{\nabla} \vec{\nabla} \cdot \frac{\vec{v}(\vec{x}')}{|\vec{x}' - \vec{x}|}}_{\vec{v}_l(\vec{x})} + \underbrace{\frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{\nabla} \times \left(\vec{\nabla} \times \frac{\vec{v}(\vec{x}')}{|\vec{x}' - \vec{x}|} \right)}_{\vec{v}_t(\vec{x})}. \tag{1.1}$$

Or one calculates the two parts of the vector field from the respective potentials existing for them,

$$\vec{v}_l(\vec{x}) = -\vec{\nabla} \phi_{\mathrm{H}}(\vec{x}), \qquad \vec{v}_t(\vec{x}) = \vec{\nabla} \times \vec{A}_{\mathrm{H}}(\vec{x}). \tag{1.2}$$

These potentials are defined by the divergence and the rotation of the vector field

$$\phi_{\mathrm{H}}(\vec{x}) = -\frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{v}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|}, \tag{1.3}$$

$$\vec{A}_{\mathrm{H}}(\vec{x}) = \frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{v}(\vec{x}') \times \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|}. \tag{1.4}$$

As concerns validity, the uniqueness of decomposition and the existence of the respective potentials, one finds different conditions.

In fact, Helmholtz was largely anticipated by George Stokes (presented in 1849 and published in 1856 in [3], see p. 10, item 8), so it is also called Helmholtz-Stokes theorem [4], especially in hydrodynamics, where the theorem is of particular relevance. There, the fluid fields of decomposition have physical properties of freedom of vorticity and incompressibility, which for each field makes the analysis simpler [5]. In his discussion of the theorem, Lamb [5] states the conditions for divergency and vorticity of the vector field in infinity in order to prove the theorem: they should be of the order of $1/r^n$ with $n > 3$.

Föppl introduced the decomposition theorem into German textbooks on electrodynamics [6]. In the first chapter he presents the appropriate tools of vector analysis since they were already used in hydrodynamics. Regarding the theorem he assumed a finite extension of the sources and vortices and, therefore, assumed a behavior for the corresponding vector field of the form $|\vec{v}| \sim 1/r^2$ for $|\vec{x}| = r \to \infty$. However, his proof permits less restrictive conditions, namely an asymptotic decay of the field only somewhat stronger than $1/r$. The decomposition theorem can be found in one of these formulations in most textbooks or lecture notes on electrodynamics.

The main point made after presenting the theorem is in most cases the advantage of introducing a scalar and vector potentials. It is applied in electrostatics and magnetostatics for cases where the extension of the sources is restricted to a finite region (see for example [7]). However, even in electro- and magnetostatics there exist configurations with slow decreasing fields. The electric field of an infinite straight wire, which bears an electric charge, decays as $\sim 1/\rho$, where ρ is the distance to the wire. If, on the other hand, the wire carries a current, then the magnetic field decays as $\sim 1/\rho$. In both cases, a regularization is appropriate to get the potentials from finite integrals over the sources without using symmetry arguments, which are not applicable in more complicated geometries. A less restrictive formulation is found in [8] (Appendix B as an interesting corollary) stating that the field should go in infinity faster to zero than $1/r$.

Already in Aachen in 1905 professor Otto Blumenthal together with Sommerfeld, proved [9] that any vector field that goes to zero asymptotically can be decomposed in a curl-free and a divergence-free part (weak version). Blumental's formulation reads as follows (see [9], p. 236):

"Let $\vec{v}$ be a vector, which is, in addition to arbitrary many derivatives, everywhere finite and continuous and vanishes at infinity with its derivatives; then one can always decompose this vector into two vectors, a curl-free $\vec{v}_l$ and a divergence-free $\vec{v}_t$, such that

$$\vec{v} = \vec{v}_l(\vec{x}) + \vec{v}_t(\vec{x}). \tag{*}$$

The vectors $\vec{v}_l$ and $\vec{v}_t$ diverge asymptotically weaker than $\ln r$.
In addition, one has the following proposition for uniqueness: $\vec{v}_l$ and $\vec{v}_t$ are unique up to an additive constant vector because of the given properties." No further specification for the behavior of the vector field was given.

This formulation was taken over in its essential statements by Sommerfeld in 1944 [10]. He noted further that the fundamental theorem of vector analysis, as he called it, was already proven by Stokes [3] in 1849 and in a more complete form by Helmholtz paper of 1858. In a footnote he cites the paper of Blumenthal: *For a rigorous proof see: O. Blumenthal, Ueber die Zerlegung unendlicher Vektorfelder, (Math. Ann., 1905, 61, 235). His only restriction is that V and its first derivative vanish at infinity while no additional assumption is made how quickly they vanish. It turns out that the component fields V_l and V_t need not vanish themselves, they may even become in a restricted way infinite. In the following we shall make the somewhat vague assumption that V vanishes "sufficiently strongly" at infinity.*

Later on it was shown that the conditions of continuity and differentiability can be weakened [11, 12] and that the theorem can be applied to vector fields behaving according to a certain power law [13]. Based on Blumenthal's method of regularization of the Green function, Neudert and Wahl [14] among other things investigated the asymptotic behavior of a vector field $\vec{v}$ if its sources $\operatorname{div}\vec{v}$ and vortices $\operatorname{curl}\vec{v}$ fulfill some conditions including differentiability and asymptotic decay.

These developments remained to a large extent unnoticed in the physical literature and in mathematical physics (for an exception see [15]). Thus, it seemed to be necessary to show the validity of the decomposition theorem for electromagnetic radiation fields that decay asymptotically like $1/r$. In fact

there were several items to clarify for time dependent vector fields, especially the question of retardation, its connection to causality and the choice of gauge.

The paper is organized as follows: first we develop a systematic method of regularization, then we reformulate the decomposition theorem including all potentials for such cases and finally we give two applications of the theorem.

2. Regularization Method

The regularization method, which is the basis of Blumenthal's proof, was not explicated in its generality and in its improvement in order to be applicable to vector fields, which decay asymptotically with a specified power law (or even increase as we shall see below). The idea is as follows: Since the property of the vector field cannot be changed in order to make the involved integrals finite, one tries to change the weighting function $1/|\vec{x}' - \vec{x}|$ appearing in the solution for the two fields. Going back to the construction of these solutions, one used the Green function of the Poisson equation.

The solution $\phi_0(\vec{x})$ of the Poisson equation

$$\Delta \phi_0(\vec{x}) = -4\pi \rho(\vec{x}) \tag{2.1}$$

with the source density $\rho(\vec{x})$ is found by introducing its Green function

$$G_0(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}' - \vec{x}|}, \tag{2.2}$$

$$\phi_0(\vec{x}) = \int \mathrm{d}^3 x' \, \rho(\vec{x}') \, G_0(\vec{x}, \vec{x}'). \tag{2.3}$$

If the solution exists in the whole domain of $\mathbb{R}^3$, the integral should be finite. This is guaranteed by a sufficient decay of the integrand, either by a sufficient strong decay of the source density and/or by a sufficient decrease of the Green function.

In his work on the Helmholtz decomposition theorem [9], Blumenthal presented a method to make this solution finite (regularizing the solution) by changing the Green function of the Poisson equation, without changing the Poisson equation (which means without changing the source density). He mentioned on p. 236 of [9] the similarity of his method to the "convergence generating" terms in the theorem of Mittag-Leffler. Thus, one can prove the existence of the potential for cases where the source density is less strongly decreasing. From this method it becomes clear how a systematic extension of the decomposition theorem is possible.

Introduction of an arbitrary point $\vec{x}_0$ [apart from the condition that $\rho(\vec{x}_0)$ is finite at this point; regularization point or regulator] and noting that $G_0(\vec{x}, \vec{x}') = G_0(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0)$, we expand G_0 in a power series in $\vec{x} - \vec{x}_0$

$$G_0(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}' - \vec{x}_0|} - (\vec{x} - \vec{x}_0) \cdot \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}_0|} + \dots. \tag{2.4}$$

A stronger decrease for large $|\vec{x}'|$ of the Green function is now reached by subtraction of the corresponding expansion terms. We get the following set of stronger decreasing Green functions

$$G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_0(\vec{x}, \vec{x}') - \frac{1}{|\vec{x}' - \vec{x}_0|}, \tag{2.5}$$

$$G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) - \frac{(\vec{x} - \vec{x}_0) \cdot (\vec{x}' - \vec{x}_0)}{|\vec{x}' - \vec{x}_0|^3}. \tag{2.6}$$

The asymptotic decrease of these modified Green functions is as $\sim 1/(r')^{1+i}$. For $i \leqslant 2$, the subtracted terms do not change the source density

$$\Delta G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -4\pi \delta(\vec{x}' - \vec{x}) \qquad \text{for} \qquad 0 \leqslant i \leqslant 2. \tag{2.7}$$

However, they make it possible to extend the range of the validity for which the existence of the potential (and the decomposition) can be proven

$$\phi_i(\vec{x}) = \int d^3x' \, \rho(\vec{x}') \, G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \qquad \text{and} \qquad \Delta\phi_i(\vec{x}) = -4\pi\rho(\vec{x}) \qquad \text{for} \qquad i \leqslant 2. \tag{2.8}$$

The solutions $\phi_i(\vec{x})$ differ only by a (divergence- and curl-free) solution of the Laplace equation, i.e., $\phi_0(\vec{x})$ differs from $\phi_1(\vec{x})$ by a constant value and from $\phi_2(\vec{x})$ by a linear function, both depending on $\vec{x}_0$.

Trying to extend the range of validity even further, one may subtract the next (third) term in the expansion (2.4) from G_2 and obtain

$$G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) - \frac{1}{2}\big((\vec{x} - \vec{x}_0) \cdot \vec{\nabla}'\big)^2 \frac{1}{|\vec{x}' - \vec{x}_0|}. \tag{2.9}$$

However, now G_3 fulfills the Poisson equation

$$\Delta G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -4\pi\big[\delta(\vec{x}' - \vec{x}) - \delta(\vec{x}' - \vec{x}_0)\big] \tag{2.10}$$

from which it follows that G_3 leads to a solution of a modified Poisson equation

$$\Delta\phi_3(\vec{x}) = -4\pi\big[\rho(\vec{x}) - \rho(\vec{x}_0)\big]. \tag{2.11}$$

Thus, the method described here is not suitable for Green functions G_i with $i > 2$. This means (as we will see later) that vector fields which increase linearly or even stronger will not be decomposed by the regularization method described here.

Nevertheless, one should note that the Poisson equation can be solved even with G_3 if we subtract the solution for the inhomogeneity $\rho(\vec{x}_0)$

$$\bar{\phi}_3(\vec{x}) = \int d^3x' \, \rho(\vec{x}')G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) - 2\pi\frac{\rho(\vec{x}_0)}{3}|\vec{x} - \vec{x}_0|^2. \tag{2.12}$$

The relation

$$\vec{\nabla}G_{i+1}(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -\vec{\nabla}'G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0), \qquad i \geqslant 0, \tag{2.13}$$

can be derived from (2.5), (2.6) and (2.9). They are used a few times, mainly to compute the vector fields $\vec{v}_l$ and $\vec{v}_t$ and to establish relations between them.

We would like to note that in higher order iterations of the regularization beyond $i = 3$ for the singularity of $G_i \sim |\vec{x}' - \vec{x}_0|^{-i}$ at $\vec{x}' = \vec{x}_0$, a convergence of the solution can only be reached if the sources vanish sufficiently strongly at the regularization points. In the following examples we will restrict ourselves to a regularization for $i \leqslant 2$ at the point $\vec{x}_0 = 0$, because the Green functions are simpler without loss of generality. In this case, the scalar potential is fixed to $\phi_i(\vec{x} = 0) = 0$ for $i = 1, 2$. We will keep this choice in the remaining part of the paper as far as possible.

3. The extended fundamental theorem of vector analysis

It has already been noted that today the formulation of the fundamental theorem rests in its form on the work of Blumenthal. However, there are several reasons not to take the formulations of Blumenthal resp. Sommerfeld literally. For instance, the uniqueness of the decomposition into the fields of the sources and vortices was only shown up to a constant vector. We will formulate the conditions in such a form that a strict uniqueness of the decomposition is given. Furthermore, in the proof that will be given below, the potentials by which the decomposed fields are calculated are part of the theorem (strong version). It is common in electrodynamics to calculate the physical fields via the introduction of potentials.

Thus, we formulate the theorem in the following way:

Let $\vec{v}(\vec{x})$ be piecewise continuous differentiable vector field, then the decomposition

$$\vec{v}(\vec{x}) = \vec{v}_l + \vec{v}_t = -\vec{\nabla}\phi_H(\vec{x}) + \vec{\nabla} \times \vec{A}_H(\vec{x}) \tag{3.1}$$

reads

$$\phi_{\mathrm{H}}(\vec{x}) = -\frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{v}(\vec{x}') \cdot \vec{\nabla}' G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0), \tag{3.2}$$

$$\vec{A}_{\mathrm{H}}(\vec{x}) = \frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{v}(\vec{x}') \times \vec{\nabla}' G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0), \tag{3.3}$$

where i is taken for the asymptotic behavior $\lim_{r \to \infty} v(r) \, r^{1-i+\epsilon} < \infty$, with $\epsilon > 0$. This decomposition is unique for $i = 0, 1$ and unique apart for a constant vector field for $i = 2$.

Remarks:

- Curl- and divergence-free fields $\vec{v}_h$ can be added to $\vec{v}_l$ if they are subtracted from $\vec{v}_t$ without affecting the boundary conditions of $\vec{v}$. Such harmonic vector fields are suppressed if one explicitly demands that $\vec{v}_l$ and/or $\vec{v}_t$ should vanish asymptotically and establish a strict uniqueness of the decomposition.

- Usually, the potentials $\phi_{\mathrm{H}}(\vec{x})$ and $\vec{A}_{\mathrm{H}}(\vec{x})$ are defined with the Green function G_0 (2.2). If they are finite, then there is no need for G_1 (2.5). However, if the vector field $\vec{v}$ decays asymptotically as $1/r$ or weaker, one generally should use the Green function G_1 as shown in (3.2) and (3.3) in order to avoid divergences in the potentials $\phi_{\mathrm{H}}(\vec{x})$ and $\vec{A}_{\mathrm{H}}(\vec{x})$.

- As already mentioned in section 2, the potentials are for $i = 1, 2$ fixed to the values $\phi_{\mathrm{H}}(\vec{x}_0) = 0$ and $\vec{A}_{\mathrm{H}}(\vec{x}_0) = 0$ by the choice of the regularization point $\vec{x}_0$, and for $i = 1$ this choice does not affect the vector fields $\vec{v}_l$ and $\vec{v}_t$, whereas for $i = 2$ $\vec{v}_l$ and $\vec{v}_t$ they vanish at the regularization point.

- The vector potential $\vec{A}$ by its definition is purely transversal, $\vec{\nabla} \cdot \vec{A}_{\mathrm{H}} = 0$.

- In the special case of the theorem where $\vec{v}$ approaches zero at infinity weaker than any power of $1/r$ (the case $\epsilon = 1$), then v_l and v_t may diverge logarithmically although the sum of the two parts decays to zero [9].

- We want to stress the point that the decomposition theorem holds for any vector field independent of the type of physical equations that the vector field might fulfill. On the other hand, if one thinks of the electric field or the magnetic field as examples of the theorem, due to the Maxwell equations, these fields turn out to be connected although in relation to the decomposition theorem they are independent. However, the potentials for the decomposed parts can be identified with these fields.

Let us define the source density $\rho_{\mathrm{H}}(\vec{x})$ and the vortex density $\vec{j}_{\mathrm{H}}(\vec{x})$ as

$$\rho_{\mathrm{H}}(\vec{x}) = \frac{\vec{\nabla} \cdot \vec{v}(\vec{x})}{4\pi}, \qquad \vec{j}_{\mathrm{H}}(\vec{x}) = \frac{\vec{\nabla} \times \vec{v}(\vec{x})}{4\pi}, \tag{3.4}$$

then decomposition of the corresponding vector field in its irrotational (curl-free) and solenoidal (divergence-free) parts leads to the following result:

$$\vec{\nabla} \cdot \vec{v}_l(\vec{x}) = 4\pi \rho_{\mathrm{H}}(\vec{x}) \quad \text{and} \quad \vec{\nabla} \times \vec{v}_l(\vec{x}) = 0, \tag{3.5}$$

$$\vec{\nabla} \times \vec{v}_t(\vec{x}) = 4\pi \vec{j}_{\mathrm{H}}(\vec{x}) \quad \text{and} \quad \vec{\nabla} \cdot \vec{v}_t(\vec{x}) = 0. \tag{3.6}$$

The potentials, (3.2) and (3.3), can be rewritten by partial integration if the vector fields are everywhere continuously differentiable

$$\phi_{\mathrm{H}}(\vec{x}) = \int \mathrm{d}^3 x' \, \rho_{\mathrm{H}}(\vec{x}') G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0), \qquad \vec{A}_{\mathrm{H}}(\vec{x}) = \int \mathrm{d}^3 x' \, \vec{j}_{\mathrm{H}}(\vec{x}') G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0). \tag{3.7}$$

The main advantage of the extended theorem lies in the resulting systematic procedure of calculating the respective quantities. This is done in the following way: One may start the integration with G_0 for a **finite** volume V. If the integral does not converge in the limit $V \to \infty$, one should subtract the value of the

already calculated quantity for the finite volume taken at the regularization point and then perform the limit $V \to \infty$, and so on. Computing the scalar potential ϕ_{H} in (3.2) with G_2 needs no further integration

$$\phi_{\mathrm{H}}(\vec{x}, \vec{x}_0) = \lim_{V \to \infty} \left[\hat{\phi}_0(\vec{x}) - \hat{\phi}_0(\vec{x}_0) - (\vec{x} - \vec{x}_0) \cdot \vec{\nabla}_0 \hat{\phi}_0(\vec{x}_0) \right], \tag{3.8}$$

where $\hat{\phi}_0(\vec{x})$ is the scalar potential calculated with G_0. It turns out that although the vector field is decaying at infinity of an order where a regularization seems to be necessary, the integrals might still converge and a further regularization is not necessary. The radiation field is such an example (see section 4.1).

3.1. Sketch of the proof

We do not present explicit steps of the proof (for this see [16]), but in order to show that the different integrals, which arise in (3.2) and (3.3), exist and are finite, we separate the volume of integration into an inner volume of a sphere with radius $R \gg r$ and the outer domain $r' \geqslant R$. Now, the large r behavior of the corresponding G_i is taken into account to prove the convergence. We note that the singularities at $\vec{x}$ and at zero do not lead to a diverging contribution to the integral, as long as $i \leqslant 2$. If the contribution of the outer domain to the potential vanishes, then the existence of $\phi_{\mathrm{H}}(\vec{x})$ has been proved.

If the finiteness of the scalar potential (3.2) is affirmed, one gets the field $\vec{v}_l$ by calculating the gradient of ϕ_{H}. This field has the same sources as $\vec{v}$ ($\vec{\nabla} \cdot \vec{v}_l = \vec{\nabla} \cdot \vec{v}$) and it is irrotational because the curl of a gradient field always vanishes. Subsequently, one proceeds quite similarly for the vortex field $\vec{A}_{\mathrm{H}}(\vec{x})$.

Finally, we check that the sum $\vec{v}_l + \vec{v}_t = \vec{v}$ apart from a constant vector for $i = 2$: At first we switch in (3.2) and (3.3) from $\vec{\nabla}' G_i$ to $-\vec{\nabla} G_{i+1}$ according to (2.13). One obtains for the sum of $\vec{v}_l + \vec{v}_t$ using (2.7) and (2.10) for $x_0 = 0$

$$\vec{v}(\vec{x}) + \vec{v}_t(\vec{x}) = -\frac{1}{4\pi} \int \mathrm{d}^3 x' \left[\vec{\nabla}\vec{\nabla} \cdot \vec{v}(\vec{x}') - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{v}(\vec{x}') \right) \right] G_{i+1}(\vec{x}, \vec{x}') = -\frac{1}{4\pi} \int \mathrm{d}^3 x' \, \vec{v}(\vec{x}') \Delta G_{i+1}(\vec{x}, \vec{x}')$$
$$= \vec{v}(\vec{x}) - \vec{v}(0) \delta_{i,2} \,. \tag{3.9}$$

3.2. Comments on the uniqueness

We have decomposed the vector field $\vec{v}$ in a source field $\vec{v}_l$ and a vortex field $\vec{v}_t$, under the boundary condition that the total field $|\vec{v}|$ vanishes going to infinity. In order to reach a uniqueness of the decomposition, we demand that $|\vec{v}_l|$ and consequently also $|\vec{v}_t|$ vanish going to infinity. The respective differences of the longitudinal and transversal decomposition parts are divergence- and curl-free and, hence, the harmonic solutions of the Laplace equation. Due to the boundary condition in infinity, they should be zero and the differences of the vector fields are zero and the decomposition is unique.

An exception should be made in the case when $i = 2$ is chosen. Then, the difference in the vector fields could be a linear harmonic function resulting in a uniqueness up to a linear term (see again for more details in [16]).

4. Application to time dependent fields and diverging fields

4.1. The radiation field

When Blumenthal published his extension of the Helmholtz theorem, he pointed to the field of electromagnetic waves, noting that it is of the $\mathrm{O}(1/r)$ and remarked: *In consequence, for vector fields of this kind, the theorem in his present formulation would not be applicable.*[1] Due to Blumenthal's proof, however, the theorem is applicable to such vector fields.

Usually, the theorem is not applied to time dependent problems in textbooks on electrodynamics, whereas it is used in textbooks on hydrodynamics. One reason might have been that the vector fields are solutions of Maxwell's equations which are relativistic contrary to the equations of classical fluid dynamics.

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[1]*Auf derartige Vektoren wäre also, der Satz in seiner bisherigen Ausdehnung bereits nicht anwendbar.*

As late as the beginning of the 21st century, this problem with the conventional formulation of Helmholtz theorem was taken up, without knowledge of Blumenthal's paper. In literature one can find a discussion of the question whether the theorem can be applied to retarded fields. It was thought that this mathematical theorem could come into conflict with causality in the case of the propagation of time dependent vector field with finite velocity. The appearance of the quasistatic potentials has led to this discussion in the case of Coulomb gauge. However it was recognized earlier [17] and confirmed later [18], that the physical quantities are causal and the decomposition is valid also for time dependent (retarded) fields. Rohrlich [18] (see also [17] and references therein) argued that the theorem can be applied to vector field of any time dependence, without referring to Blumenthal's paper and without mentioning the weaker decay of the radiation field.

However, the discussion went on considering the expressions of different options to choose the potentials for electromagnetic fields and it was shown by Jackson (see [19] and references therein) that quasistatic potentials can also be used. Nevertheless, the question was taken up again quite recently in a paper by Stewart [20] with the title *"Does the Helmholtz theorem of vector decomposition apply to the wave fields of electromagnetic radiation?"*. Since also in this paper Blumenthal's proof is not mentioned, the validity of Helmholtz decomposition is performed explicitly. This explicit calculation shows, on the other hand, that no regularization is necessary due to the appearance of e^{ikr}/r terms in the integrals. Unfortunately, the author takes this property, which comes from the retardation, as an argument for nonconvergence of the integrals appearing in the Helmholtz decomposition for vector fields behaving as $1/r$.

Radiation fields, which decay asymptotically as $1/r$, are rarely connected with the decomposition theorem. If one starts with the assumption that the asymptotic behavior of the field should be stronger than $1/r$, additional properties of the field are needed in order to prove the decomposition of the radiation fields [20]. Let us now show decomposition as an example of an oscillating point dipole. We also point to the differences in the meaning of different quantities such as sources and potentials within the theorem, (they are subscripted by "H") and those quantities appearing in Maxwell's equations and the potentials introduced to solve these equations. Strictly speaking, the conditions of the theorem are not fulfilled if the vector field has singularities. This also holds for the radiation fields considered below. However, the integration over the sources in (3.2) and (3.3) remains finite.

The periodically moving charge densities $\rho(\vec{x}, t) = \rho(\vec{x}) e^{-i\omega t}$ of frequency ω emit a radiation field of the same frequency. For simplicity, we use the complex notation supposing that the physical quantities (charge density, potential, fields) are always real parts of the corresponding complex quantities. The radiation fields factorize in the same way as the sources $\vec{v}(\vec{x}, t) = \vec{v}(x) e^{-i\omega t}$, where in $\vec{v}(\vec{x})$, the dependence on the frequency ω resp. wave number $k = \omega/c$ has been suppressed.

The electric radiation field $\vec{E}(\vec{x})$ of an oscillating point dipole $\vec{p}(t) = \vec{p} e^{-i\omega t}$ reads in Gaussian units {see [21], (8.4.5) and (8.4.6), p. 294. The time dependence is in this case contained in the Fourier factor $e^{-i\omega t}$}

$$\vec{v}_E(\vec{x}) \equiv \vec{E}(\vec{x}) = \frac{e^{ikr}}{r} \left\{ k^2 \vec{e}_r \times (\vec{p} \times \vec{e}_r) + \frac{1}{r^2} (1 - ikr) \left[3(\vec{p} \cdot \vec{e}_r) \vec{e}_r - \vec{p} \right] \right\}. \tag{4.1}$$

$\vec{e}_r = \vec{x}/r$ is the unit vector in the direction of $\vec{x}$, and $\vec{v}_E(\vec{x})$ is the spatial part of the electric field. We add the magnetic radiation field

$$\vec{v}_B(\vec{x}) \equiv \vec{B}(\vec{x}) = k^2 \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right) (\vec{e}_r \times \vec{p}). \tag{4.2}$$

Both fields are a solution of Maxwell's equations and fulfill

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = 4\pi \rho(\vec{x}), \qquad \vec{E}(\vec{x}) = \frac{i}{k} \left[\vec{\nabla} \times \vec{B}(\vec{x}) - \frac{4\pi}{c} \vec{j}(\vec{x}) \right], \qquad \vec{\nabla} \cdot \vec{B}(\vec{x}) = 0, \qquad \vec{\nabla} \times \vec{E}(\vec{x}) = ik\vec{B}(\vec{x}), \tag{4.3}$$

the source and vortex density should be [see (3.4)]

$$\rho_{EH}(\vec{x}) \,\widehat{=}\, \rho_p(\vec{x}), \qquad \rho_p(\vec{x}) = -\vec{p} \cdot \vec{\nabla} \delta(\vec{x}), \qquad \vec{j}_{EH}(\vec{x}) = \frac{ik}{4\pi} \vec{B}(\vec{x}), \qquad \vec{j}_p(\vec{x}) = -ick\vec{p}\delta(\vec{x}). \tag{4.4}$$

$\rho_p(\vec{x})$ is a localized charge of the static dipole and $\vec{j}_p(\vec{x})$ is the current density of the local oscillating dipole. The Helmholtz vortex density is extended in the whole domain decreasing for $r \to \infty$ with $1/r$. It can be identified with the spatial part of the magnetic radiation field $\vec{B}$ {see [21], (8.4.5) and (8.4.6), p. 294} apart from a factor, as expected from Faraday's law of induction. The wave number dependence in different quantities is caused by retardation. Surprisingly, in $\rho_{EH}(\vec{x})$, contrary to $\vec{j}_{EH}(\vec{x})$, it drops out. This asymmetry has already been discussed by Brill and Goodman [17]. Due to the absence of retardation in ρ_{EH}, the scalar potential is quasistatic

$$\phi_{EH}(\vec{x}) = -\frac{1}{4\pi} \int d^3 x' \, \vec{v}_E(\vec{x}') \cdot \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|} = \int d^3 x' \frac{\rho_{EH}(\vec{x}')}{|\vec{x}' - \vec{x}|} = \frac{\vec{p} \cdot \vec{e}_r}{r^2} = \phi^{\text{qstat}}(\vec{x}) = \phi_C(\vec{x}). \qquad (4.5)$$

Multiplying by the factor $e^{-i\omega t}$, one obtains the quasistatic (acausal) dipole potential $\phi_C(\vec{x}, t)$ as it is known using the Coulomb gauge. Hence, it is clear that the longitudinally decomposed vector field $\vec{v}_l$ is the quasistatic electric field of a point dipole

$$\vec{v}_{El}(\vec{x}) = -\vec{\nabla}\phi_{EH}(\vec{x}) = \left[-\vec{p} + 3(\vec{p} \cdot \vec{e}_r)\vec{e}_r \right] \frac{1}{r^3} = \vec{E}^{\text{qstat}}(\vec{x}) \qquad (4.6)$$

and does not contribute in the radiation zone to the electric field, which is purely transversal. The decomposition is finally shown by calculating the transversal part $\vec{v}_{Et} = \vec{\nabla} \times \vec{A}_{EH}$ according to the theorem of the vector potential

$$\vec{A}_{EH}(\vec{x}) = \frac{1}{4\pi} \int d^3 x' \, \vec{v}_E(\vec{x}') \times \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|} = \frac{i}{4\pi k} \int d^3 x' \left[\vec{B}(\vec{x}') \times \vec{\nabla}' - \frac{4\pi}{c} \vec{j}(\vec{x}') \right] \times \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|}$$

$$= \frac{i}{k}\vec{B}(\vec{x}) + \frac{\vec{e}_r}{r^2} \times \vec{p} = \frac{i}{k}\left[\vec{B}(\vec{x}) - \vec{B}^{\text{qstat}}(\vec{x}) \right], \qquad (4.7)$$

where $\vec{B}^{\text{qstat}}(\vec{x})$ is the quasistatic magnetic field of a point dipole. We have again used Maxwell's equations (4.3).

In electrodynamics, one never defines a vector potential for the electric field, but it is known from the Ampère-Maxwell-equation that the electric field outside the sources can be calculated via the curl of $\vec{B}$. However, this is just the way we can calculate the transverse vector field

$$\vec{v}_{Et}(\vec{x}) = \vec{\nabla} \times \vec{A}_{EH}(\vec{x}) = \vec{E}(\vec{x}) - \vec{E}^{\text{qstat}}(\vec{x}) = \vec{E}(\vec{x}) - \vec{v}_{El}(\vec{x}). \qquad (4.8)$$

The causal character of the total electric radiation field $\vec{v}_E(\vec{x})$ is restored [18, 19]. This way of calculation is quite general and it is not only restricted to point sources.

The same decomposition may be done for the magnetic radiation field $\vec{B}(\vec{x})$, which, however, is trivial since the field is only transversal [see (4.3)]. The Helmholtz vortex density of the magnetic field is presented by (4.4)

$$\vec{j}_{BH}(\vec{x}) = \frac{1}{4\pi}\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{1}{c}\vec{j}(\vec{x}) - \frac{ik}{4\pi}\vec{E}(\vec{x}), \qquad (4.9)$$

which is the total electric current (including the displacement current).

The Helmholtz vector potential fulfills $\vec{\nabla} \cdot \vec{A}_{BH} = 0$ and $\vec{\nabla} \times A_{BH} = \vec{B}$, the same conditions as for the vector potential $\vec{A}_C$ in the Coulomb gauge. We indeed obtain for this example $\vec{A}_{BH}(\vec{x}) = \vec{A}_C(\vec{x})$ [16, 21].

$$\vec{A}_{BH}(\vec{x}) = \frac{1}{4\pi} \int d^3 x' \, \vec{v}_B(\vec{x}') \times \vec{\nabla}' \frac{1}{|\vec{x}' - \vec{x}|} = \frac{1}{ik}\left[\vec{E}(\vec{x}) - \vec{E}^{\text{qstat}}(\vec{x}) \right]. \qquad (4.10)$$

Thus, all the fields, the vector potential $\vec{A}_{BH}(\vec{x})$, the vortex field $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}_{BH}(\vec{x})$ and the vortex density $\vec{j}_{BH}(\vec{x}) = \vec{\nabla} \times \vec{B}(\vec{x})/4\pi$ decay asymptotically as $1/r$. This is a consequence of retardation. In the Helmholtz vector potentials for both radiation fields, one explicitly sees that the corresponding quasistatic parts without retardation are subtracted.

One may be surprised that all calculations for the radiation field could be performed without regularization as expected according to the order of the decay of the vector field. Anyway, the integrals converge in an explicit calculation [20]. This might happen in other cases too (G_0 instead of G_1 etc.). One reason lies in the symmetries of the sources and circulations. For instance, fields like $\vec{v}(\vec{x}) = \vec{p}/r$ need no regularization. On the contrary, for a vector field like $\vec{v}(\vec{x}) = \vec{e}_r/r$, the regularization term is necessary to reach convergence, but the regularization point $\vec{x}_0$ should be different from zero. In such a case, we get for the potential $\phi(\vec{x}) = \ln r_0 - \ln r$.

4.2. Finite or diverging fields in infinity

In order to demonstrate the extended theorem, we present two mathematical examples. Both of them have the same vectorial structure as the radiation field. Let us take first the vector field

$$\vec{v}(\vec{x}) = \vec{e}_r \times (\vec{a} \times \vec{e}_r), \tag{4.11}$$

where $\vec{a}$ is a constant vector. $\vec{v}$ is finite but nonzero in the limit $r \to \infty$. It seems to be more convenient to firstly determine sources and vortices and then to calculate the fields belonging to these

$$\rho_{\mathrm{H}}(\vec{x}) = -\frac{1}{4\pi}\frac{2}{r}(\vec{a}\cdot\vec{e}_r), \qquad \vec{j}_{\mathrm{H}}(\vec{x}) = \frac{1}{4\pi}\frac{1}{r}(\vec{e}_r \times \vec{a}). \tag{4.12}$$

We would like to note that the example includes some subtle items: (1) Since the vector field is not continuous at the origin, we should take $\vec{x}_0$ different from zero as regularization point. (2) Due to the vectorial character of the field, the singularity at zero in the source and vortices is approached differently. This is no obstacle for applying the theorem. Then, we get ϕ_{H} from (3.2) as follows:

$$\phi_{\mathrm{H}}(\vec{x}, \vec{x}_0) = \int \mathrm{d}^3 x' \, \rho_{\mathrm{H}}(\vec{x}') G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -\frac{2}{3}\left\{(\vec{a}\cdot\vec{x})(\ln r_0 - \ln r) - (\vec{a}\cdot\vec{e}_{r_0})\left[r_0 - (\vec{x}\cdot\vec{e}_{r_0})\right]\right\}. \tag{4.13}$$

To calculate the integrals, it is useful to introduce spherical coordinates. The analogous calculation for the vector potential (3.3) yields

$$\vec{A}_{\mathrm{H}}(\vec{x}, \vec{x}_0) = \frac{1}{3}\left\{(\vec{x} \times \vec{a})(\ln r_0 - \ln r) - (\vec{e}_{r_0} \times \vec{a})\left[r_0 - (\vec{x}\cdot\vec{e}_{r_0})\right]\right\}. \tag{4.14}$$

In the last step, i.e., the calculation of the decomposed vector fields, we get

$$\vec{v}_l(\vec{x}, \vec{x}_0) = -\vec{\nabla}\phi_{\mathrm{H}}(\vec{x}, \vec{x}_0) \equiv \vec{v}_l(\vec{x}) - \vec{v}_l(\vec{x}_0) \qquad \text{with} \qquad \vec{v}_l(\vec{x}) = \frac{2}{3}\left[-\ln r\,\vec{a} + \vec{v}(\vec{x})\right], \tag{4.15}$$

$$\vec{v}_t(\vec{x}, \vec{x}_0) = \vec{\nabla} \times \vec{A}_{\mathrm{H}}(\vec{x}, \vec{x}_0) \equiv \vec{v}_t(\vec{x}) - \vec{v}_t(\vec{x}_0) \qquad \text{with} \qquad \vec{v}_t(\vec{x}) = \frac{1}{3}\left[2\ln r\,\vec{a} + \vec{v}(\vec{x})\right]. \tag{4.16}$$

Thus, we have demonstrated that the vector field can be decomposed in its irrotational and solenoidal components, both diverging logarithmically. However, these terms cancel in the sum and it is indeed $\vec{v}_l(\vec{x}) + \vec{v}_t(\vec{x}) = \vec{v}(x)$. A similar calculation can be performed for the example of a sublinearly diverging vector field

$$\vec{v}(\vec{x}) = \sqrt{r}\,\vec{e}_r \times (\vec{a} \times \vec{e}_r). \tag{4.17}$$

Now, the vector field is continuous at the origin and, therefore, one is allowed to choose the regularization point $\vec{x}_0 = 0$. The decomposition reads

$$\vec{v}_l(\vec{x}) = -\frac{4\sqrt{r}}{7}\left[2\vec{a} + (\vec{a}\cdot\vec{e}_r)\vec{e}_r\right], \qquad \vec{v}_t(\vec{x}) = \frac{4\sqrt{r}}{7}\left[2\vec{a} + (\vec{a}\cdot\vec{e}_r)\vec{e}_r\right] + \vec{v}(\vec{x}). \tag{4.18}$$

5. Conclusion

We have presented the fundamental theorem of vector analysis (Helmholtz decomposition theorem) for vector fields decaying weakly and extended it to even sublinearly diverging vector fields by a systematic regularization procedure. Contrary to the original proof [9], we can distinguish between different cases. Note, however, that not only the decay of the vector field is important for introducing a regularization but also its symmetry. So, it might be the case that due to symmetry reasons, a lower level of regularization can be used in the decomposition as might have been expected just looking at the order of the decay of the vector field.

Thus, considering the validity of Helmholtz decomposition theorem, there is no doubt that the theorem can be quite generally applied to electromagnetic fields either static or dynamic. Because of the

relevance of this extension of the Helmholtz decomposition theorem for textbooks on electrodynamics and on mathematical physics, a pedagogical version has been given by one of the authors [22].

There are physical examples in electro- and magnetostatics with sources which extend to infinity and strength does not decay to zero there, like a charged straight wire. Usually, the fields of highly symmetric examples can be calculated in reduced geometry (e.g., in two dimensions). We only mention that the vector field for a charged xy-half plane $[\rho_H(\vec{x}) = \sigma \delta(z) \theta(x)]$

$$\vec{v}(\vec{x}) = \frac{\sigma}{4\pi} \left[\vec{e}_x \ln\left(x^2 + z^2\right) + \vec{e}_z \left(\pi \operatorname{sgn} z + 2 \arctan \frac{x}{z}\right) \right] \tag{5.1}$$

diverges logarithmically in the x-direction, but can be calculated using the formalism of the Helmholtz decomposition theorem.

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Теорема про розвинення Гельмгольца і регуляризаційне розширення Блюменталя

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Теорема про розвинення Гельмгольца для векторного поля зазвичай представляється з сильними обмеженнями на поле і лише для незалежних від часу полів. У 1905 р. Блюменталь показав, що розвинення є можливим для любого асимптотично слабоспадного векторного поля. Він використав у доведенні регуляризаційний метод, який можна було розширити для доведення теореми для векторних полів, що є асимптотично сублінійно висхідними. Результат Блюменталя застосовано до часовозалежних полів дипольного випромінення і штучного сублінійно висхідного поля.

Ключові слова: теорема Гельмгольца, векторне поле, електромагнітне випромінення